Task 9. Calculation of the matrix eigenvector for the given eigenvalue

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Task 9

Write a program that calculates eigenvector of the matrix **A** for the given eigenvalue.

Eigenvalue problem

- Eigenvalue equation for the square matrix $n \times n$

$$\boldsymbol{A}\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k \quad \Leftrightarrow \quad (\boldsymbol{A} - \lambda_k 1_n)\boldsymbol{x}_k = 0 \quad k = 1, 2, ..., n$$

  where $\boldsymbol{x}_i$ is the eigenvector $(n \times 1)$ of $\mathbf{A}$ matrix belonging to λ_i eigenvalue being the number, in general, the complex number; 1_n is the $n \times n$ unit matrix
- Definition: $\boldsymbol{X} = [\boldsymbol{x}_1 \ ... \ \boldsymbol{x}_n]$ is the matrix $n \times n$, where $\boldsymbol{x}_k$ its columns $n \times 1$ are the eigenvectors of the matrix A corresponding to consecutive eigenvalues λ_1,...,λ_n. It is noted that $\det \boldsymbol{X} \neq 0$, because eigenvectors $\mathbf{x}_k$ are linearly independent
- Spectral decomposition of the matrix $\mathbf{A}$

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1},$$

  where $\mathbf{D}$ is the diagonal matrix $(\mathbf{D})_{ij} = \lambda_i \delta_{ij}$ and δ_{ij} is the Kronecker delta
- Spectral decomposition of the inverse matrix $\mathbf{A}^{-1}$

$$\boldsymbol{A}^{-1} = \boldsymbol{X}\boldsymbol{D}^{-1}\boldsymbol{X}^{-1},$$

  where $\mathbf{D}^{-1}$ is the diagonal matrix and $(\mathbf{D}^{-1})_{ij} = \lambda_i^{-1}\delta_{ij}$

(In the context of) shifted inverse iteration method

- Eigenvalue equation of the shifted matrix $\boldsymbol{A}_s = \boldsymbol{A} - s\mathbb{1}_n$, where $s \in \mathbb{R}$ and $\boldsymbol{A}\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$

$$\boldsymbol{A}_s\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_k - s\mathbb{1}_n\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k - s\boldsymbol{x}_k = (\lambda_k - s)\,\boldsymbol{x}_k$$

  Shifted matrix $\boldsymbol{A}_s$ possesses the same eigenvectors as the matrix $\mathbf{A}$, and the eigenvalues of $\boldsymbol{A}_s$ are "shifted" by s, i.e. $\lambda_s = \lambda_k - s$

- Spectral decomposition of $\mathbf{A}_s$ and $\mathbf{A}_s^{-1}$ matrices

$$\boldsymbol{A}_s = \boldsymbol{X}\boldsymbol{D}_s\boldsymbol{X}^{-1} \qquad (\boldsymbol{D}_s)_{ij} = (\lambda_i - s)\,\delta_{ij}$$

$$\boldsymbol{A}_s^{-1} = \boldsymbol{X}\boldsymbol{D}_s^{-1}\boldsymbol{X}^{-1} \qquad (\boldsymbol{D}_s^{-1})_{ij} = \frac{\delta_{ij}}{\lambda_i - s}$$

- Consider *sample* vector $\boldsymbol{x}^{(0)} = \sum_{k=1}^{n} c_k\boldsymbol{x}_k$ ([1]), then

$$\boldsymbol{A}_s^{-1}\boldsymbol{x}_k = \frac{1}{\lambda_k - s}\boldsymbol{x}_k \quad \Rightarrow \quad \boldsymbol{A}_s^{-1}\boldsymbol{x}^{(0)} = \sum_{k=1}^{n}\frac{c_k}{\lambda_k - s}\boldsymbol{x}_k$$

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[1] Eigenvectors $\boldsymbol{x}_k$ are linearly independent, thus any vector can be expressed as their linear combination.

(In the context of) shifted inverse iteration method - cont.

- If matrix **A** possesses different eigenvalues, i.e. $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, then for $s \approx \lambda_i$ we have

$$\left|\frac{1}{\lambda_i - s}\right| \gg \left|\frac{1}{\lambda_k - s}\right| \quad k \neq i$$

$$\boldsymbol{A}_s^{-1}\boldsymbol{x}^{(0)} = \sum_{k=1}^{n} \frac{c_k}{\lambda_k - s}\boldsymbol{x}_k \approx \frac{c_i}{\lambda_i - s}\boldsymbol{x}_i \equiv \frac{c_i}{\epsilon}\boldsymbol{x}_i,$$

  where $\lambda_i - s \equiv \epsilon$; it is crucial that $c_i \neq 0$

- If $|c_i| \sim 1$, $|\epsilon| \ll 1$ and $\|\boldsymbol{x}^{(0)}\| \sim 1$, then for $s \to \lambda_i$, i.e. $\epsilon \to 0$, we have

$$\boldsymbol{A}_s\boldsymbol{x}_i = \frac{\epsilon}{c_i}\boldsymbol{x}^{(0)} \longrightarrow 0_{n\times 1},$$

  because this limit leads to $\boldsymbol{A}_s \to \boldsymbol{A}_{\lambda_i}$ and, as a consequence,

$$\boldsymbol{A}_{\lambda_i}\boldsymbol{x}_i = (\boldsymbol{A} - \lambda_i 1_n)\boldsymbol{x}_i = 0_{n\times 1}$$

Algorithm

1. Calculate $\boldsymbol{A}_s = \boldsymbol{A} - s1_n$, where $s = \lambda_i - \epsilon$ and $\epsilon \sim 0$
2. Perform decomposition $\boldsymbol{A}_s = \boldsymbol{LUP}$ (Doolittle's method)
3. For the sample vector $\boldsymbol{x}^{(0)}$, where $\|\boldsymbol{x}^{(0)}\| = 1$ solve the equation

$$\boldsymbol{A}_s \boldsymbol{y}^{(1)} = \boldsymbol{x}^{(0)}$$

   (apply LUP decomposition)
4. Normalize $\boldsymbol{y}^{(1)}$, i.e. $\boldsymbol{x}^{(1)} = \frac{\boldsymbol{y}^{(1)}}{\|\boldsymbol{y}^{(1)}\|}$, then $\|\boldsymbol{x}^{(1)}\| = 1$; $\boldsymbol{x}^{(1)}$ is the eigenvector $\boldsymbol{x}_i$ (or its subsequent approximation)
5. In general, points 3 and 4 should be repeated

Remarks

- Matrix $\mathbf{A}_s$ is almost singular for $\epsilon \sim 0$
- $\det \boldsymbol{A}_s \sim 0$, thus in Doolittle's method $\det \boldsymbol{L} = 1$ and $\det \boldsymbol{U} \sim 0$
- Minimal value of $\epsilon \sim \lambda_i \epsilon_{mach} \neq 0$, because we want to avoid $\det \boldsymbol{U} = 0$; on the other hand we want to have single-step method, which is possible for $\epsilon \sim 0$
- $\|\boldsymbol{y}\| \sim \frac{1}{|\epsilon|} \gg 1$, e.g. in double precision $\|\boldsymbol{y}\| \sim 10^{15}$, then $\|\boldsymbol{A}_s \boldsymbol{x}^{(1)}\| = \frac{\|\boldsymbol{x}^{(0)}\|}{\|\boldsymbol{y}^{(1)}\|} \sim 10^{-15}$
- Sample input vector has to be such that $\boldsymbol{x}^{(0)\,Tr} \boldsymbol{x}_i \neq 0$

Other approach

- Assume that **A** possesses different eigenvalues
- $\boldsymbol{A}_{\lambda_i}\boldsymbol{x}_i = 0_{n\times 1}$
- $\boldsymbol{A}_{\lambda_i}$ is singular, i.e. $\det \boldsymbol{A}_{\lambda_i} = 0$
- Doolittle's LUP decomposition with partial pivoting

  $$\boldsymbol{A}_{\lambda_i} = \boldsymbol{LUP},$$

  where $\det \boldsymbol{L} = 1$, $\det \boldsymbol{U} = 0$, the last row in $\boldsymbol{U}$ is filled with zeros
- System of linear equations for $\boldsymbol{x}_i$, i.e. $\boldsymbol{A}_{\lambda_i}\boldsymbol{x}_i = 0_{n\times 1}$, possesses infinite number of solutions and (at least) one arbitrary parameter
- Denote $\boldsymbol{Px}_i = \boldsymbol{x}'_i$, $\boldsymbol{Ux}'_i = \boldsymbol{y}_i$, then

  $$\boldsymbol{A}_{\lambda_i}\boldsymbol{x}_i = \boldsymbol{Ly}_i = 0_{n\times 1} \;\Rightarrow\; \boldsymbol{Ux}'_i = 0_{n\times 1}$$

  (this is because $\boldsymbol{L}$ is not singular and, as a consequence, $\boldsymbol{y}_i = 0_{n\times 1}$)

Algorithm "2"

1. Calculate $\boldsymbol{A}_{\lambda_i} = \boldsymbol{A} - \lambda_i 1_n$
2. Decompose $\boldsymbol{A}_{\lambda_i} = \boldsymbol{LUP}$ by means of Doolittle's method with partial pivoting
3. Solve the system of linear equations

   $$\boldsymbol{U}\boldsymbol{x}_i' = 0_{n\times 1}$$

   by setting $(\boldsymbol{x}_i')_n = 1$ the remaining elements $(\boldsymbol{x}_i')_k$, $k = 1, ..., n-1$ may by calculated with backward substitution method
4. Eigenvector $\boldsymbol{x}_i = \boldsymbol{P}^{Tr}\boldsymbol{x}_i'$
5. We can normalize the eigenvector, i.e. $\boldsymbol{x}_i \to \frac{\boldsymbol{x}_i}{\|\boldsymbol{x}_i\|}$

TO DO

1. Read the real matrix $\mathbf{A}_{n\times n}$ and its eigenvalues
2. Calculate eigenvectors $\boldsymbol{x}_i$ of $\mathbf{A}$ matrix for the corresponding eigenvalues λ_i
3. Check $\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$
4. Calculate eigenvectors for symmetric and non-symmetric matrices
5. Check the performance of the method for the matrix that possesses to the same eigenvalues

Examples

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix}$$

$$\lambda = 2.76300328661375, -1.72368589498208, -5.03931739163167$$

$$A = \begin{pmatrix} 3 & 0 & 2 & -2 \\ 2 & 0 & -2 & 2 \\ 0.475 & -0.65 & 4.5 & -1.625 \\ 1.1 & -1.4 & 0 & 2.5 \end{pmatrix}$$

$$\lambda = 1,\ 2,\ 3,\ 4$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 5 & 5 & 6 & 8 \end{pmatrix}$$

$$\lambda = -1.7292612617663754, -0.0437773119849116, 0.7322067668156992, 18.0408318069355893$$

$$A = \begin{pmatrix} 2 & 13 & -14 & 3 \\ -2 & 25 & -22 & 4 \\ -3 & 31 & -27 & 5 \\ -2 & 34 & -32 & 7 \end{pmatrix}$$

$$\lambda = 1,\ 1,\ 2,\ 3$$