Task 5. Systems of linear equations

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Task 5

Write a program that solves a system of linear equations $\mathbf{Ax} = \mathbf{b}$ with the aid of LU decomposition of $\mathbf{A}$ matrix. The program gets the number of equations, the elements of $\mathbf{A}$ matrix and the elements of the right-hand-side column $\mathbf{b}$ from the input.

Układ równań liniowych

- System of n of the linear equations with *unknowns* x_i, $i = 1, \ldots, n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \quad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

- Matrix form of the system of linear equations

$$\mathbf{Ax} = \mathbf{b},$$

where **A** is a matrix of factors, **x** - column of unknowns, **b** - column of *right-hand-sides*

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Solution of the system of linear equations with triangle matrix

- Consider the system of n linear equations $\mathbf{L}\mathbf{x} = \mathbf{b}$, where $\mathbf{L}$ is the lower triangle matrix
- We can solve such system of linear equations with forward-substitution method

$$
\begin{aligned}
x_1 &= b_1/\ell_{11} \\
x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22} \\
&\vdots \\
x_i &= (b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j)/\ell_{ii} \\
&\vdots \\
x_n &= (b_n - \sum_{j=1}^{n-1} \ell_{nj}x_j)/\ell_{nn}
\end{aligned}
$$

Solution of the system of linear equations with triangle matrix

- Consider the system of n linear equations $\mathbf{U}\mathbf{x} = \mathbf{b}$, where $\mathbf{U}$ is the upper triangle matrix
- We can solve such system of linear equations with backward-substitution method

$$
\begin{aligned}
x_n &= b_n / u_{nn} \\
x_{n-1} &= (b_{n-1} - u_{n-1,n} x_n) / u_{n-1,n-1} \\
&\vdots \\
x_i &= (b_i - \sum_{j=i+1}^{n} u_{ij} x_j) / u_{ii} \\
&\vdots \\
x_1 &= (b_1 - \sum_{j=2}^{n} u_{1j} x_j) / u_{11}
\end{aligned}
$$

Method of solution of the system of linear equations using LU decomposition

- Aim: solve the system of linear equations $\mathbf{Ax} = \mathbf{b}$
- We have performed $\mathbf{A} = \mathbf{LU}$ decomposition
- Substituting $\mathbf{LUx} = \mathbf{b}$ and by denoting $\mathbf{Ux} \equiv \mathbf{y}$, where $\mathbf{y}$ is a column $n \times 1$, then we have to solve two systems of linear equations with triangle matrices

  $$\mathbf{Ly} = \mathbf{b}$$

  and

  $$\mathbf{Ux} = \mathbf{y}$$

  with forward- and backward-substitution methods, respectively.

Inclusion of partial pivoting

- Doolittle method with partial pivoting leads to the following decomposition
$$\mathbf{A} = \mathbf{LUP},$$
where permutation matrix $\mathbf{P}$ is orthogonal, i.e. $\mathbf{P}^{-1} = \mathbf{P}^{Tr}$
- Solution of the system of linear equations
$$\mathbf{Ax} = \mathbf{b} \quad \Rightarrow \quad \mathbf{LUPx} = \mathbf{b}$$
  1. Solve $\mathbf{Ly} = \mathbf{b}$ with forward-substitution method
  2. Solve $\mathbf{Ux}' = \mathbf{y}$ with backward-substitution method, where $\mathbf{x}' = \mathbf{Px}$
  3. Solution $\mathbf{x} = \mathbf{P}^{Tr}\mathbf{x}'$

Inclusion of partial pivoting

- *Metoda Crouta z częściowym wyborem elementu głównego prowadzi do rozkładu*
$$A = PLU,$$
gdzie P *jest ortogonalną* ($P^{-1} = P^{Tr}$) *macierzą permutacji*
- *Rozwiązanie układu równań liniowych*
$$Ax = b \Rightarrow PLUx = b$$

1. *Rozwiązujemy* $Ly = b'$ *metodą podstawiania wprzód, gdzie* $b' = P^{Tr}b$
2. *Rozwiązujemy* $Ux = y$ *metodą podstawiania wstecz*

TO DO - modification of the program with LU decomposition (see Task 4)

1. From the input[1] read size n and the elements of $\mathbf{A}$ matrix
2. Perform LU decomposition of $\mathbf{A}$ matrix using Doolittle or Crout method with partial pivoting
3. Solve the system of linear equations $\mathbf{Ax} = \mathbf{b}$ using LU decomposition of $\mathbf{A}$ matrix with partial pivoting
4. Check $\mathbf{Ax} = \mathbf{b}$
5. Compare the results using existing functions (e.g. $x = A \backslash b$ in Matlab)

Perform calculations in single precision

[1] Or from file

Examples

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -3 \\ 2 & -2 & 2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 21 & 0 & 770 & 0 & 50666 \\ 0 & 770 & 0 & 50666 & 0 \\ 770 & 0 & 50666 & 0 & 3956810 \\ 0 & 50666 & 0 & 3956810 & 0 \\ 50666 & 0 & 3956810 & 0 & 335462666 \end{pmatrix} \quad b_2 = \begin{pmatrix} 152789 \\ 102102 \\ 11921866 \\ 7964286 \\ 1010395474 \end{pmatrix}$$

(see `http://www.fizyka.umk.pl/~tecumseh/EDU/MNII/inp`)