Task 6. Condition number of matrix

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Task 6

Write a program that calculates the inverse matrix $\mathbf{A}^{-1}$ and its condition number $\mathrm{cond}(\mathbf{A})$ using the LU decomposition. Check the effect of scaling of the system of linear equations $\mathbf{Ax} = \mathbf{b}$ on the condition number $\mathrm{cond}(\mathbf{A})$ and on the accuracy of the solution of the system of linear equations.

Vector norm

- Properties of vector norm

$$
\begin{aligned}
&\|\mathbf{x}\| \geqslant 0 \\
&\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \\
&\forall_\gamma \; \|\gamma \mathbf{x}\| = |\gamma| \, \|\mathbf{x}\| \\
&\|\mathbf{x_1} + \mathbf{x_2}\| \leqslant \|\mathbf{x}_1\| + \|\mathbf{x}_2\|
\end{aligned}
$$

- Various definitions of vector norms

$$
\begin{aligned}
\|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i| \\
\|\mathbf{x}\|_2 &= \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \\
\|\mathbf{x}\|_\infty &= \max_{1 \leqslant i \leqslant n} |x_i|
\end{aligned}
$$

Matrix norm consistent with vector norm

$$\|\mathbf{Ax}\| \leqslant \|\mathbf{A}\|\|\mathbf{x}\|$$

Thus

$$\|\mathbf{A}\| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}$$

Properties of matrix norm

$$\begin{aligned}
&\forall_{\mathbf{x}} \|\mathbf{Ax}\| \leqslant \|\mathbf{A}\|\|\mathbf{x}\| \\
&\|\mathbf{A}\| \geqslant 0 \\
&\forall_{\gamma} \|\gamma\mathbf{A}\| = |\gamma|\|\mathbf{A}\| \\
&\|\mathbf{A}+\mathbf{B}\| \leqslant \|\mathbf{A}\| + \|\mathbf{B}\| \\
&\|\mathbf{AB}\| \leqslant \|\mathbf{A}\|\|\mathbf{B}\|
\end{aligned}$$

Definitions of matrix norms consistent with the particular vector norm

$$\|\mathbf{A}\|_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} |a_{ij}| \qquad \|\mathbf{A}\|_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|\mathbf{A}\|_2 = ?$$

Error analysis of system of linear equations: matrix condition number

If $\mathbf{A}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}+\delta\mathbf{b}$, to $\|\mathbf{b}\| \leqslant \|\mathbf{A}\|\|\mathbf{x}\|$ i $\|\delta\mathbf{x}\| \leqslant \|\mathbf{A}^{-1}\|\|\delta\mathbf{b}\|$, then

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leqslant \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} = \mathrm{cond}(\mathbf{A})\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

If $(\mathbf{A}+\delta\mathbf{A})(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}$, to $\|\delta\mathbf{x}\| \leqslant \|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\|\|\mathbf{x}+\delta\mathbf{x}\|$, then

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}+\delta\mathbf{x}\|} \leqslant \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} = \mathrm{cond}(\mathbf{A})\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}$$

- Matrix condition number $\mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$
- In the case of singular matrix $\mathrm{cond}(\mathbf{A}) = \infty$

Taken from: J. Kobus, Metody numeryczne, 2014/2015

Inverse matrix

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{1}$$

1. It is seen from the above definition that k-th column $\mathbf{x}_k$ of inverse matrix $\mathbf{A}^{-1}$ satisfies the following equation (system of linear equations)

$$\mathbf{A}\mathbf{x}_k = \mathbf{e}_k, \quad k = 1, 2, ..., n$$

   where $\mathbf{e}_k$ is k-th column of the unit matrix. Using LU decomposition we can calculate each column of the inverse matrix.

2. Alternatively, we can use the LU decomposition $\mathbf{A} = \mathbf{LUP}$ in the following way $\mathbf{A}^{-1} = \mathbf{P}^{-1}\mathbf{U}^{-1}\mathbf{L}^{-1}$. In such a case the columns of the inverse matrices $\mathbf{L}^{-1}$ and $\mathbf{U}^{-1}$ may be calculated applying forward- and backward-substitution methods for the following equations (systems of linear equations)

$$\mathbf{L}\mathbf{y}_k = \mathbf{e}_k \quad \mathbf{U}\mathbf{z}_k = \mathbf{e}_k \quad k = 1, 2, ..., n$$

   where $\mathbf{y}_k$ and $\mathbf{z}_k$ are the k-th columns of the $\mathbf{L}^{-1}$ and $\mathbf{U}^{-1}$, respectively, and $\mathbf{P}^{-1} = \mathbf{P}^{Tr}$.

Matrix condition number

- From definition

$$\mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$$

- In practice usually we try to avoid the calculation of the matrix inverse, in such a case we estimate the norm of matrix inverse $\|\mathbf{A}^{-1}\|$
If $\mathbf{Ax} = \mathbf{b}$, then $\|\mathbf{x}\| \leqslant \|\mathbf{A}^{-1}\|\|\mathbf{b}\|$. Thus

$$\|\mathbf{A}^{-1}\| \geqslant \frac{\|\mathbf{x}\|}{\|\mathbf{b}\|}$$

and

$$\mathrm{cond}(\mathbf{A}) \geqslant \|\mathbf{A}\|\frac{\|\mathbf{x}\|}{\|\mathbf{b}\|}$$

Scaling of the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 & \Big| : \max_{1 \leqslant j \leqslant n} |a_{1j}| \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 & \Big| : \max_{1 \leqslant j \leqslant n} |a_{2j}| \\ \quad\vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots & \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n & \Big| : \max_{1 \leqslant j \leqslant n} |a_{nj}| \end{cases}$$

$$\Downarrow$$

$$\mathbf{A}'\mathbf{x} = \mathbf{b}'$$

TO DO - modification of the program solving the system of linear equations (see Task 5)

1. From the input read size n and the elements of $\mathbf{A}$ matrix and the right-hand-side column $\mathbf{b}$
2. Perform LU decomposition of $\mathbf{A}$ matrix using Doolittle or Crout method with partial pivoting
3. Solve the system of linear equations $\mathbf{Ax} = \mathbf{b}$ and the scaled system of linear equations $\mathbf{A'x} = \mathbf{b'}$ using LU decomposition of $\mathbf{A}$ matrix with partial pivoting
4. Calculate the inverse matrices $\mathbf{A}^{-1}$ and $\mathbf{A'}^{-1}$ (check $\mathbf{AA}^{-1} = \mathbf{1}$ and $\mathbf{A'A'}^{-1} = \mathbf{1}$)
5. Cacluate condition number of the matrices **A** and **A'**
   - from definition
   - estimating
6. Check $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{A'x} = \mathbf{b'}$
7. Compare the results using existing functions (e.g. $x = A\backslash b$ and cond(A) in Matlab)

Perform calculations in single precision

Examples

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix} \qquad \mathbf{b}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -3 \\ 2 & -2 & 2 \end{pmatrix} \qquad \mathbf{b}_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{A}_3 = \begin{pmatrix} 21 & 0 & 770 & 0 & 50666 \\ 0 & 770 & 0 & 50666 & 0 \\ 770 & 0 & 50666 & 0 & 3956810 \\ 0 & 50666 & 0 & 3956810 & 0 \\ 50666 & 0 & 3956810 & 0 & 335462666 \end{pmatrix} \qquad \mathbf{b}_2 = \begin{pmatrix} 152789 \\ 102102 \\ 11921866 \\ 7964286 \\ 1010395474 \end{pmatrix}$$

(see `http://www.fizyka.umk.pl/~tecumseh/EDU/MNII/inp`)