Mody wyższych rzędów

Dotąd omawialiśmy rozwiązanie najprostsze:

$$\Psi_0(r,z) = e^{-i\left[p(z)+\frac{k}{2q(z)}r^2\right]} \qquad \Psi_0(x,y,z) = e^{-i\left[p(z)+\frac{k}{2q(z)}(x^2+y^2)\right]}$$

Rozwiązanie ogólne dla symetrii „kwadratowej" ma postać:

$$\Psi_{nm}(x,y,z) = H_n\left(\sqrt{2}\frac{x}{w(z)}\right) H_m\left(\sqrt{2}\frac{y}{w(z)}\right) e^{-i(n+m)\arctan\frac{z}{z_0}} \cdot \Psi_0(x,y,z)$$

albo:

$$\Psi_{nm}(x,y,z) = H_n\left(\sqrt{2}\frac{x}{w(z)}\right) H_m\left(\sqrt{2}\frac{y}{w(z)}\right) \frac{w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} e^{-i\left[kz-(n+m+1)\arctan\frac{z}{z_0}+\frac{k}{2}\frac{x^2+y^2}{R(z)}\right]}$$

gdzie $q(z)$ i $w(z)$ oraz w_0 i z_0 mają dokładnie taką samą postać jak dotąd:

$H_n(x)$ są wielomianami Hermite'a odpowiedniego rzędu:

$H_0(x) = 1$

$H_1(x) = 2x$

$H_2(x) = 4x^2 - 2$

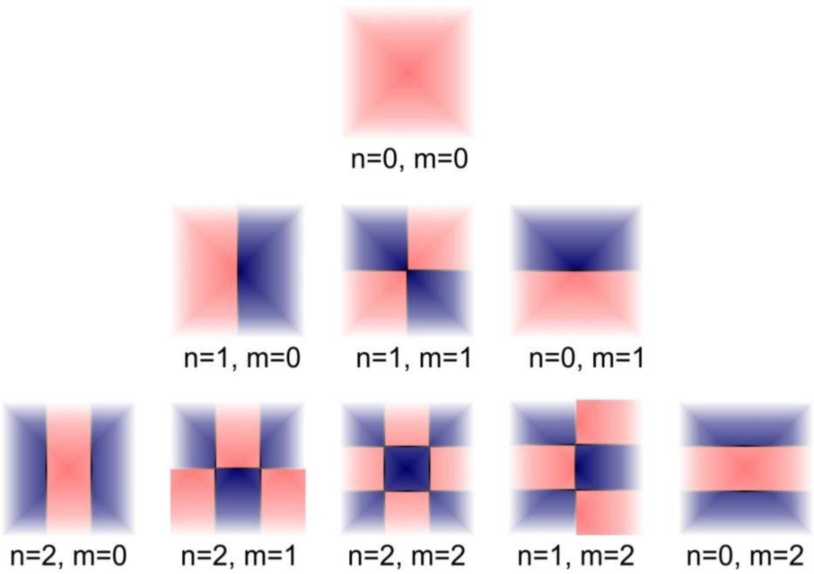
$H_3(x) = 8x^3 - 12x$

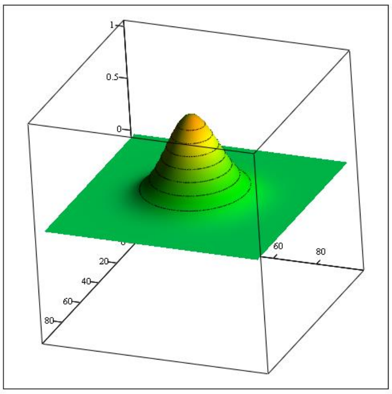
n jest ilością zer wielomianu

będące rozwiązaniami równania: $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \quad n = 0,1,2..$

ponadto: $\int_{-\infty}^{\infty} H_n(p)\, H_m(p) e^{-p^2} dp = \sqrt{\pi} 2^n n!\, \delta_{nm}$

Rozkład amplitudy pola jest teraz następujący:





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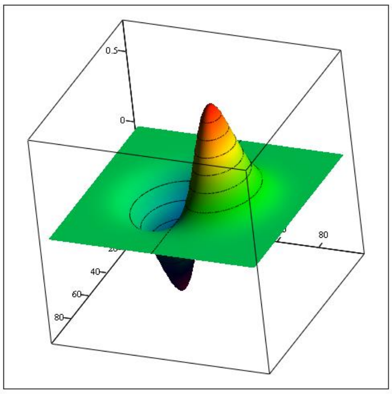

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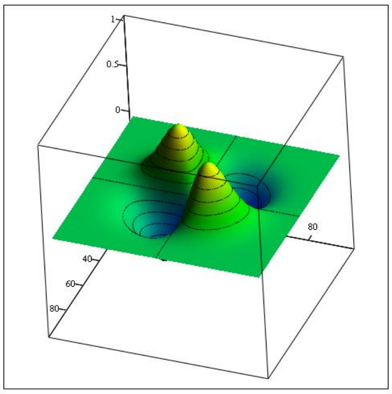

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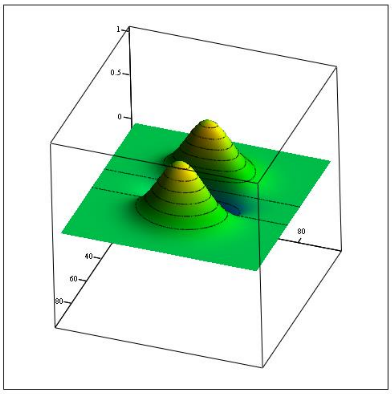

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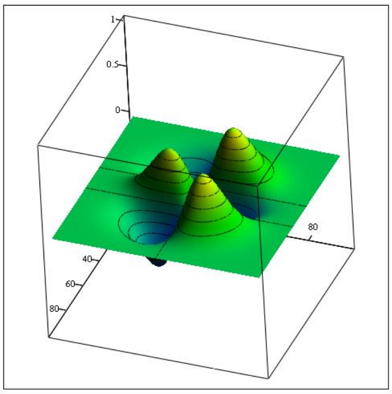

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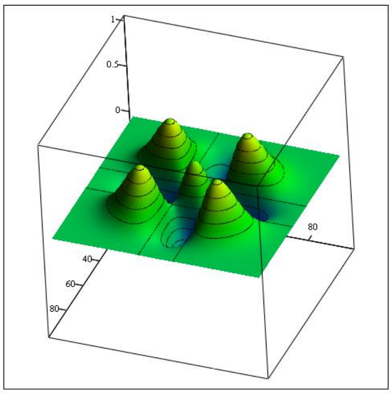

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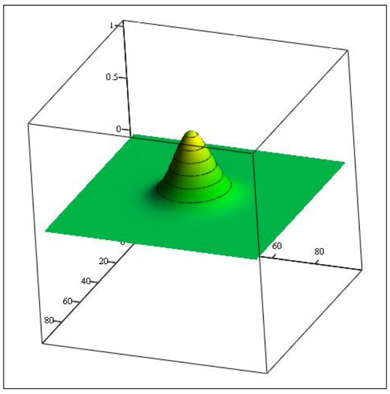

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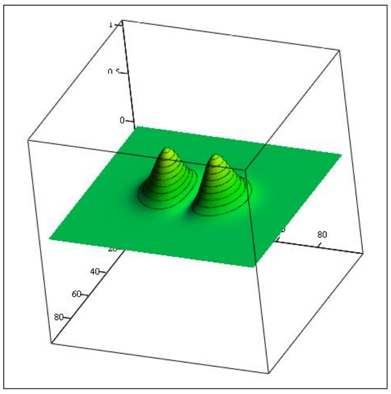

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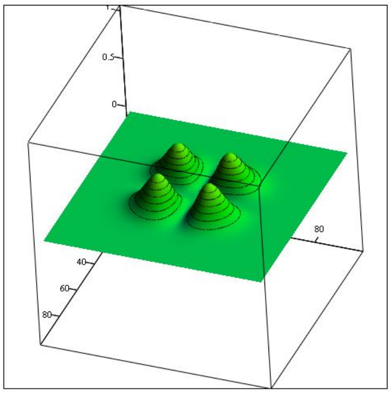

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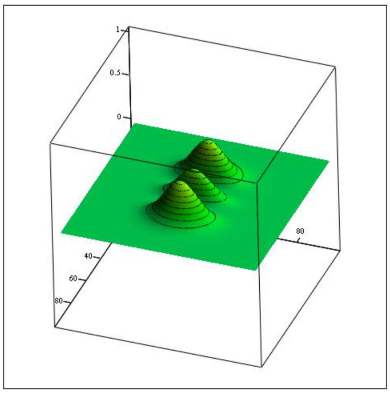

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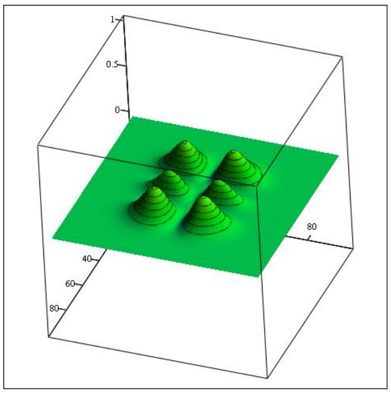

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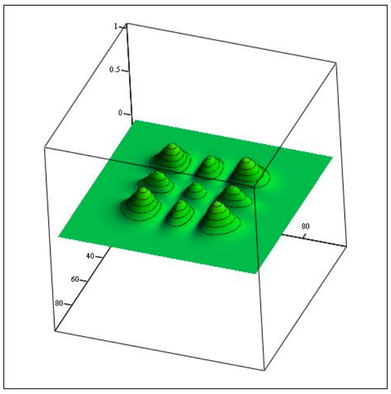

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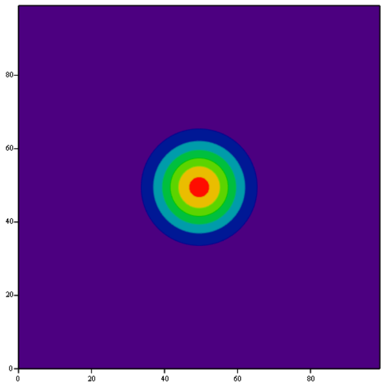
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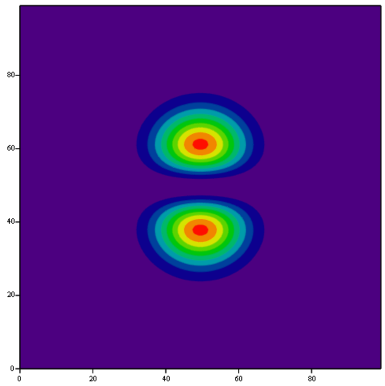
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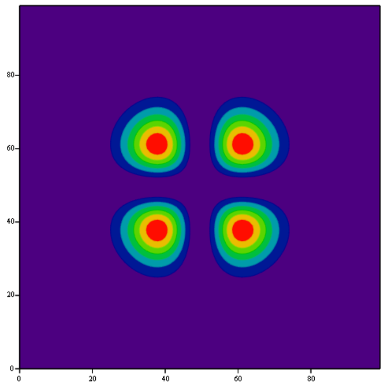
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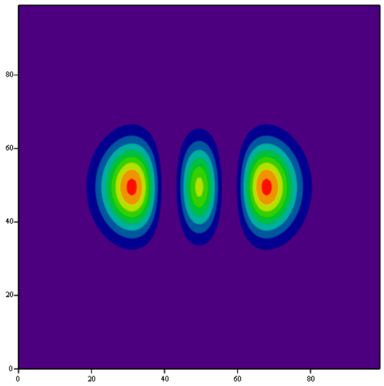
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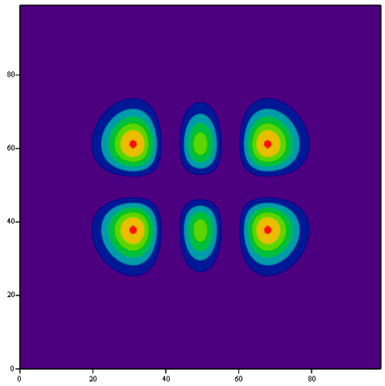
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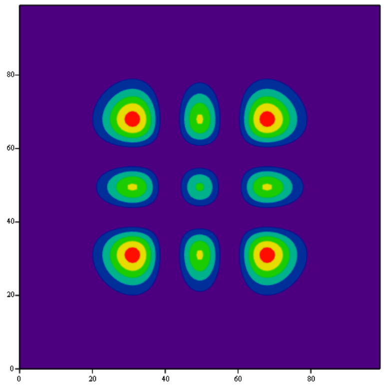
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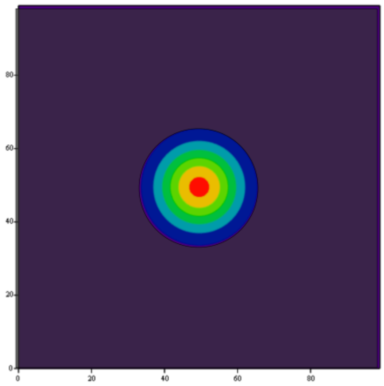
Przykład selekcji modów za pomocą przysłony: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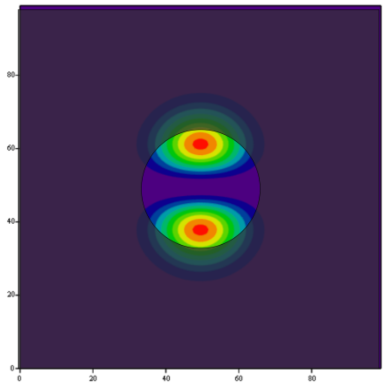
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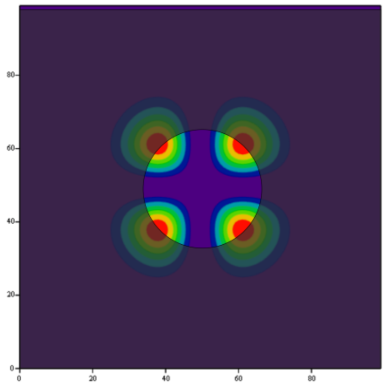
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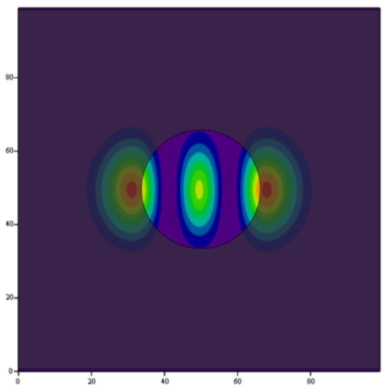
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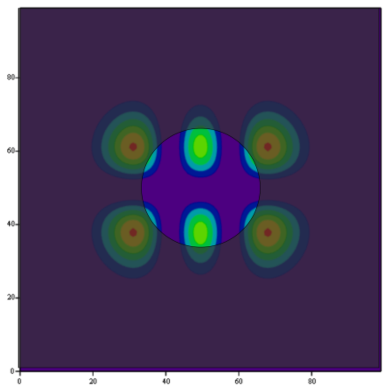
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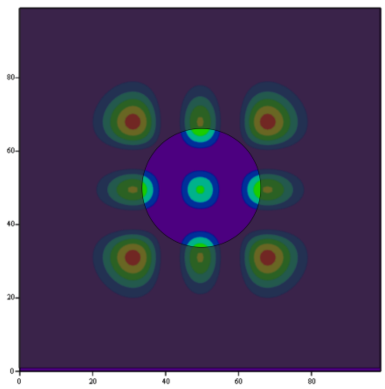
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Rozwiązanie ogólne dla symetrii obrotowej ma postać:

$$\Psi_{nm}(r,\phi,z) = \left(\sqrt{2}\frac{r}{w(z)}\right)^m L_n^m\left(2\frac{r^2}{w^2(z)}\right) e^{-i(2n+m)\arctan\frac{z}{z_0}} \cdot \cos(m\phi)\,\Psi_0(r,z)$$

albo

$$\Psi_{nm}(r,\phi,z) = \left(\sqrt{2}\frac{r}{w(z)}\right)^m L_n^m\left(2\frac{r^2}{w^2(z)}\right)\cos(m\phi)\frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\left[kz-(2n+m+1)\arctan\frac{z}{z_0}+\frac{k}{2}\frac{r^2}{R(z)}\right]}$$

gdzie $q(z)$ i $w(z)$ oraz w_0 i z_0 mają dokładnie taką samą postać jak dotąd,

$L_n^m(x)$ – uogólniony wielomian Laguerre'a:

$$L_0^m(x) = 1$$

$$L_1^m(x) = m+1-x$$

$$L_2^m(x) = \frac{1}{2}(m+1)(m+2)-(m+2)x+\frac{1}{2}x^2$$

n jest ilością zer wielomianu

Uwaga: uogólniony wielomian Laguerre'a jest rozwiązaniem równania:

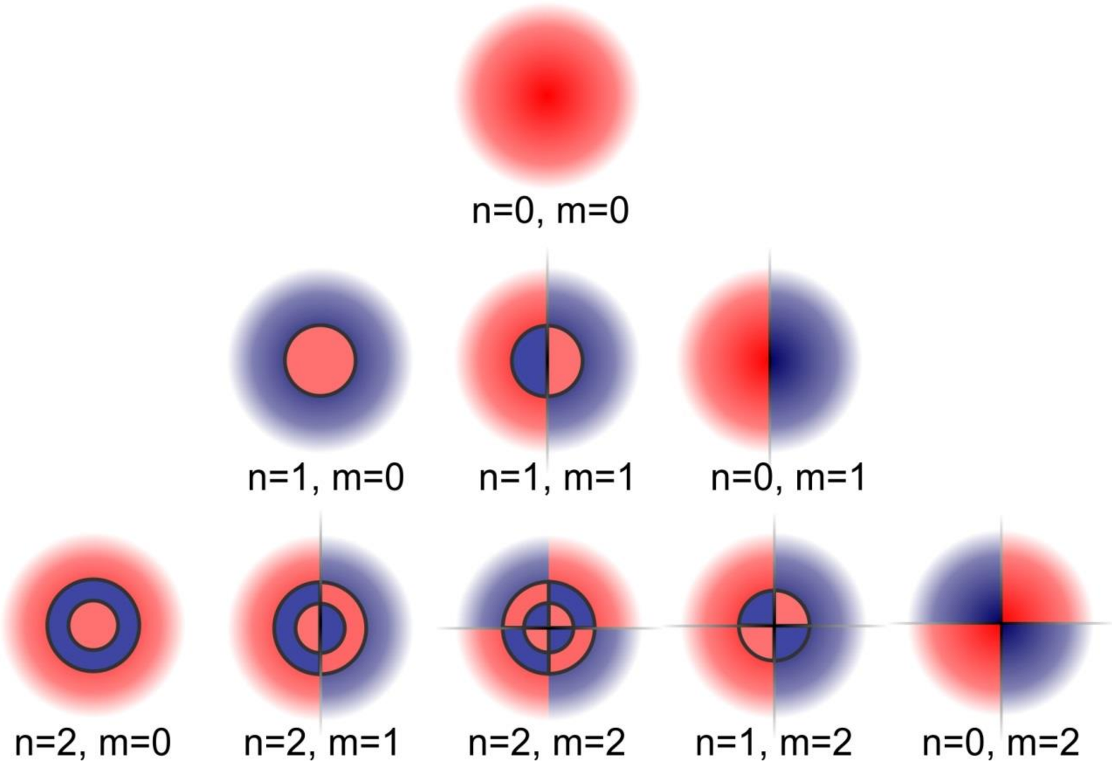
$$x^2\frac{d^2}{dx^2}L_n^m(x) + (m+1-x)\frac{d}{dx}L_n^m(x) + nL_n^m(x) = 0$$

ponadto:

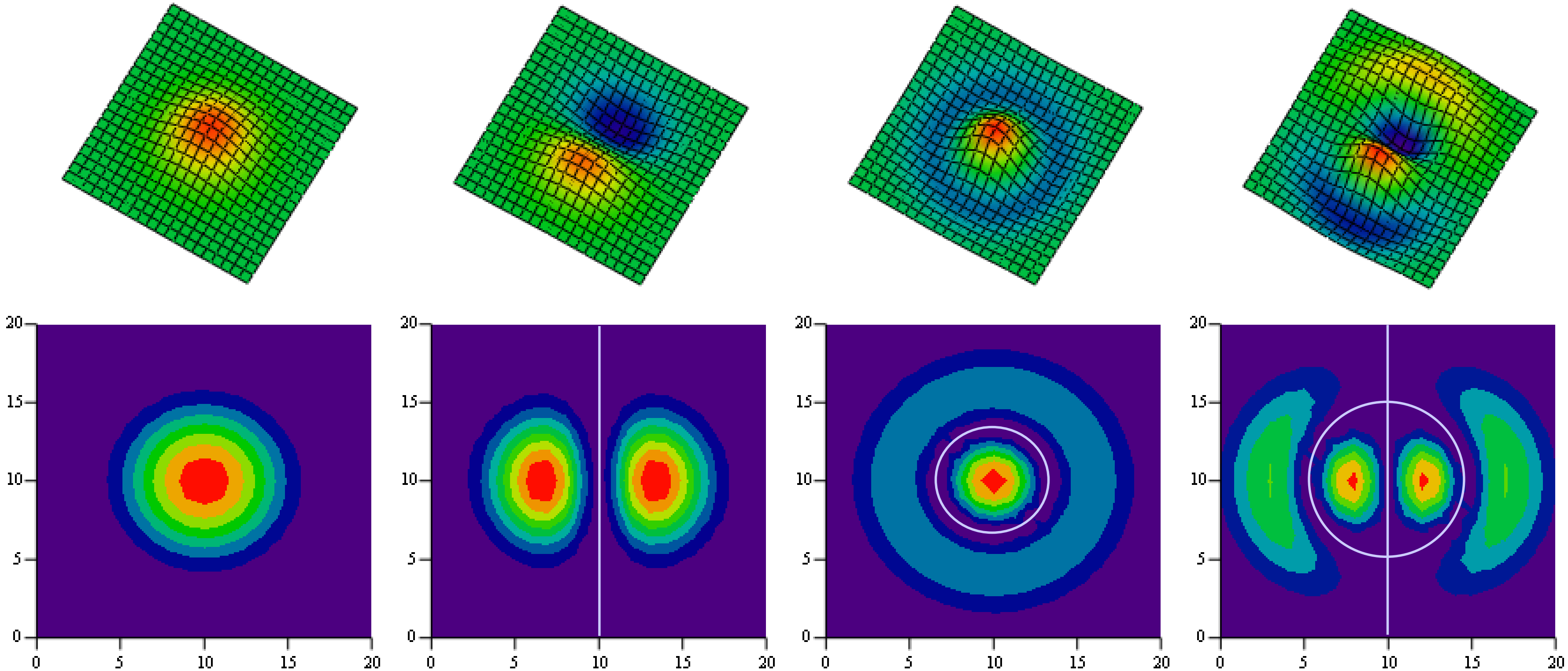
$$\int_0^\infty x^m e^{-x} L_n^m(x) L_p^m(x)\,dx = \frac{(n+m)!}{n!}\delta_{np}$$

$$\Psi_{nm}(r,\phi,z) = \left(\sqrt{2}\frac{r}{w(z)}\right)^m L_n^m\left(2\frac{r^2}{w^2(z)}\right)\cos(m\phi)\frac{w_0}{w(z)}e^{-\frac{r^2}{w^2(z)}}e^{-i\left[kz-(2n+m+1)\arctan\frac{z}{z_0}+\frac{k}{2}\frac{r^2}{R(z)}\right]}$$

Rząd modu jest ilością zer w rozkładzie amplitudy pola:



$$A(z,r,\phi,n,m)=\left(\sqrt{2}\,\frac{r}{w(z)}\right)^{m}\cdot L\left(\frac{2r^{2}}{w^{2}(z)},n,m\right)\cdot\cos(m\phi)\cdot\frac{w_{0}}{w(z)}\,e^{-\left[\frac{r}{w(z)}\right]^{2}}$$



$n=0$, $m=0$ | $n=0$, $m=1$ | $n=1$, $m=0$ | $n=1$, $m=1$