Teoria i metody optymalizacji

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Zasada minimum Pontriagina

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$\mathbf{x}(t=0) = \mathbf{x}_0,\quad \mathbf{x}(t=t_f) = \mathbf{x}_f$ dowolny, t_f dowolny

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*$$

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*$$

$$\left[\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}(t)\right]_{*t_f} \delta\mathbf{x}_f + \left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f = 0$$

Zasada minimum Pontriagina

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$||\mathbf{u}(t)|| \leq \mathbf{U}$$

$$|u_j(t)| \leq U_j \longrightarrow U_j^- \leq u_j(t) \leq U_j^+$$

Sterowanie optymalne

$$\mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t)$$

$$\Delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) = J(\mathbf{u}(t)) - J(\mathbf{u}^*(t)) \geq 0 = \delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) + O(*)$$

$$\delta J = \frac{\partial J}{\partial \mathbf{u}} \delta\mathbf{u}(t) \longrightarrow \delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) \geq 0.$$

Przypomnienie

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* \right.$$
$$\left. + \boldsymbol{\lambda}'(t) \left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\right] dt$$

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) dt$$

$$\Delta J = J_a(\mathbf{u}(t)) - J_a(\mathbf{u}^*(t))$$
$$\doteq \int_{t_0}^{t_f} (\mathcal{L}^\delta - \mathcal{L}) dt + \mathcal{L}|_{t_f} \delta t_f$$

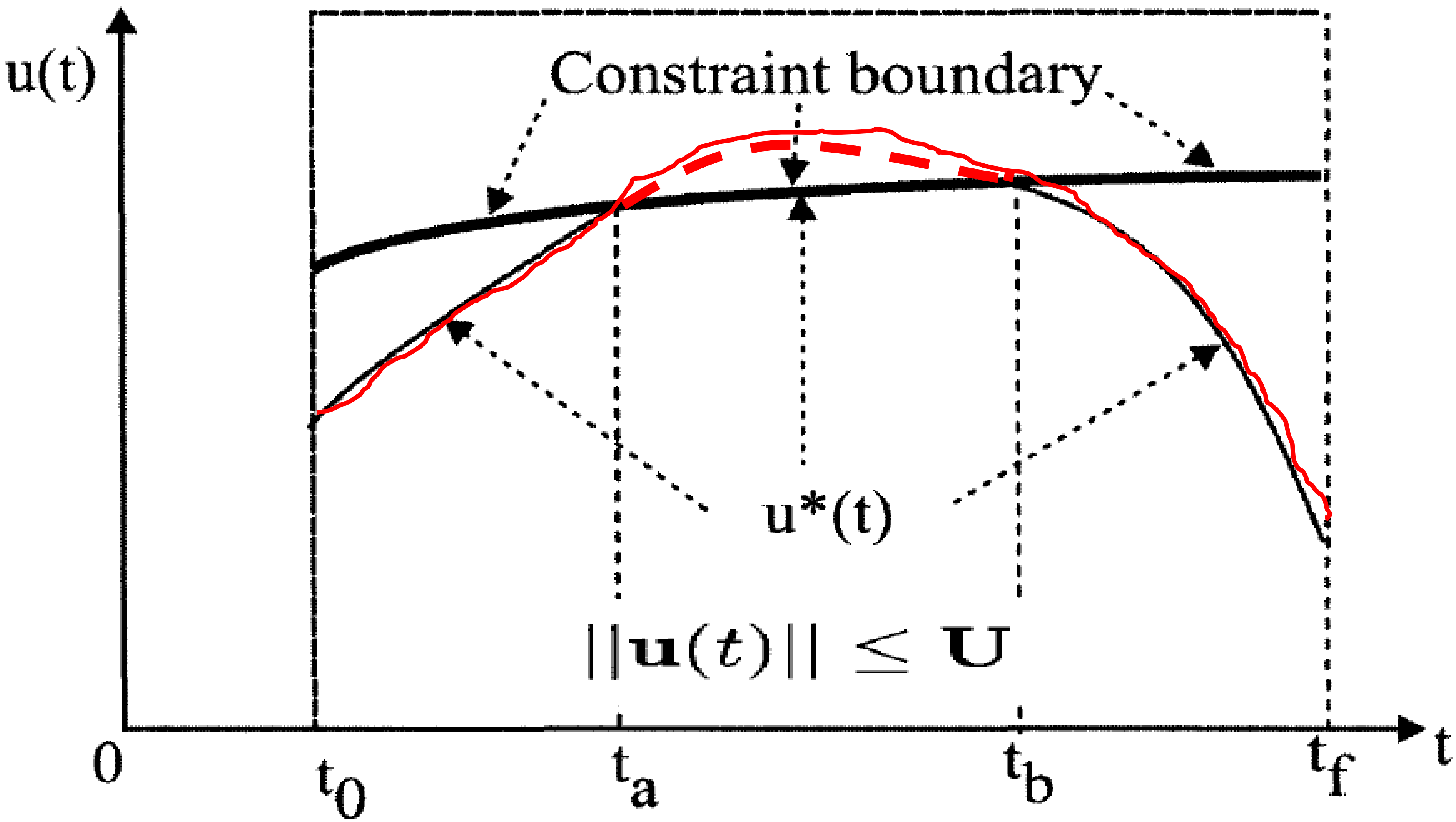
$$\delta J = \int_{t_0}^{t_f} \left\{ \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t) \right\} dt$$
$$+ \mathcal{L}|_{t_f} \delta t_f.$$

Zasada minimum Pontriagina

$$\delta J(\mathbf{u}^*(t), \delta\mathbf{u}(t)) = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial \mathcal{H}}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}(t) \right]_* \delta\mathbf{x}(t) + \left[\frac{\partial \mathcal{H}}{\partial \mathbf{u}} \right]_*' \delta\mathbf{u}(t) + \left[\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}} - \dot{\mathbf{x}}(t) \right]_*' \delta\boldsymbol{\lambda}(t) \right\} dt + \left[\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}(t) \right]_{*t_f}' \delta\mathbf{x}_f + \left[\mathcal{H} + \frac{\partial S}{\partial t} \right]_{*t_f} \delta t_f .$$

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Warunek $||\mathbf{u}(t)|| \leq \mathbf{U}$



Warunek konieczny zasady minimum Pontriagina

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \leq \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t)$$

Najlepsze z możliwego, aczkolwiek nie optymalne

$$\min_{|\mathbf{u}(t)| \leq \mathbf{U}} \left\{ \mathcal{H}\left(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t\right) \right\} = \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t).$$

Podsumowanie

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$\mathbf{x}(t=0) = \mathbf{x}_0, \quad \mathbf{x}(t=t_f) = \mathbf{x}_f$ wolny, t_f wolny

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$0 = +\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*$$

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*$$

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \le \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t)$$

$$\left[\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}(t)\right]_{*t_f} \delta \mathbf{x}_f + \left[\mathcal{H} + \frac{\partial S}{\partial t}\right]_{*t_f} \delta t_f = 0$$

Przykład

$$H = u^2 - 6u + 7 \qquad \frac{\partial H}{\partial u} = 0 \longrightarrow 2u^* - 6 = 0 \longrightarrow u^* = 3$$

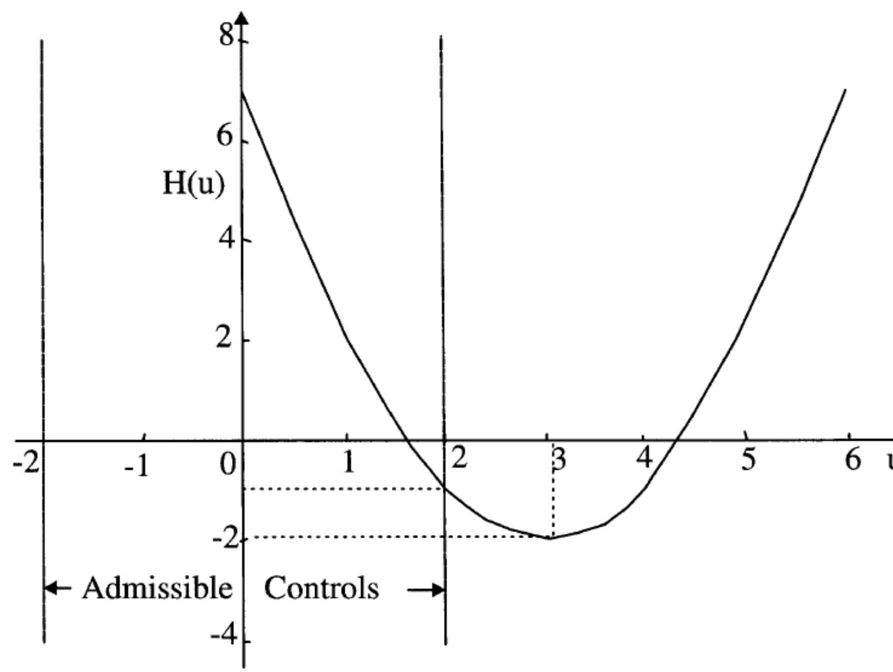
$$|u| \leq 2, \longrightarrow -2 \leq u \leq +2. \qquad H^* = 3^2 - 6\text{x}3 + 7 = -2.$$

$$H(u^*) \leq H(u),$$

$$H({u^*}^2 - 6u^* + 7) \leq H(u^2 - 6u + 7)$$

$$u^* = +2$$

$$H^* = 2^2 - 6\text{x}2 + 7 = -1$$



Przykład 2

$$\dot{x} = u \quad x(0) = 0 \quad \boxed{x(2) \to \max}$$

$$J = \int_0^2 u^2 dt \quad \boxed{|u| < 0.3}$$

$$F = u^2 = \dot{x}^2$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \lambda} = 0$$

$$-2\ddot{x} = 0$$

$$\dot{x} = C_1 = 0.5$$

$$x = C_1 t$$

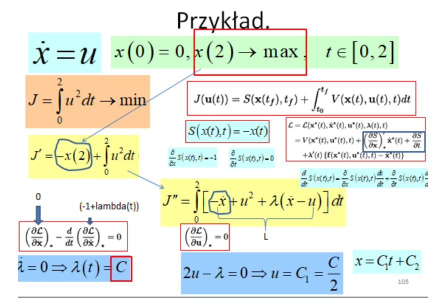
$$x(2) = 1$$

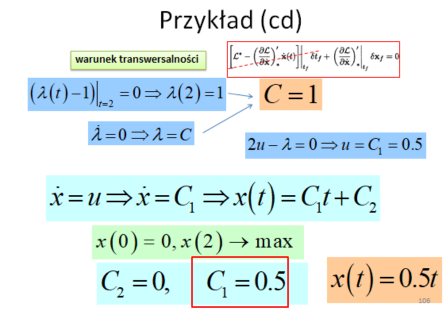
$$C_2 = 0$$

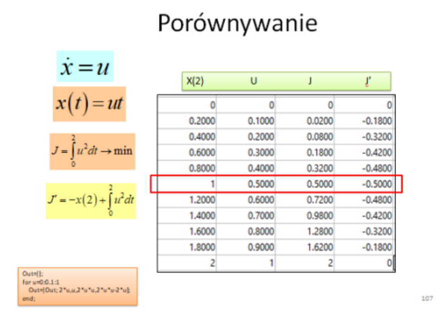
$$J = \int^2 0.5^2 dt = 0.5$$

$$J = \int_0^2 (0.3)^2 dt = 0.18$$

$$x(2) = 0.6$$







Przykład 3

$$\dot{x} = u \qquad \boxed{x\left(t_f\right) = 0 \quad \forall x(0)}$$

$$J = \int_0^{t_f} u^2 dt$$

$$t_f \to \min \Rightarrow S(x,t) = \frac{1}{2}t^2$$

$$J' = \int_0^{t_f} u^2 dt + \frac{1}{2}t^2$$

$$J = \int_0^{t_f} \left[t + u^2 + \lambda(\dot{x} - u)\right]dt = \int_0^{t_f} F dt$$

$$\frac{\partial F}{\partial u} = 2u - \lambda = 0$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \lambda = c$$

$$u = C_1 = \frac{1}{2}\lambda$$

$$x\left(t\right) = C_1 t + C_2$$

$$x(t_f) = 0 \Rightarrow C_1 t_f + C_2 = 0$$

$$\left[F - \frac{\partial F}{\partial \dot{x}}\dot{x}\right]_{t_f} = 0$$

$$t_f + C_1^2 - 2C_1^2 = 0$$

$$C_1^2 = t_f$$

$$C_1 t_f + C_2 = 0$$

PRZYKŁAD

$$\underline{x(0) = 10} = C_2$$

$$C_1^3 = -10$$

$$C_n = -\sqrt[3]{10} = -2.1544 = u$$

$$t_f = C_1^2 = 4.614\text{s}$$

$$x(t) = -2.1544t + 10$$

$$x\left(0\right) = 10$$

$$x(4.6414) = -2.1944 \cdot 4.6414 + 10 \approx 9.9994$$

Zagadnienie 1

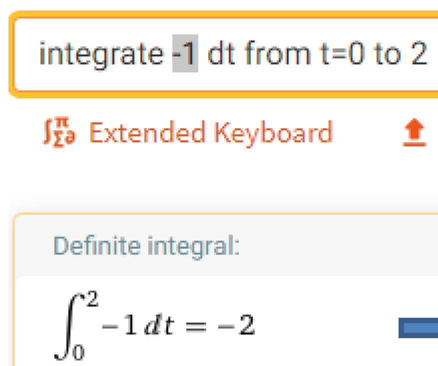
$$\dot{x} = u$$

$$x(0) = 0, x(2) \to \max, \quad t \in [0, 2]$$

$$-1 \le u \le 1$$

~~$J = \int_0^2 u^2 dt \to \min$~~

$$J = -x(2) = -x(0) + \int_0^2 (-\dot{x})\, dt = \int_0^2 [-\dot{x}]\, dt = \int_0^2 [-u(t)]\, dt \to \min$$



U=1

Zagadnienie 1(cd)

$$-1 \le u \le 1 \Leftrightarrow \left(u_{max} - u\right)\left(u - u_{min}\right) = \alpha^2 \ge 0$$

$$u_{min} = -1, \quad u_{max} = 1$$

$$J' = \int_0^2 \Big[\underbrace{-\dot{x} + \lambda_1\left(\dot{x} - u\right) + \lambda_2\left(\left(u_{max} - u\right)\left(u - u_{min}\right) - \alpha^2\right)}_{F}\Big]dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \qquad \dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = C$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}} = 0 \qquad -\lambda_1 + \lambda_2\left(2u\right) = 0$$

$$\frac{\partial F}{\partial \alpha} - \frac{d}{dt}\frac{\partial F}{\partial \dot{\alpha}} = 0 \qquad \lambda_2 \alpha = 0$$

warunek transwersalności (L)

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=2} = 0$$

$$\left.\left(\lambda_1(t) - 1\right)\right|_{t=2} = 0 \Rightarrow \lambda_1(2) = 1$$

$$\Rightarrow C = 1 \Rightarrow \lambda_1 = 1$$

Zagadnienie 1 (cd)

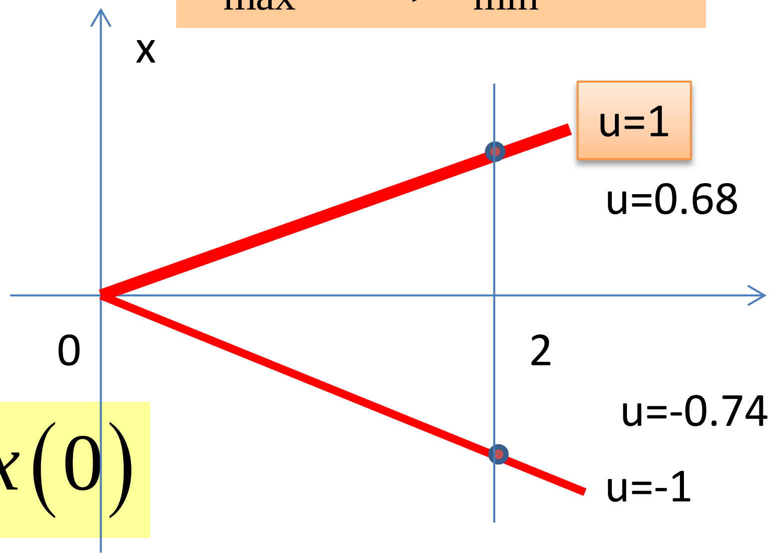
$$\begin{cases} \lambda_2 (2u) = 1 \\ \lambda_2 \alpha = 0 \xrightarrow{\times \alpha} \lambda_2 \alpha^2 = 0 \Rightarrow \lambda_2 (u_{\max} - u)(u - u_{\min}) = 0 \end{cases}$$

$$\lambda_2(t) \neq 0 \Rightarrow u = u_{\max} \quad \text{lub} \quad u = u_{\min}$$

$$u_{\max} = 1, u_{\min} = -1$$

$$x(0) = 0, x(2) \to \max, \quad t \in [0, 2]$$

?



$$x(t) = u \cdot t, u = -1, \Rightarrow x(2) = -2 < x(0)$$

$$x(t) = u \cdot t, u = 1, \Rightarrow x(2) = 2 > x(0)$$

$$u = 1$$

Czasooptymalne systemy. Przykład.

$$|u|<1$$

$$\dot{x}=u$$

$$x\left(t_f\right)=0\ ,\ \ \forall x(0)$$

$$t_f\rightarrow \min$$

$$J=\frac{1}{2}t_f^2$$

$$J'=\int_0^{t_f}\left(t+\lambda_1(\dot{x}-u)+\lambda_2\left((u+1)(1-u)-\alpha^2\right)dt\right.$$

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}=0\Rightarrow\quad \dot{\lambda}_1=0$$

$$\frac{\partial F}{\partial u}=0\quad -\lambda_1+\lambda_2(2u)=0$$

$$\frac{\partial F}{\partial \alpha}=0\quad \lambda_2\alpha=0$$

$$\left[F-\frac{\partial F}{\partial \dot{x}}\dot{x}\right]_{t_f}=0$$

$$\left[t+\lambda_1(\dot{x}-u)+\lambda_2\left((u+1)(1-u)-\alpha^2\right)-\lambda_1\dot{x}\right]_{t_f}=0$$

$$\left[t-\lambda_1 u+\lambda_2\left(1-u^2-\alpha^2\right)\right]_{t_f}=0$$

$$\lambda_1=C\quad u\in\{-1,1\}$$

$$2\lambda_2 u=\lambda_1=C\quad t\geqslant 0$$

$$\dot{x}(t)=-1\quad x(t)=u\cdot t+C$$

lub $\dot{x}(t)=1$

PRZYKŁAD

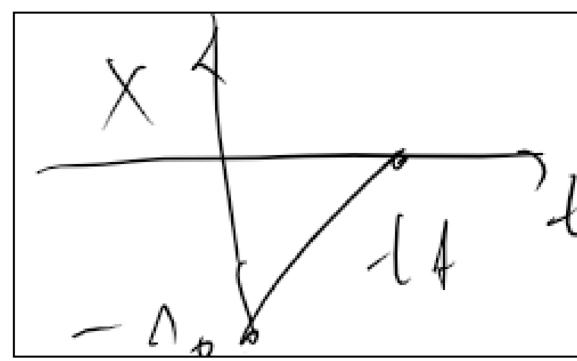
$$x\left(0\right)=-10$$

$$x(t)=1\cdot t-10$$

$$t_f=10$$

$$u=+1$$

U=-sign(x(0))



Czasooptymalne systemy

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\forall x(0) \rightarrow 0$$

System jest **całkowicie sterowalny**, jeśli rank macierzy G wynosi n (wymiar x)

$$\mathbf{G} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

$$U^- \leq \mathbf{u}(t) \leq U^+ \quad \longrightarrow \quad |\mathbf{u}(t)| \leq \mathbf{U}$$

$$|u_j(t)| \leq U_j, \qquad j = 1, 2, \cdots, r.$$

$$-1 \leq \mathbf{u}(t) \leq +1 \quad \longrightarrow \quad |\mathbf{u}(t)| \leq 1$$

$$|u_j(t)| \leq 1$$

Funkcjonał

$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f} V\left[\mathbf{x}(t), \mathbf{u}(t), t\right] dt = \int_{t_0}^{t_f} 1 dt = t_f - t_0$$

t_0 ustalony t_f dowolny

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\begin{aligned}\mathcal{H}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t)) &= 1 + \boldsymbol{\lambda}'(t)\left[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right], \\ &= 1 + \left[\mathbf{A}\mathbf{x}(t)\right]'\boldsymbol{\lambda}(t) + \mathbf{u}'(t)\mathbf{B}'\boldsymbol{\lambda}(t)\end{aligned}$$

$|\mathbf{u}(t)| \leq \mathbf{U}$

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \mathbf{A}\mathbf{x}^*(t) + \mathbf{B}\mathbf{u}^*(t),$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\mathbf{A}'\boldsymbol{\lambda}^*(t)$$

$$\mathbf{x}^*(t_0) = \mathbf{x}(t_0);$$

$$\mathbf{x}^*(t_f) = \mathbf{0}$$

Warunek konieczny zasady minimum Pontriagina

$$0 = + \left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_*$$

$$\mathcal{H}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \leq \mathcal{H}(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}^*(t), t)$$

$$1 + [\mathbf{A}\mathbf{x}^*(t)]'\boldsymbol{\lambda}^*(t) + \mathbf{u}^{*\prime}(t)\mathbf{B}'\boldsymbol{\lambda}^*(t) \leq 1 + [\mathbf{A}\mathbf{x}^*(t)]'\boldsymbol{\lambda}^*(t) + \mathbf{u}'(t)\mathbf{B}'\boldsymbol{\lambda}^*(t)$$

$$\mathbf{u}^{*\prime}(t)\underbrace{\mathbf{B}'\boldsymbol{\lambda}^*(t)} \leq \mathbf{u}'(t)\mathbf{B}'\boldsymbol{\lambda}^*(t),$$

$$\mathbf{u}^{*\prime}(t)\mathbf{q}^*(t) \leq \mathbf{u}'(t)\mathbf{q}^*(t), \quad \mathbf{q}^*(t) = \mathbf{B}'\boldsymbol{\lambda}^*(t)$$

$$\mathbf{u}'(t)\mathbf{q}^*(t) = \min_{|\mathbf{u}(t)| \leq 1} \{\mathbf{u}'(t)\mathbf{q}^*(t)\}$$

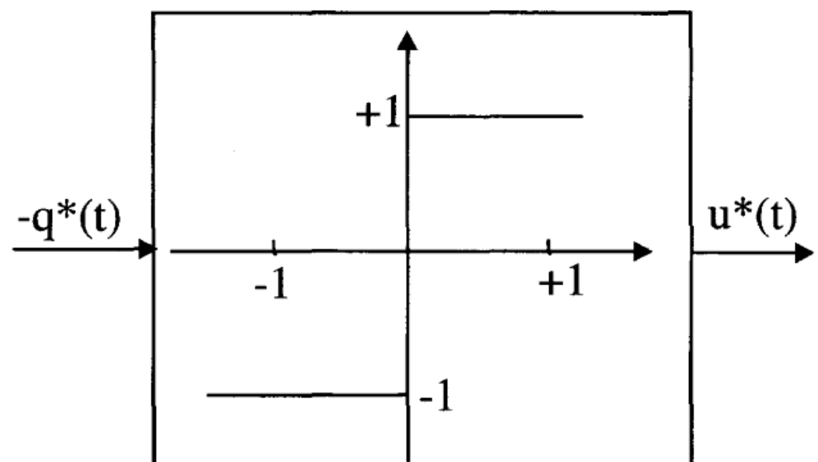
Bang-Bang Sterowanie

$$\min_{|\mathbf{u}(t)|\le 1} \{\mathbf{u}'(t)\mathbf{q}^*(t)\} = -|\mathbf{q}^*(t)|$$

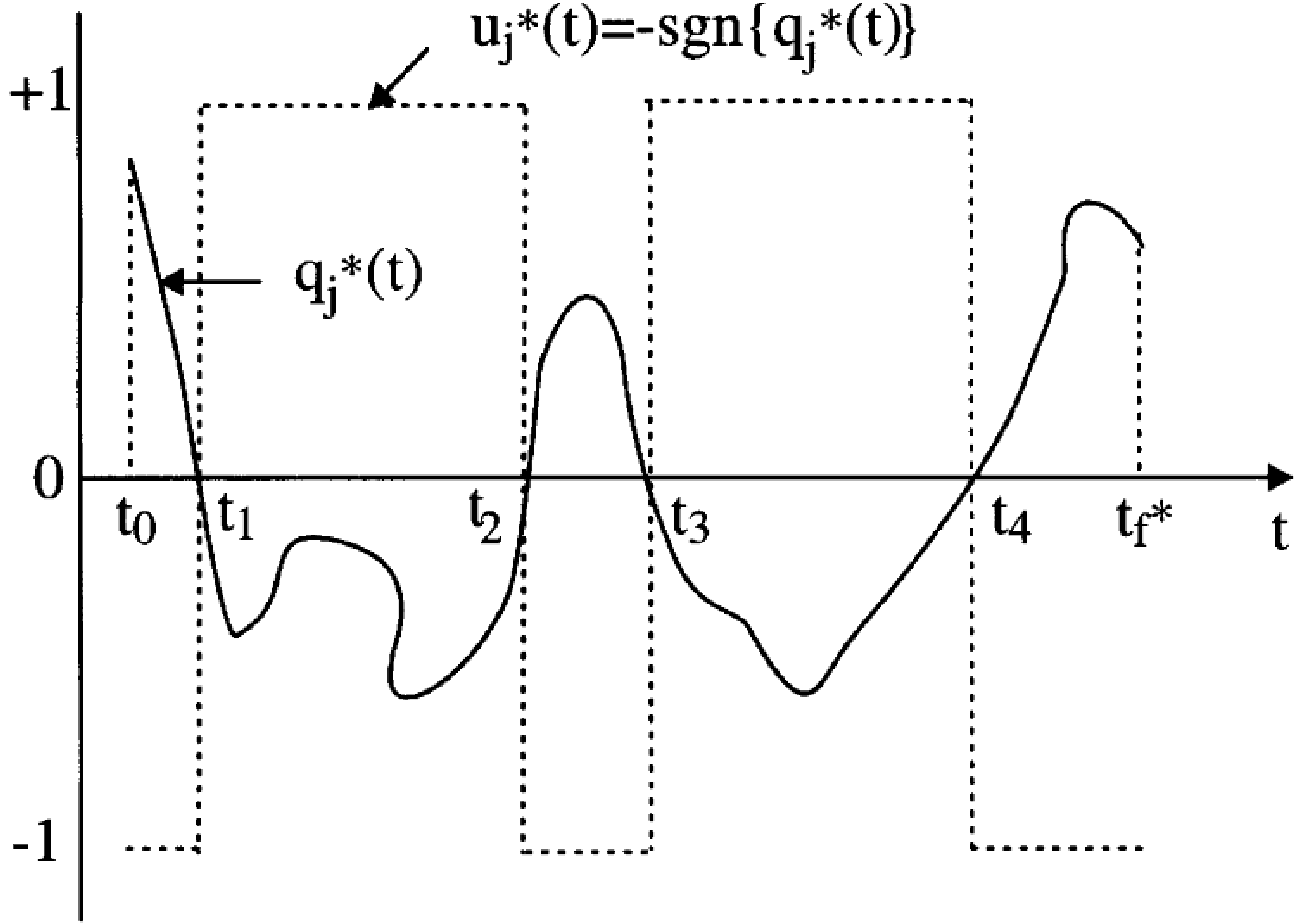
$$\mathbf{u}^*(t) = \begin{cases} +1 & \text{if } \mathbf{q}^*(t) < 0, \\ -1 & \text{if } \mathbf{q}^*(t) > 0, \\ \text{Nie wyznaczalny} & \text{if } \mathbf{q}^*(t) = 0. \end{cases}$$

$$\boxed{\mathbf{u}^*(t) = -SGN\{\mathbf{q}^*(t)\}}$$

$$u_j^*(t) = -sgn\{q_j^*(t)\} = -sgn\{\mathbf{b}_j'\boldsymbol{\lambda}^*(t)\}$$



Przykład sterowania



$$\begin{aligned}\mathcal{H}(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t)) &= 1 + \boldsymbol{\lambda}'(t)\left[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\right], \\ &= 1 + [\mathbf{A}\mathbf{x}(t)]'\boldsymbol{\lambda}(t) + \mathbf{u}'(t)\mathbf{B}'\boldsymbol{\lambda}(t)\end{aligned}$$

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \mathbf{A}\mathbf{x}^*(t) + \mathbf{B}\mathbf{u}^*(t),$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\mathbf{A}'\boldsymbol{\lambda}^*(t)$$

$$\boldsymbol{\lambda}^*(t) = \epsilon^{-\mathbf{A}'t}\boldsymbol{\lambda}^*(0)$$

$$\mathbf{u}^*(t) = -SGN\{\mathbf{B}'\epsilon^{-\mathbf{A}'t}\boldsymbol{\lambda}^*(0)\}$$

$$\begin{aligned}u_j^*(t) &= -sgn\{q_j^*(t)\}, \\ &= -sgn\left\{\mathbf{b}_j'\epsilon^{-\mathbf{A}'t}\boldsymbol{\lambda}^*(0)\right\}\end{aligned}$$

Warunek konieczny sterowalności

$\mathbf{q}^*(t)$ = 0 dla $[T_1, T_2]$

$$q_j^*(t) = \mathbf{b}_j' \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$$
$$\dot{q}^*(t) = \mathbf{b}_j' \mathbf{A}' \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$$
$$\ddot{q}^*(t) = \mathbf{b}_j' \mathbf{A}'^2 \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$$
$$\cdots\cdots\cdots\cdots$$
$$q^{(n-1)*}(t) = \mathbf{b}_j' \mathbf{A}'^{(n-1)} \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$$

$$\mathbf{G}_j' \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$$

$$\boldsymbol{\lambda}^*(0) \neq 0$$

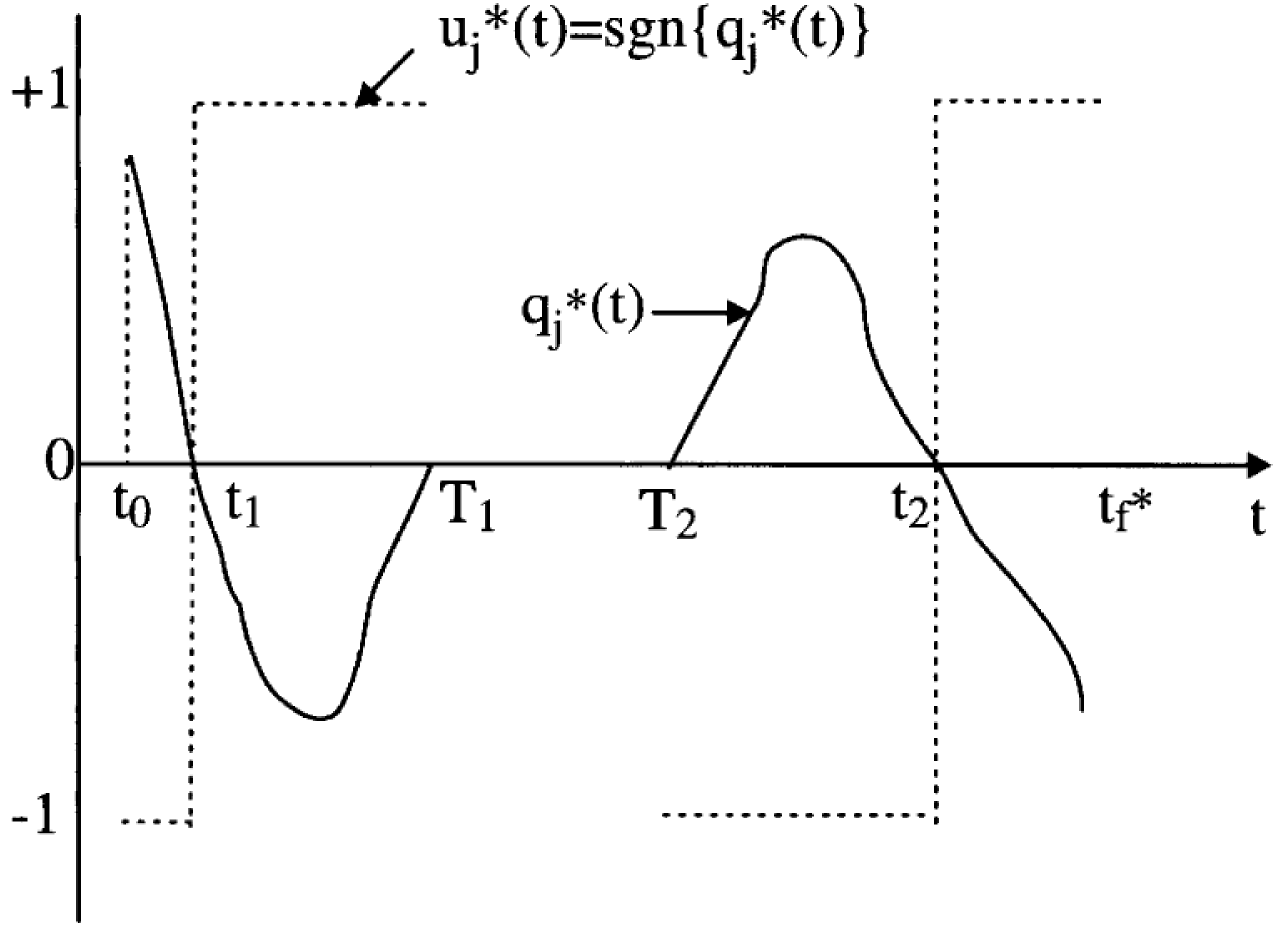
$$\mathbf{G}_j = \left[\mathbf{b}_j \;\vdots\; \mathbf{A}\mathbf{b}_j \;\vdots\; \mathbf{A}^2\mathbf{b}_j \;\vdots\; \cdots \vdots\; \mathbf{A}^{n-1}\mathbf{b}_j\right]$$
$$= \left[\mathbf{B} \;\vdots\; \mathbf{A}\mathbf{B} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots \cdots \vdots\; \mathbf{A}^{n-1}\mathbf{B}\right].$$

$\mathbf{G}_j$ całkowicie sterowalny system

time-optimal control system is *normal*

System częściowo sterowalny

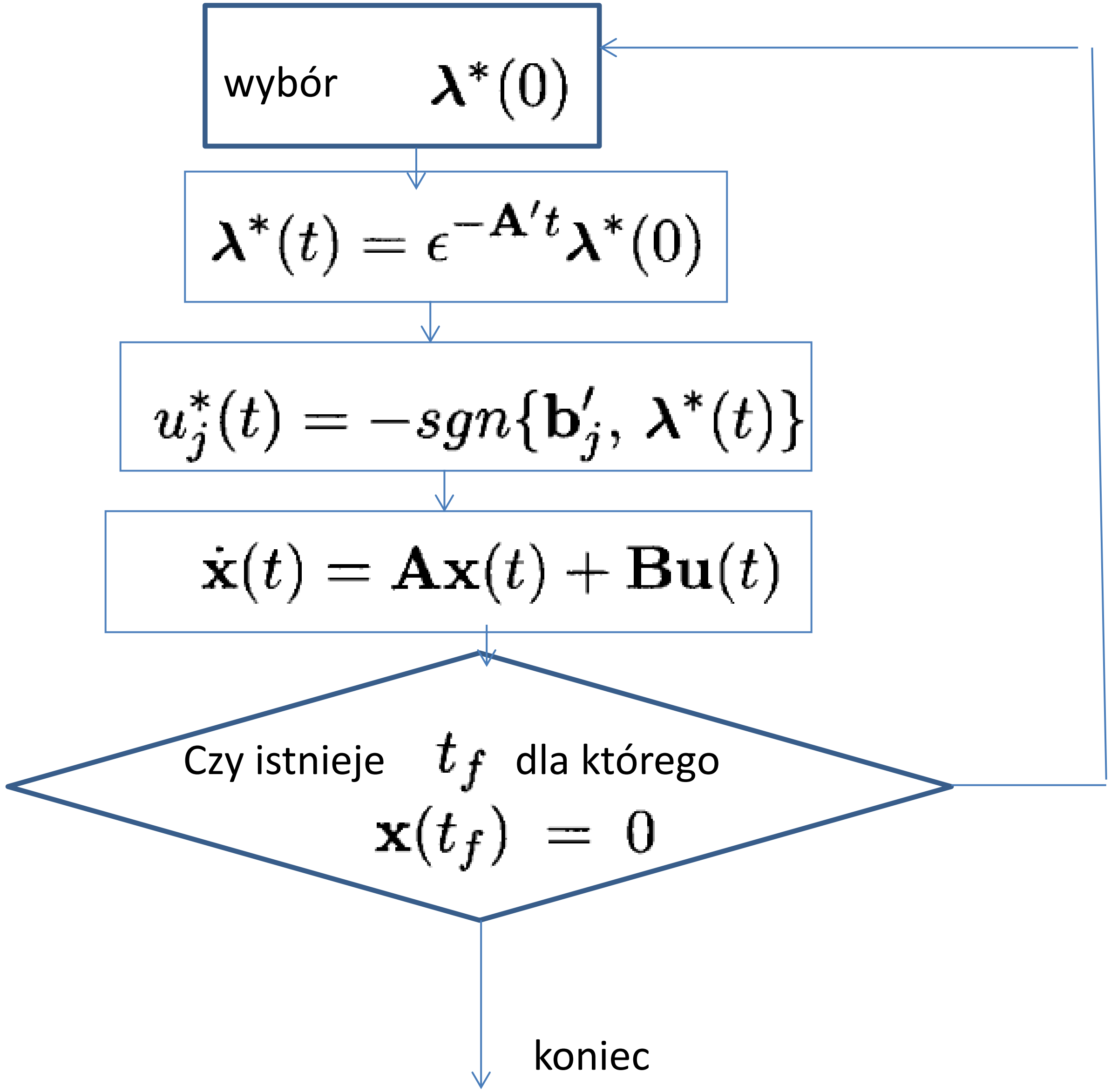
$q_j^*(t) = 0 \quad \forall\ t \in [T_1, T_2] \qquad \mathbf{G}_j' \epsilon^{-\mathbf{A}'t} \boldsymbol{\lambda}^*(0) = 0$



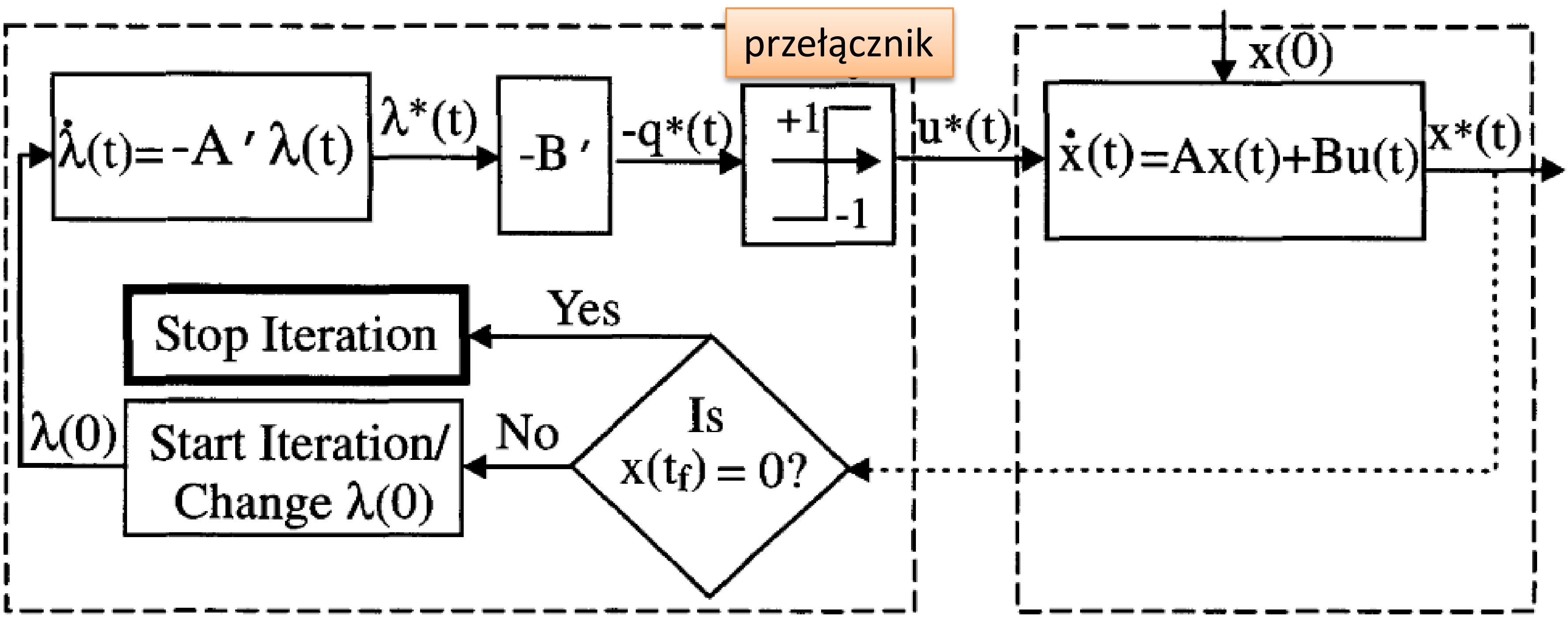
Twierdzenie

Jeśli wszystkie n wartości własne macierzy A są rzeczywiste, to optymalne sterowanie u*(t) można przełączać (od 1 do –l lub od -1 do 1) w liczbie (n - 1) razy.

Algorytm poszukiwania sterowania



Układ otwarty



Układ zamknięty

relacja

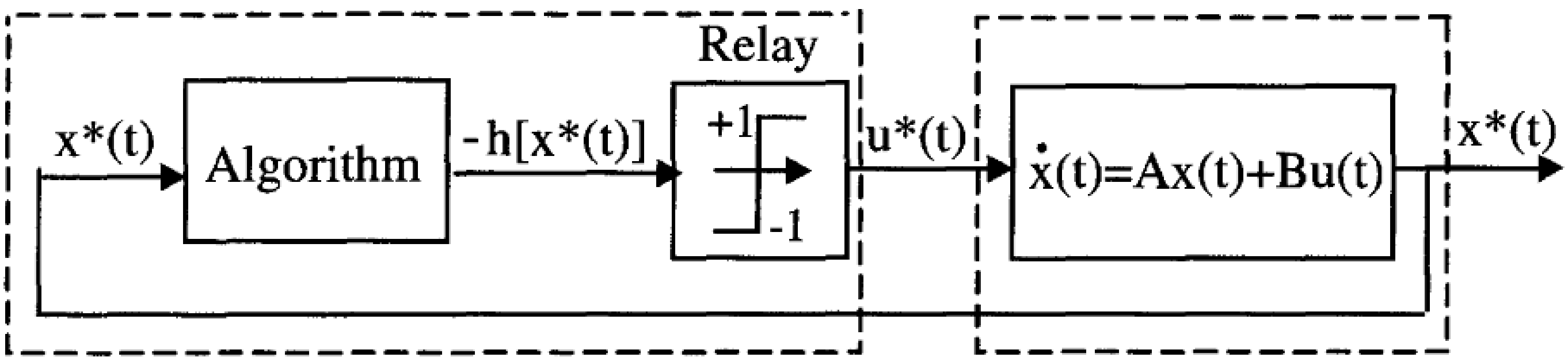
$\mathbf{u}^*(t) \Longleftrightarrow \mathbf{x}^*(t)$

równanie Riccatiego

$$\boldsymbol{\lambda}^*(t) = \mathbf{P}(t)\mathbf{x}^*(t)$$

$$\mathbf{u}^*(t) = -SGN\{\mathbf{B}'\epsilon^{-\mathbf{A}'t}\boldsymbol{\lambda}^*(0)\} \longrightarrow \boxed{\mathbf{u}^*(t) = -SGN\{\mathbf{h}(\mathbf{x}^*(t))\}}$$

$$\mathbf{h}(\mathbf{x}^*(t)) = \mathbf{B}'\boldsymbol{\lambda}^*(\mathbf{x}^*(t))$$



Przykład 7

$$m\ddot{y}(t) = \mathbf{v}(t) \qquad \ddot{y}(t) = u(t)$$

$$u(t) = \frac{\mathbf{v}(t)}{m} \qquad |u(t)| \leq 1$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$$\forall (x_{1p}, x_{2p}) \xrightarrow[\text{minimum czasu}]{} (x_{1\kappa} = 0, x_{2\kappa} = 0)$$

Przykład

$$J = \int_{t_0}^{t_f} 1 dt = t_f - t_0$$

$$\mathcal{H}(\mathbf{x}(t), \boldsymbol{\lambda}(t), u(t)) = 1 + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)$$

Zasada Pontriagina

$$\mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), u^*(t)) \leq \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), u(t),)$$

$$= \min_{|\mathbf{u}| \leq 1} \mathcal{H}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), u(t))$$

$$1 + \lambda_1^*(t) x_2^*(t) + \lambda_2^*(t) u^*(t) \ \leq \ 1 + \lambda_1^*(t) x_2^*(t) + \lambda_2^*(t) u(t)$$

$$|u(t)| \leq 1 \qquad \lambda_2^*(t) u^*(t) \leq \lambda_2^*(t) u(t)$$

$$u^*(t) = -sgn\{\lambda_2^*(t)\}$$

Przykład

$$\mathcal{H}(\mathbf{x}(t), \boldsymbol{\lambda}(t), u(t)) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t)$$

$$\dot{\lambda}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1^*} = 0,$$

$$\dot{\lambda}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2^*} = -\lambda_1^*(t)$$

$$\lambda_1^*(t) = \lambda_1^*(0),$$

$$\lambda_2^*(t) = \lambda_2^*(0) - \lambda_1(0)t$$

$$u^*(t) = -sgn\{\lambda_2^*(t)\}$$

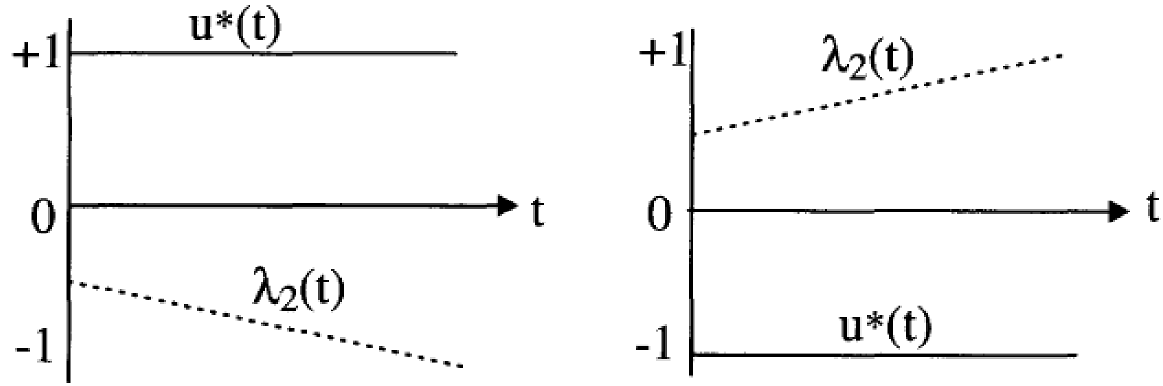
Przykład

Sterowanie

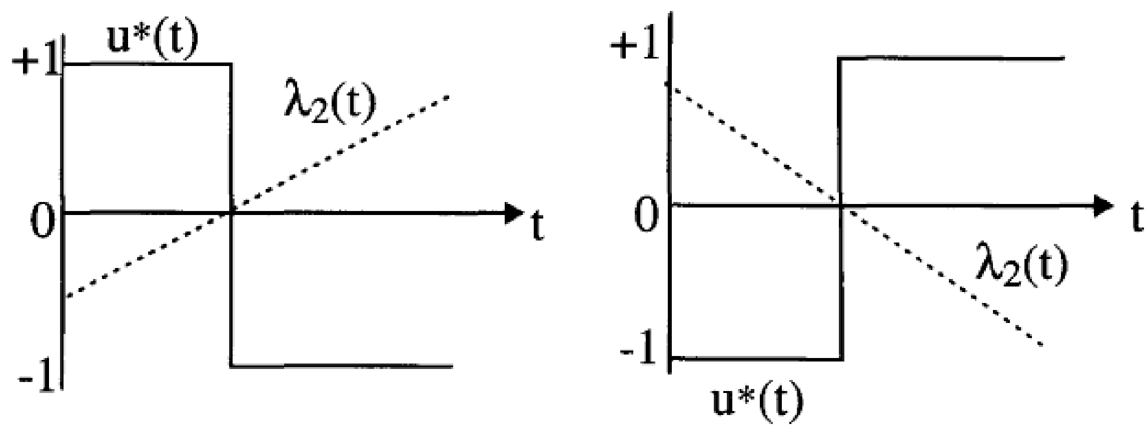
$$\{+1\},\quad \{-1\},$$

$$\{+1,-1\},\quad \{-1,+1\}$$

System 2 rzędu ma maksimum 1 przełączenie



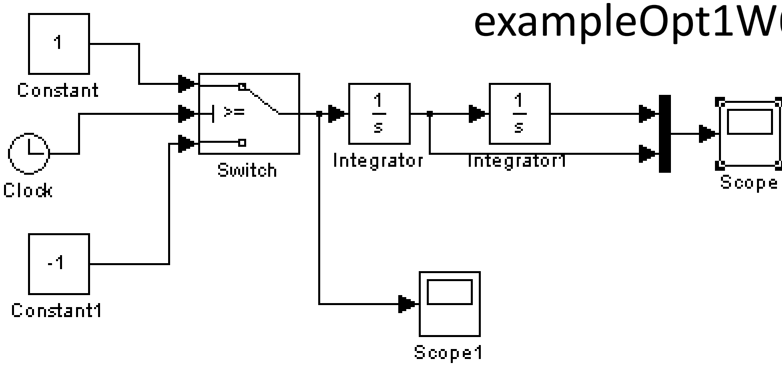
(a) $\lambda_1(0) > 0;\ \lambda_2(0) < 0$ (b) $\lambda_1(0) < 0;\ \lambda_2(0) > 0$

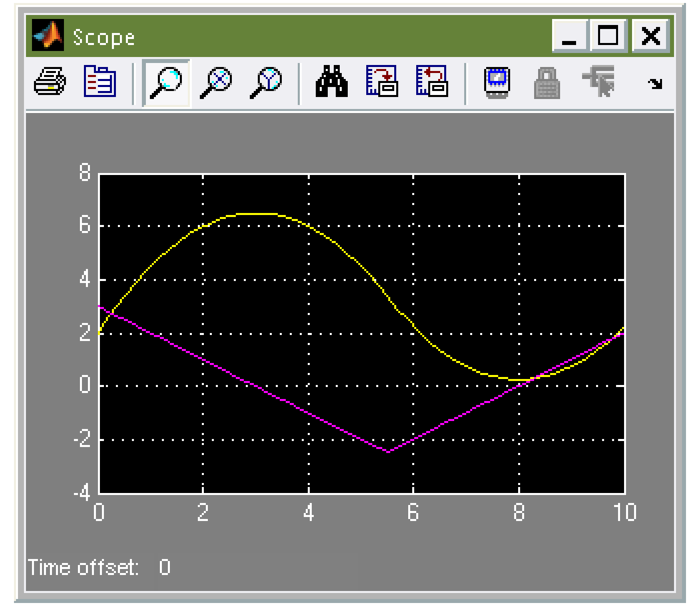


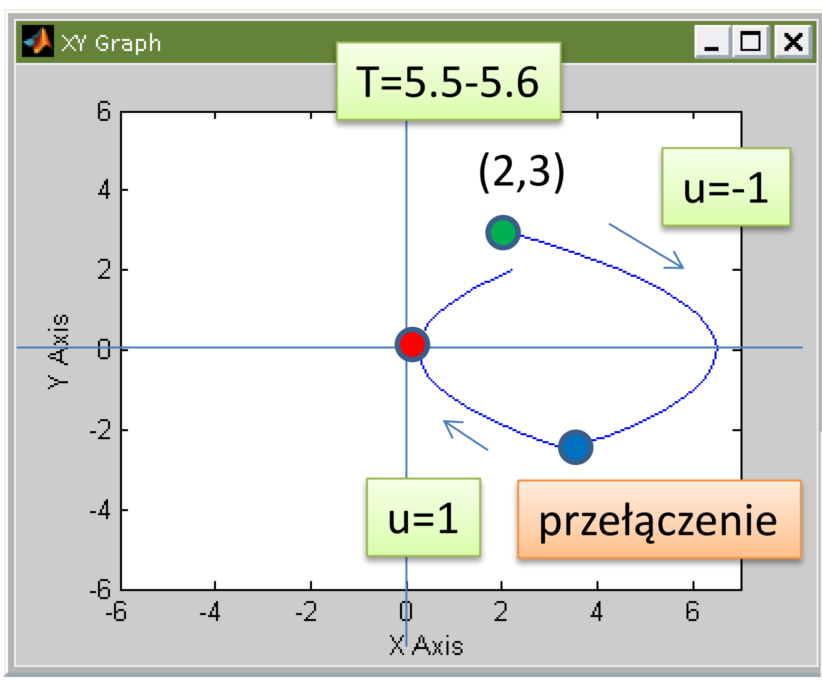
(c) $\lambda_1(0) < 0;\ \lambda_2(0) < 0$ (d) $\lambda_1(0) > 0;\ \lambda_2(0) > 0$

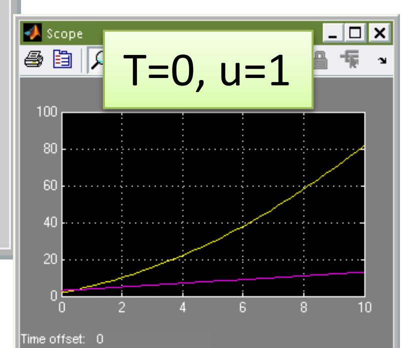
Badania

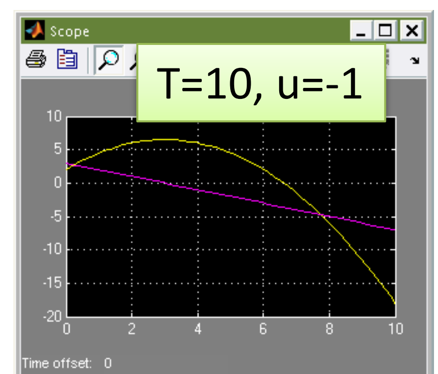
exampleOpt1W6.slx



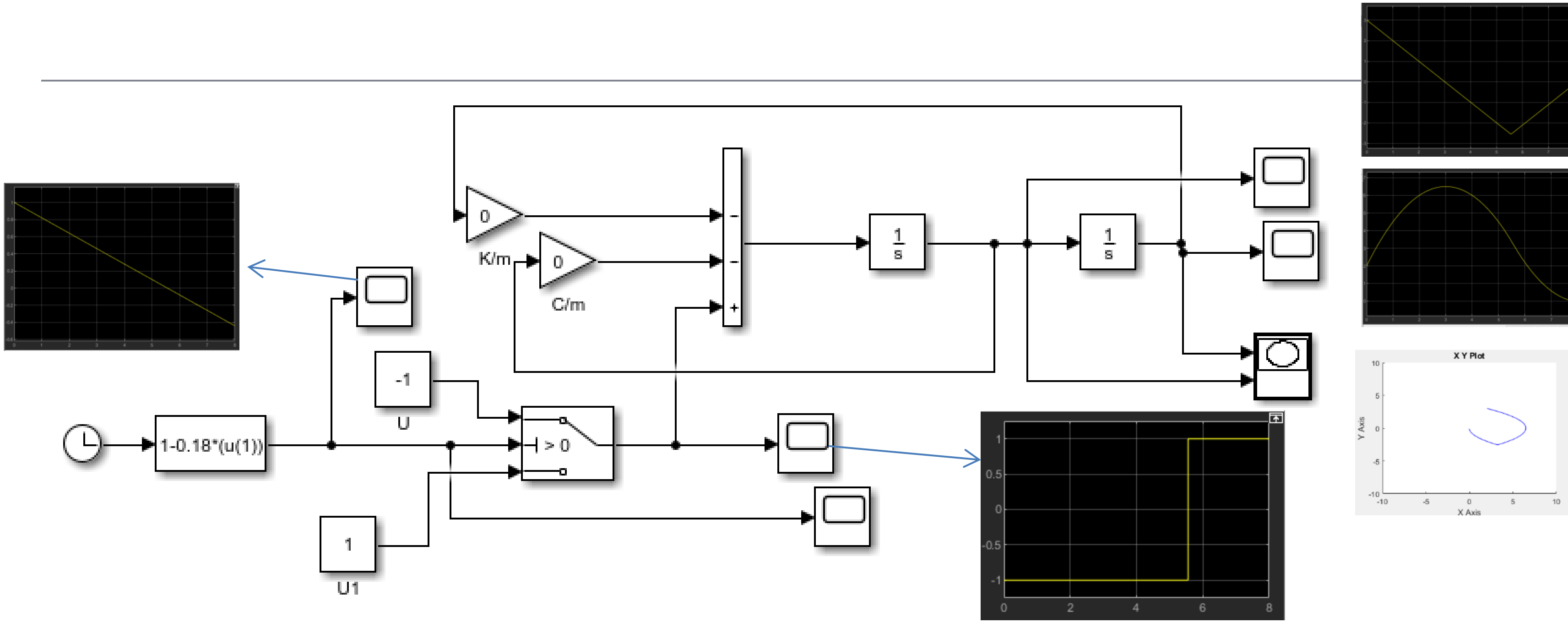


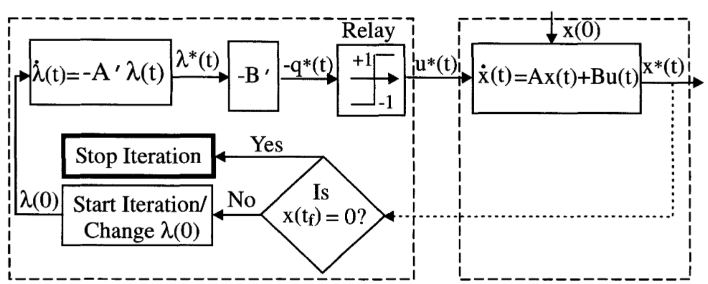






Układ otwarty





Układ zamknięty

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$U = u^*(t) = \pm 1$

$$x_1^*(t) = x_1^*(0) + x_2^*(0)t + \frac{1}{2}Ut^2,$$
$$x_2^*(t) = x_2^*(0) + Ut,$$

$$t = (x_2(t) - x_{20})/U,$$

$$x_1(0) = x_{10}, x_2(0) = x_{20}$$

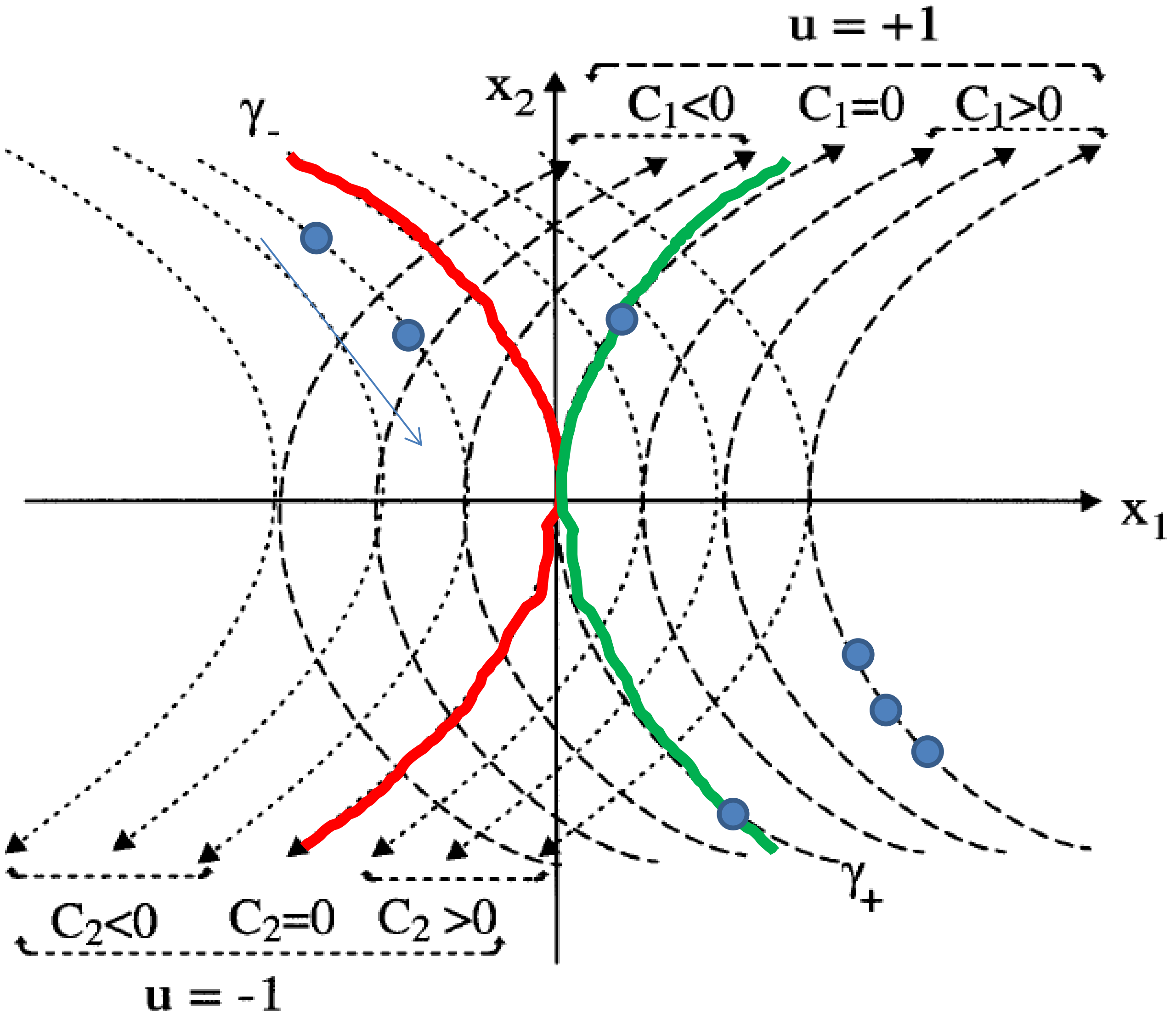
$$x_1(t) = x_{10} - \frac{1}{2}Ux_{20}^2 + \frac{1}{2}Ux_2^2(t)$$

$$u = U = +1, \quad \begin{cases} t & = x_2(t) - x_{20} \\ x_1(t) & = x_{10} - \frac{1}{2}x_{20}^2 + \frac{1}{2}x_2^2(t) = C_1 + \frac{1}{2}x_2^2(t) \end{cases}$$

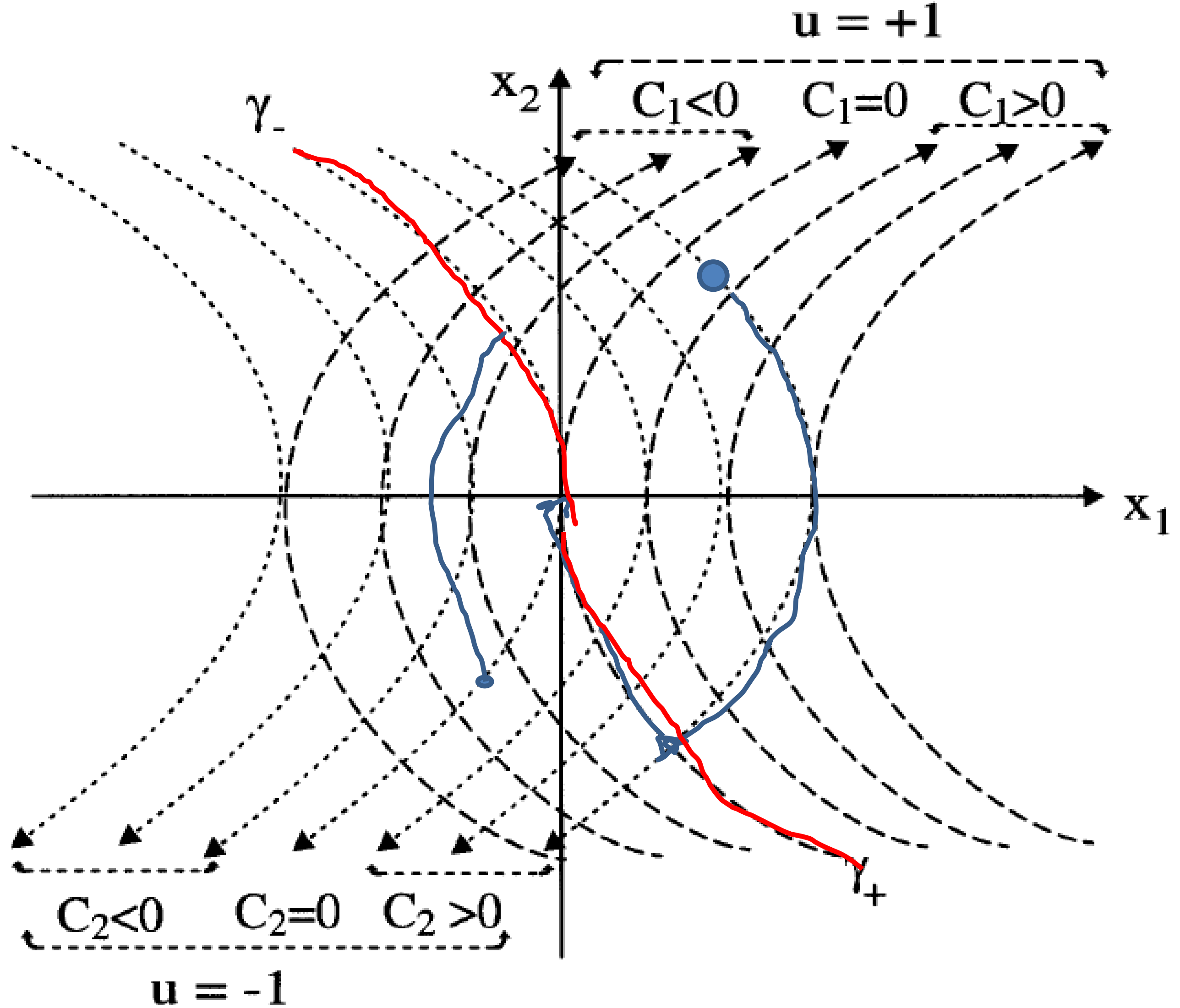
$$u = U = -1, \quad \begin{cases} t & = x_{20} - x_2(t), \\ x_1(t) & = x_{10} + \frac{1}{2}x_{20}^2 - \frac{1}{2}x_2^2(t) = C_2 - \frac{1}{2}x_2^2(t) \end{cases}$$

$$C_1 = x_{10} - \tfrac{1}{2}x_{20}^2 \qquad C_2 = x_{10} + \tfrac{1}{2}x_{20}^2$$

Portret fazowy



Portret fazowy



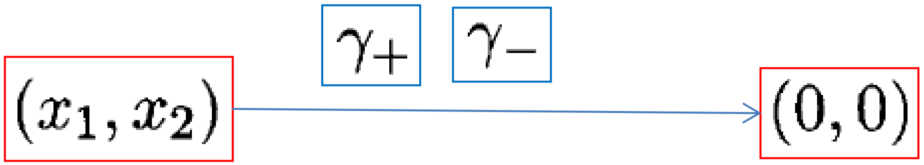
Trajektoria przełączenia

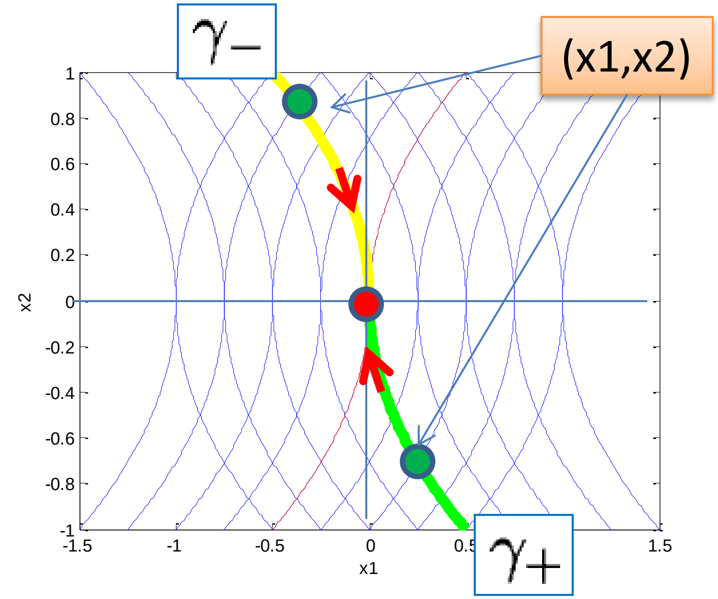
$$x_1(t = t_f) = 0; \qquad x_2(t = t_f) = 0$$

$$x_1(t) = x_{10} - \frac{1}{2}Ux_{20}^2 + \frac{1}{2}Ux_2^2(t)$$

$$0 = x_{10} - \frac{1}{2}Ux_{20}^2 + 0 \longrightarrow x_{10} = \frac{1}{2}Ux_{20}^2$$

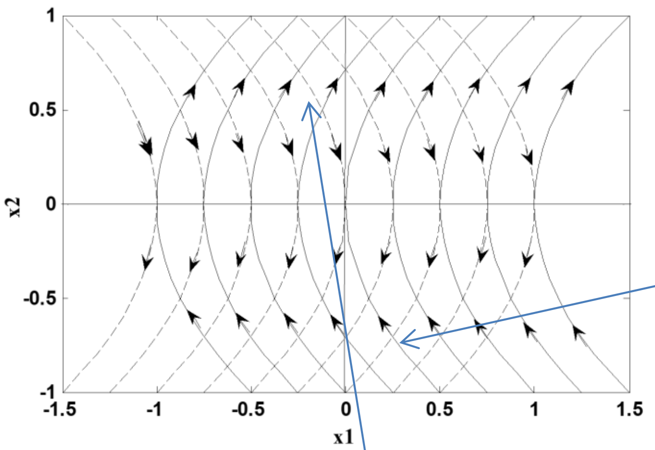
Dla dowolnych $x_1 = x_{10}, x_2 = x_{20}$ $\qquad x_1 = \frac{1}{2}Ux_2^2$





Przykład

Celem jest $(0,0)$ $\quad u = \pm 1$



Mamy 2 trajektorii

$$u = +1$$

$$\gamma_+ = \left\{ (x_1, x_2) : x_1 = \frac{1}{2} x_2^{\,2}; x_2 \le 0 \right\}.$$

$$u = -1$$

$$\gamma_- = \left\{ (x_1, x_2) : x_1 = -\frac{1}{2} x_2^{\,2}; x_2 \ge 0 \right\}.$$

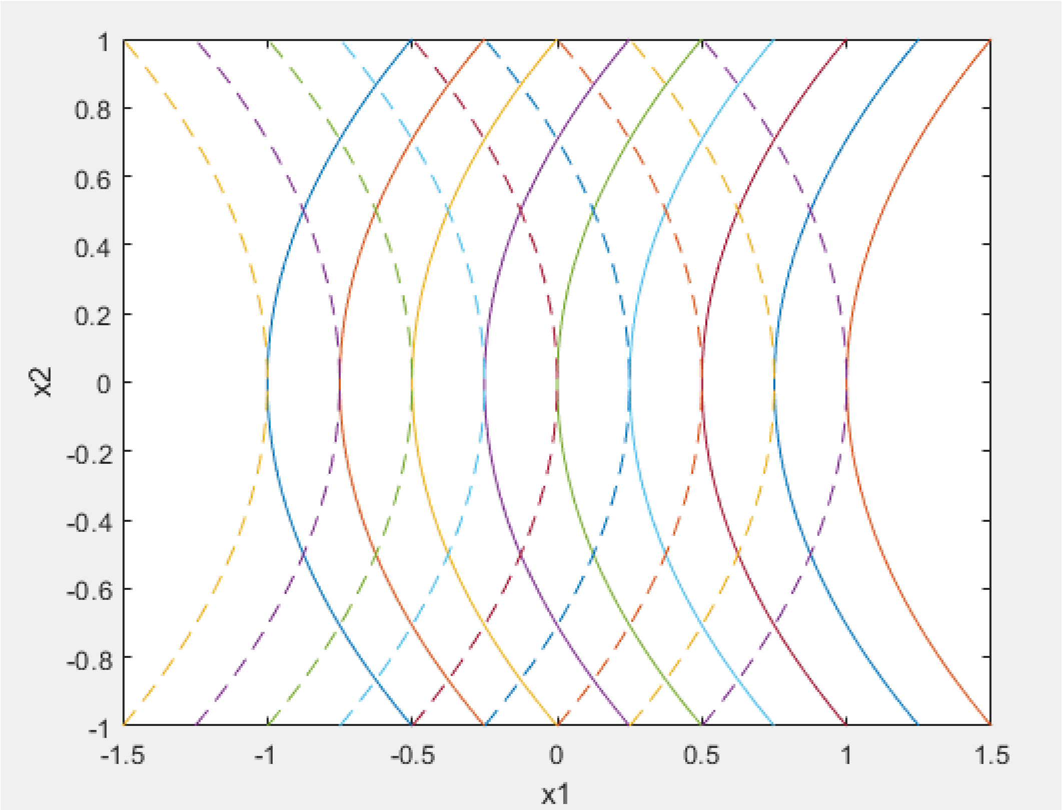
Określenie przełączania

$$\gamma = \gamma_+ \cup \gamma_- = \left\{ (x_1, x_2) : x_1 = -\frac{1}{2} x_2 \left| x_2 \right| \right\}$$

Przykład

Pr1w5.m

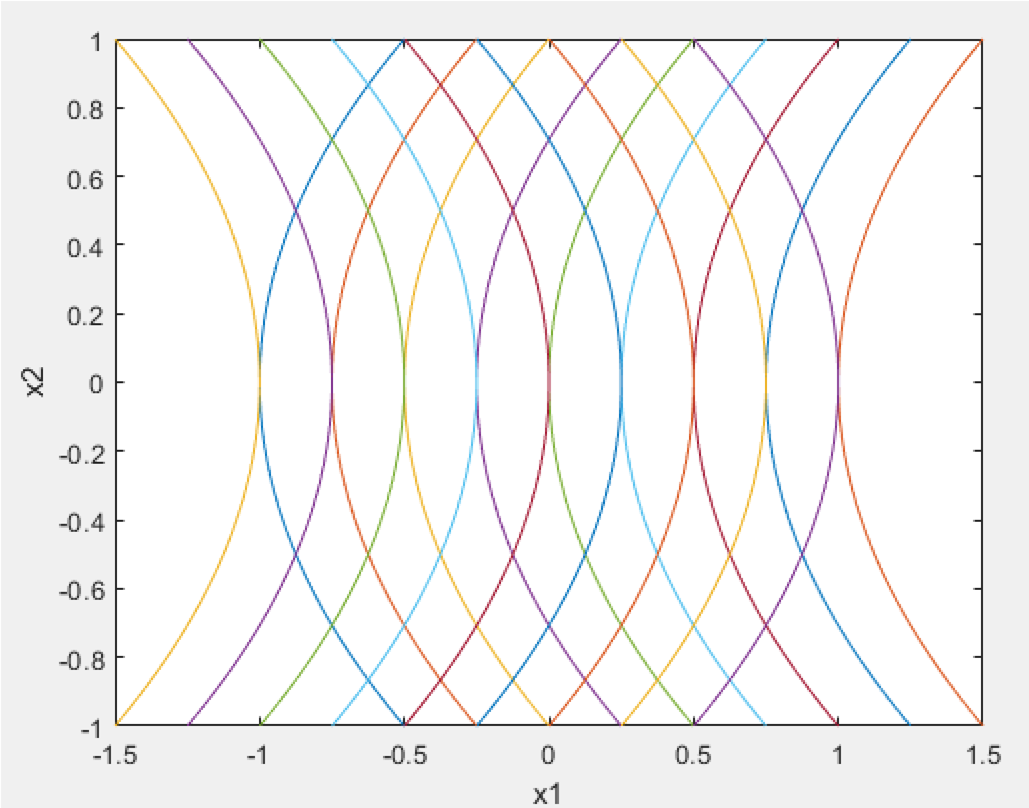
```
>> % u = +1
x10=[-1.0  -0.75  -0.5  -0.25  0  0.25  0.5  0.75
1.0];
x20=0;
for k=1:9
ksi1=x10(k)-x20.^2/2;
 x2=[-1:0.01:1];
x1=x2.^2/2+ksi1;
plot(x1,x2,'-')
hold on
end
% u = -1
for k=1:9
ksi2=x10(k)+x20.^2/2;
x2=[-1:0.01:1];
x1= - x2.^2/2+ksi2;
plot(x1,x2,'--')
xlabel('x1')
ylabel('x2')
hold on
End
```



Przykład (cd)

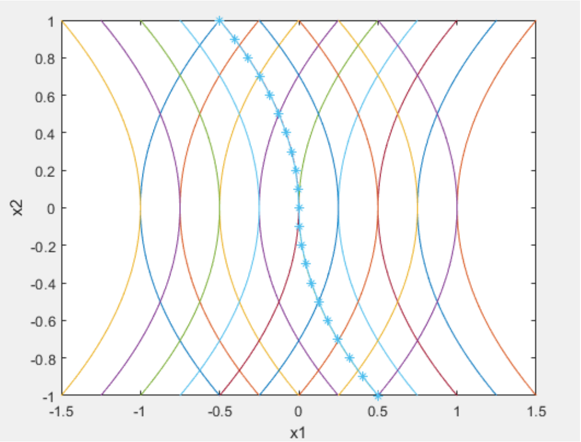
Pr2w5.m

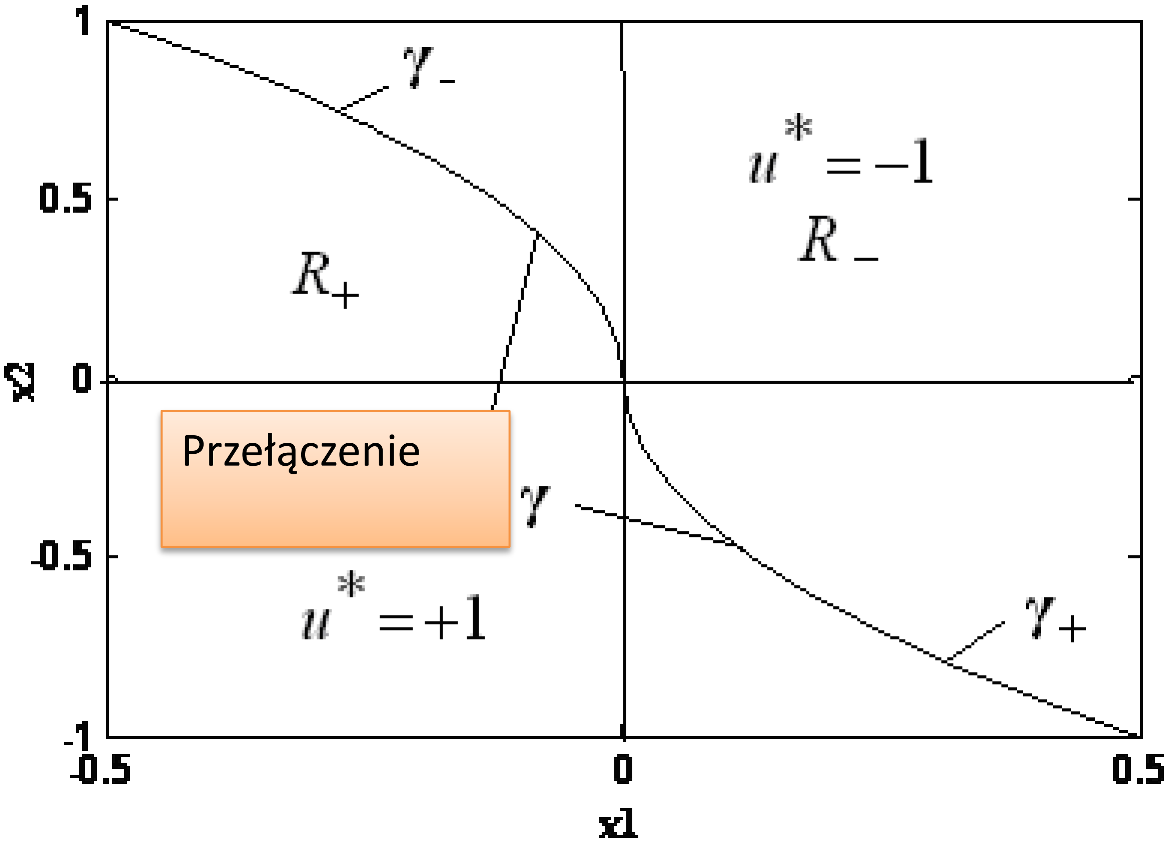
```
u =[ 1   -1];
x10=[-1.0  -0.75  -0.5  -0.25
0  0.25  0.5  0.75 1.0];
x20=0;
for i=1:2
for k=1:9
ksi=x10(k)-u(i)*x20.^2/2;
x2=[-1:0.01:1];
x1=u(i)*x2.^2/2+ksi;
plot(x1,x2,'-')
xlabel('x1')
ylabel('x2')
hold on
end
end
```



Przykład

```
>> x2 = [-1: 0.1: 1];
x1 = - 0.5*x2.*abs(x2);
plot(x1,x2,'-')
xlabel('x1')
ylabel('x2')
```





$$u^* = u^*(x_1, x_2) = +1 \;\; \forall\, (x_1, x_2) \in \gamma_+ \cup R_+;$$

$$u^* = u^*(x_1, x_2) = -1 \;\; \forall (x_1, x_2) \in \gamma_- \cup R_-.$$

Przykład

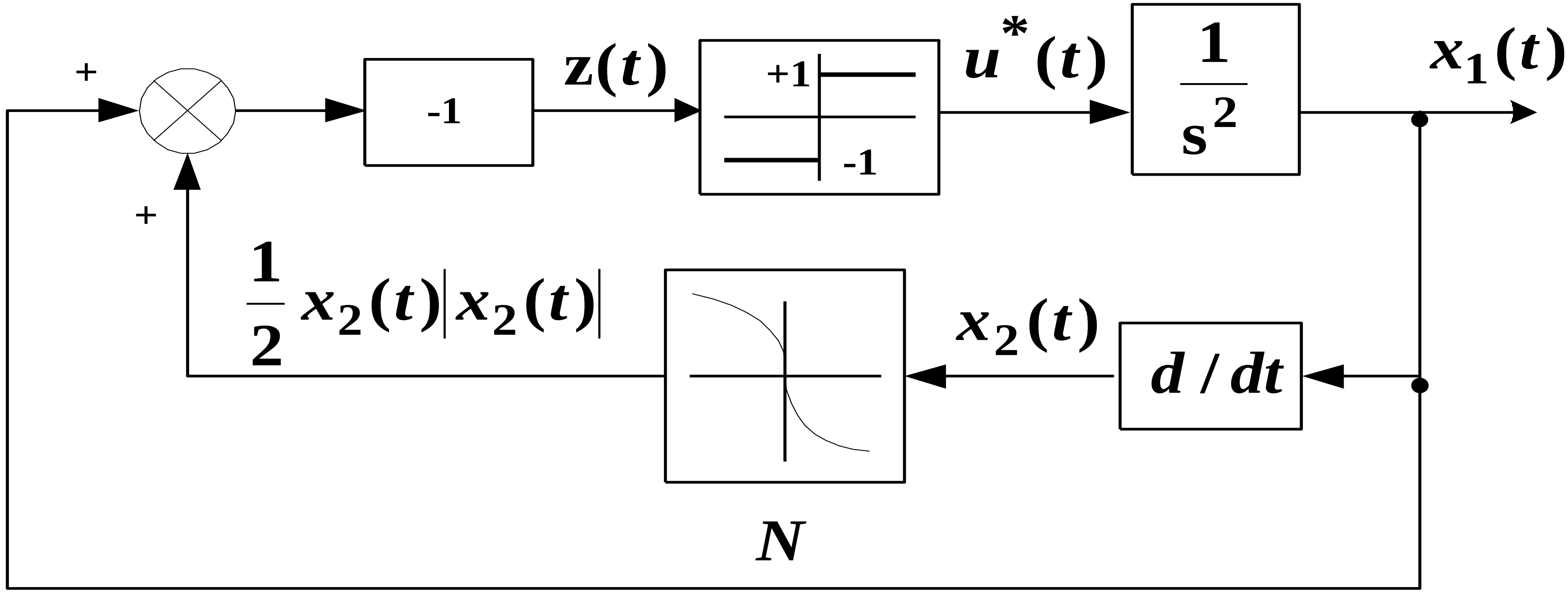
Równanie przełączenia

$$\mathbf{z}(t) = -x_1(t) - \frac{1}{2} x_2(t) \left| x_2(t) \right|$$

$$jeśli \ \ (x_1(t), x_2(t) \in R_{-,} \ to \ \ \mathrm{z}(t) > 0,$$

$$jeśli \ \ (x_1(t), x_2(t) \in R_{+,} \ to \ \ \mathrm{z}(t) < 0.$$

Przykład



Obliczenie czasu

$(x_1, x_2) \rightarrow (0,0)$

$$t_f^* = \begin{cases} x_2 + \sqrt{4x_1 + 2x_2^2} & \text{if} \quad (x_1, x_2) \in R_- \quad \text{or} \quad x_1 > -\frac{1}{2}x_2|x_2| \\ -x_2 + \sqrt{-4x_1 + 2x_2^2} & \text{if} \quad (x_1, x_2) \in R_+ \quad \text{or} \quad x_1 < -\frac{1}{2}x_2|x_2| \\ |x_2| & \text{if} \quad (x_1, x_2) \in \gamma \quad \text{or} \quad x_1 = -\frac{1}{2}x_2|x_2| \end{cases}$$

$$t = (x_2(t) - x_{20})/U,$$

$x_2 = -2 \quad x_1 = \frac{1}{2} \cdot 4 = 2$

$t = 0 + 2 \cdot (1) = 2s$

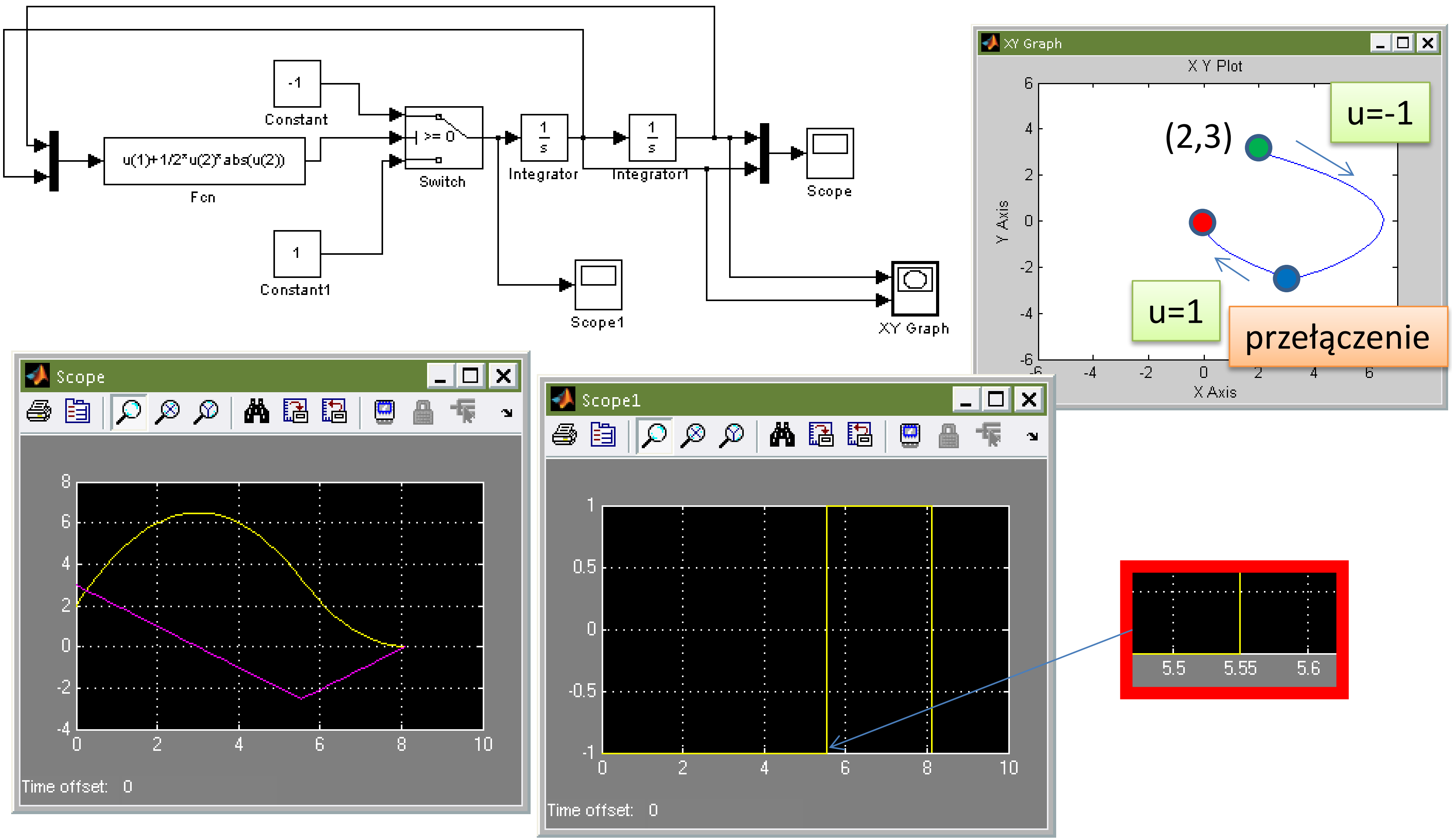
$(x_2 = 2 \quad x_1 = 2)$

$x_2 = -2 \quad x_1 = 2$

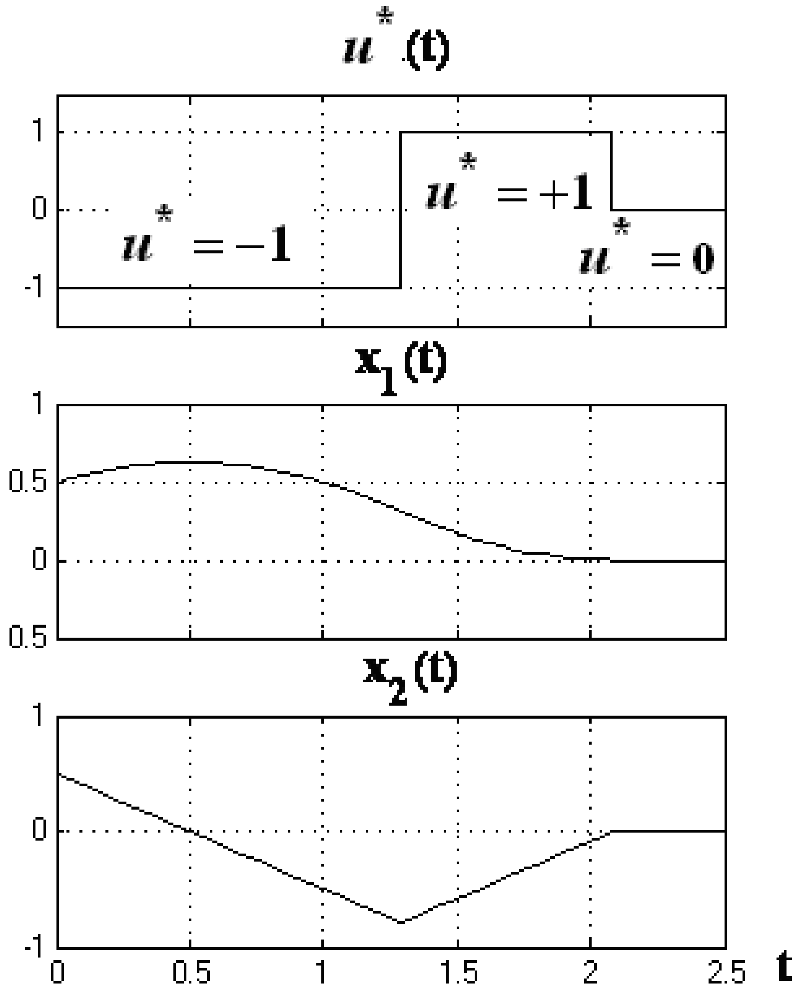
$t = (-2 - 2) \cdot (-1) = 4s$

Symulacja

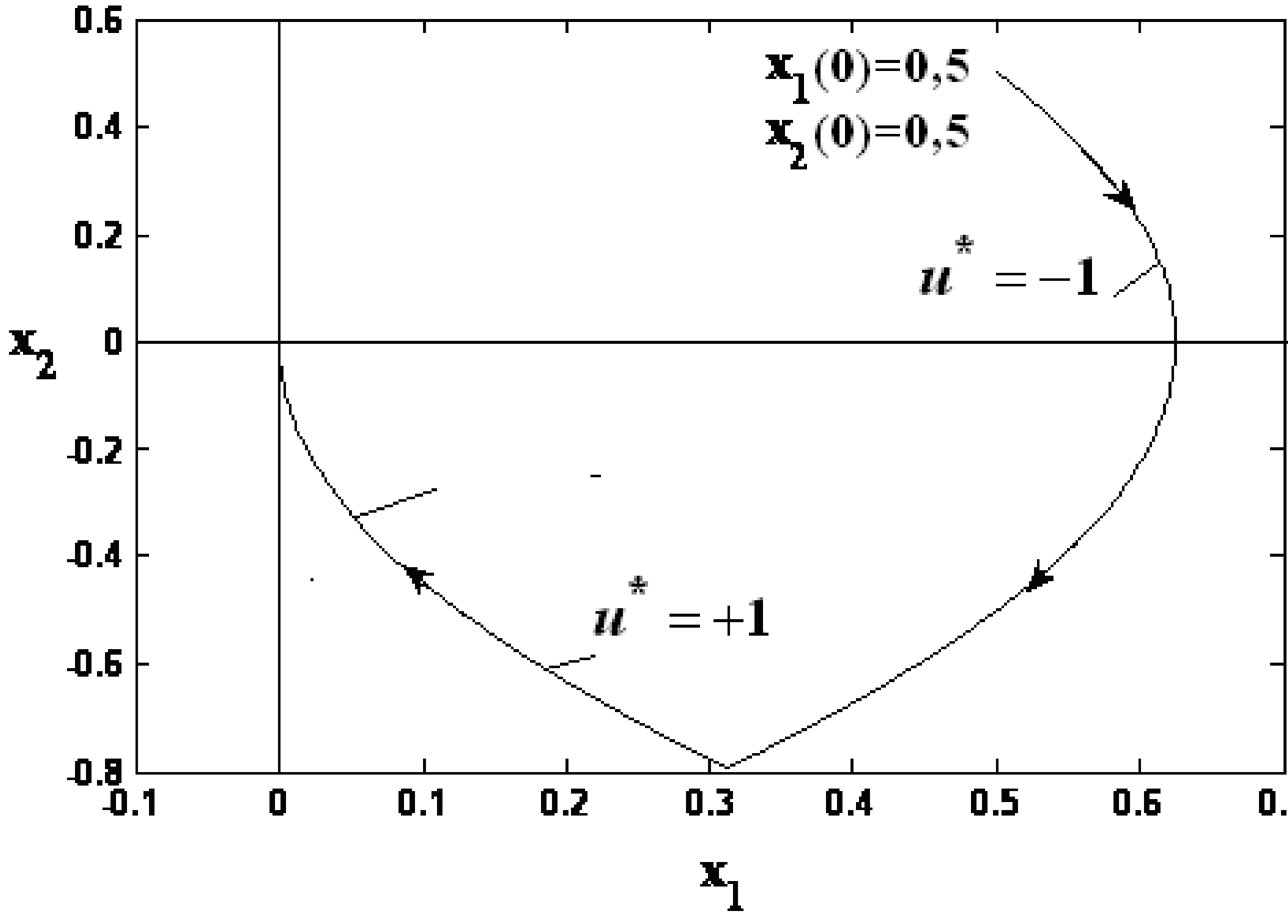
exampleOpt2W6.slx



Przykład

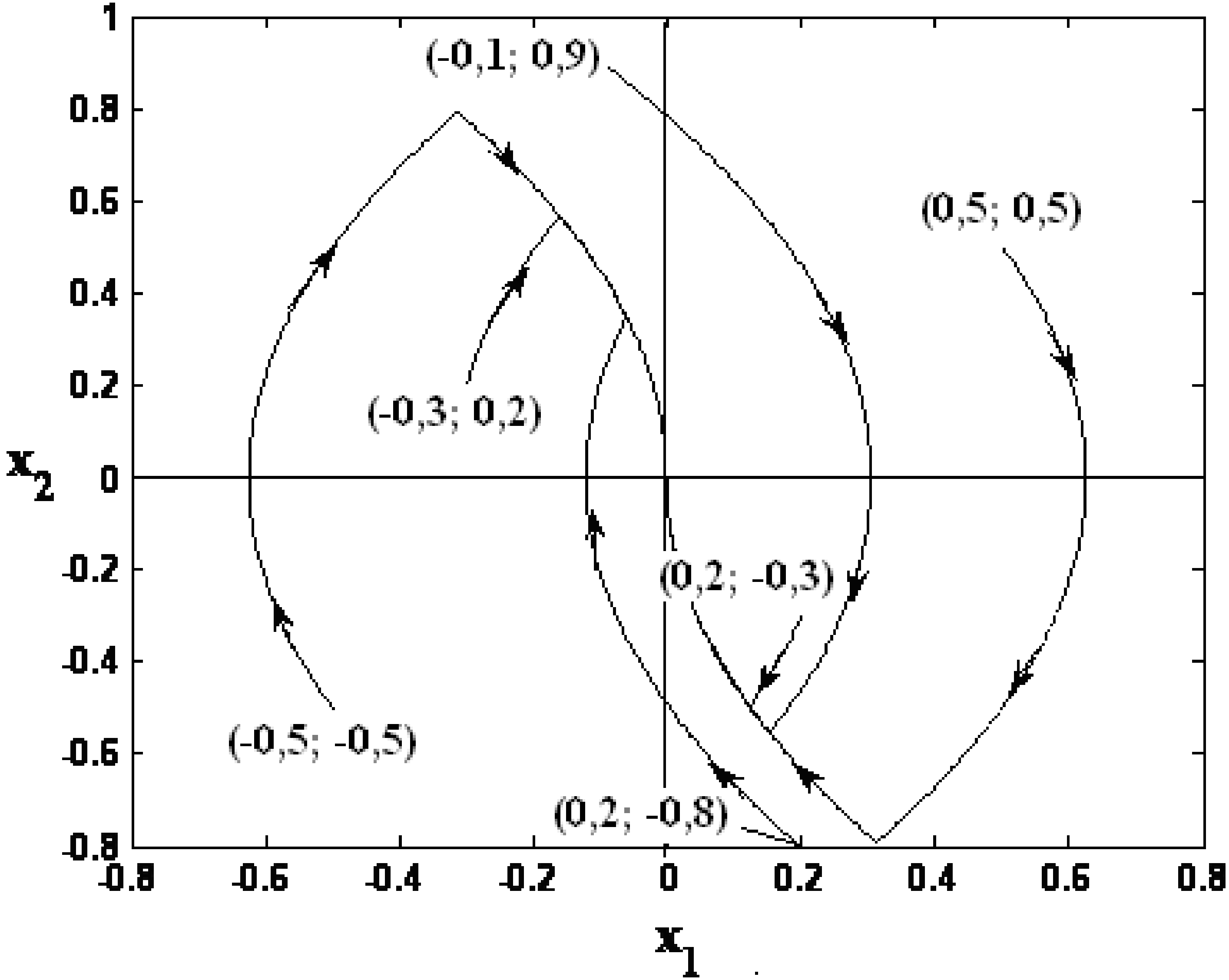


Przykład



x1(0)=0,5
x2(0)=0,5
u* = −1
u* = +1
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
x2
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
x1

Przykład


(-0,1; 0,9)
(0,5; 0,5)
(-0,3; 0,2)
(0,2; -0,3)
(-0,5; -0,5)
(0,2; -0,8)
x_2
x_1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8

Własne wartości zespolone

$$\frac{d^2 y(t)}{dt^2} + \omega^2 y(t) = Ku(t),$$

$$K > 0 \qquad |u(t)| \leq 1$$

$$\dot{y}_1(t) = y_2(t),$$

$$\dot{y}_2(t) = -\omega^2 y_1(t) + Ku(t),$$

$$x_1(t) = \frac{\omega}{K} y_1(t);$$

$$x_2(t) = \frac{1}{K} y_2(t).$$

$$\dot{x}_1(t) = \omega x_2(t),$$

$$\dot{x}_2(t) = -\omega x_1(t) + u(t),$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$A_x = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

$$\det(sI - A_x) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\right) =$$

$$= \det\left(\begin{bmatrix} s & -\omega \\ \omega & s \end{bmatrix}\right) = s^2 + \omega^2 = 0.$$

$$s_1 = j\omega, \quad s_2 = -j\omega,$$

System autonomiczny

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$x(t) = \Phi_x(t)\,x(0),$$

$$\Phi_x(t) = L^{-1}\left[\Phi_x(s)\right] = L^{-1}[sI - A_x]^{-1}$$

$$\Phi_x(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \qquad \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$

$$x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Realizacja w MatLabie

```
>> syms  s  Ax  omega
I=eye(2);
Ax=[0  omega;  -omega  0];
phix=ilaplace(inv(s*I-Ax))

phix =
[  cos(omega*t),  sin(omega*t)]
[ -sin(omega*t),  cos(omega*t)]
```

Analiza ruchu

$$x_1(t) = x_{10}\cos\omega t + x_{20}\sin\omega t;$$

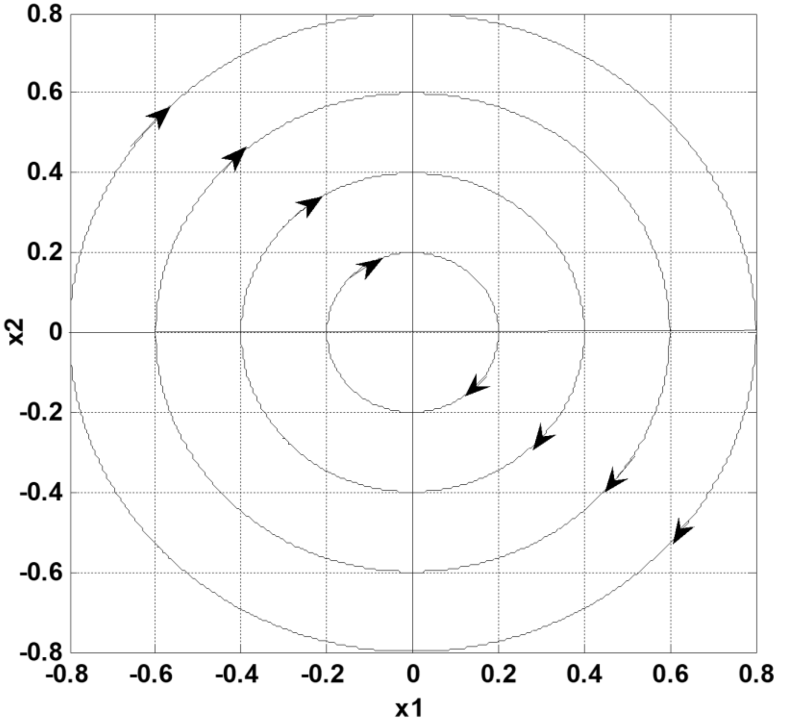
$$x_2(t) = -x_{10}\sin\omega t + x_{20}\cos\omega t.$$

$$x_1^2(t) + x_2^2(t) = x_{10}^2 + x_{20}^2 = R^2$$

$$x_2(t) = \pm\sqrt{R^2 - x_1^2(t)}$$

Realizacja w MatLabie

```
>> R=[0.2   0.4   0.6   0.8];
for i=1:4
x1=[-R(i):0.001:R(i)];
x2=sqrt(R(i).^2-x1.^2);
plot(x1,x2, '-')
hold on
x2=-sqrt(R(i).^2-x1.^2);
plot(x1,x2, '-')
xlabel('x1'), ylabel('x2')
grid on
end
```



Sterowanie czasooptymalne

$$\begin{cases} \dot{x}_1(t) = \omega\, x_2(t), \\ \dot{x}_2(t) = -\omega\, x_1(t) + u(t). \end{cases} \qquad |u(t)| \le 1.$$

$$H = \sum_{\alpha=1}^{2} \lambda_\alpha f_\alpha(x,u) = \omega\, x_2 \lambda_1 - \omega\, x_1 \lambda_2 + u \lambda_2$$

$$u = \mathbf{sign}\,\{\lambda_2\}$$

max H

Sterowanie czasooptymalne

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} = \omega\,\lambda_2,$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -\omega\,\lambda_1,$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\lambda(t) = \Phi_{\lambda}\,(t)\lambda(0)$$

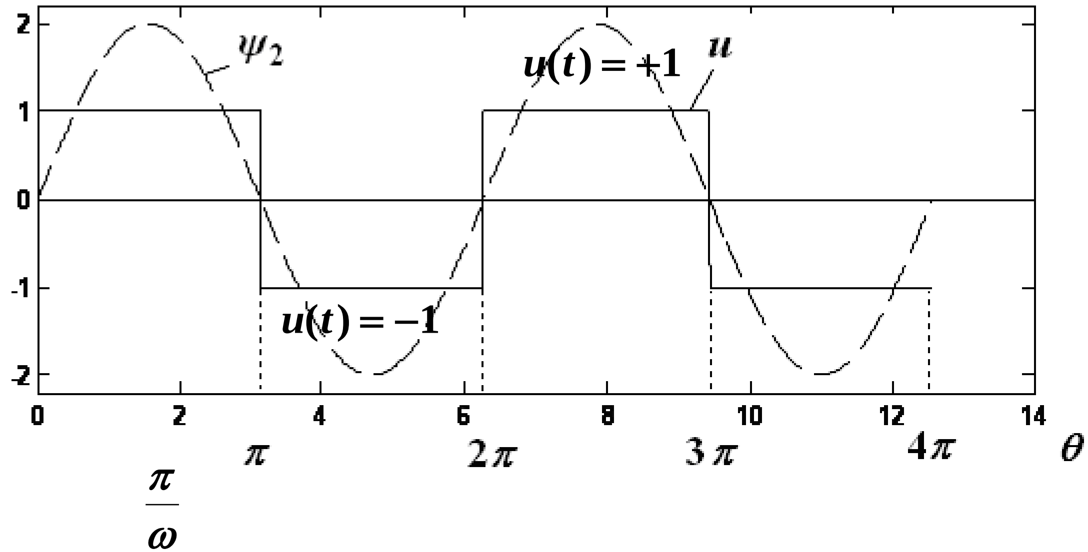
$$\Phi_{\lambda}(t) = L^{-1}\left[\Phi_{\lambda}(s)\right] = L^{-1}[sI - A_{\lambda}]^{-1} \qquad A_{\lambda} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

$$\Phi_{\lambda}(t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \qquad \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \lambda_1(0) \\ \lambda_2(0) \end{bmatrix},$$

$$\lambda_2(t) = -\lambda_1(0)\sin\omega t + \lambda_2(0)\cos\omega t \qquad \lambda_2(t) = a\,(\sin\omega t + \alpha) = a\,(\sin\theta + \alpha)$$

$$\theta = \omega t; \quad a = \sqrt{\lambda_1^2(0) + \lambda_2^2(0)}\,; \quad \alpha = \mathbf{arctg}\,\frac{-\lambda_2(0)}{\lambda_1(0)}$$

```
>> psi10=2;
psi20=0;
alpha=atan(-psi20/ psi10);
a=sqrt(psi10^2+ psi20^2);
theta=[0:0.01:4*pi];
psi2=a*(sin(theta)+alpha);
plot(theta, psi2, '--')
hold on
u=sign(psi2);
plot(theta, u, '-')
```



$$u = \mathbf{sign}\,\{\lambda_2\}$$

Rozwiązanie

$$\begin{cases} \dot{x}_1(t) = \omega\, x_2(t), \\ \dot{x}_2(t) = -\omega\, x_1(t) + u(t) \end{cases}$$

$$u = \pm 1$$

$$x_{10} = x_1(0), \quad x_{20} = x_2(0)$$

$$x_1(t) = \left(x_{10} - \frac{u}{\omega} \right) \cos \omega t + x_{20} \sin \omega t + \frac{u}{\omega},$$

$$x_2(t) = -\left(x_{10} - \frac{u}{\omega} \right) \sin \omega t + x_{20} \cos \omega t.$$

Transformacje

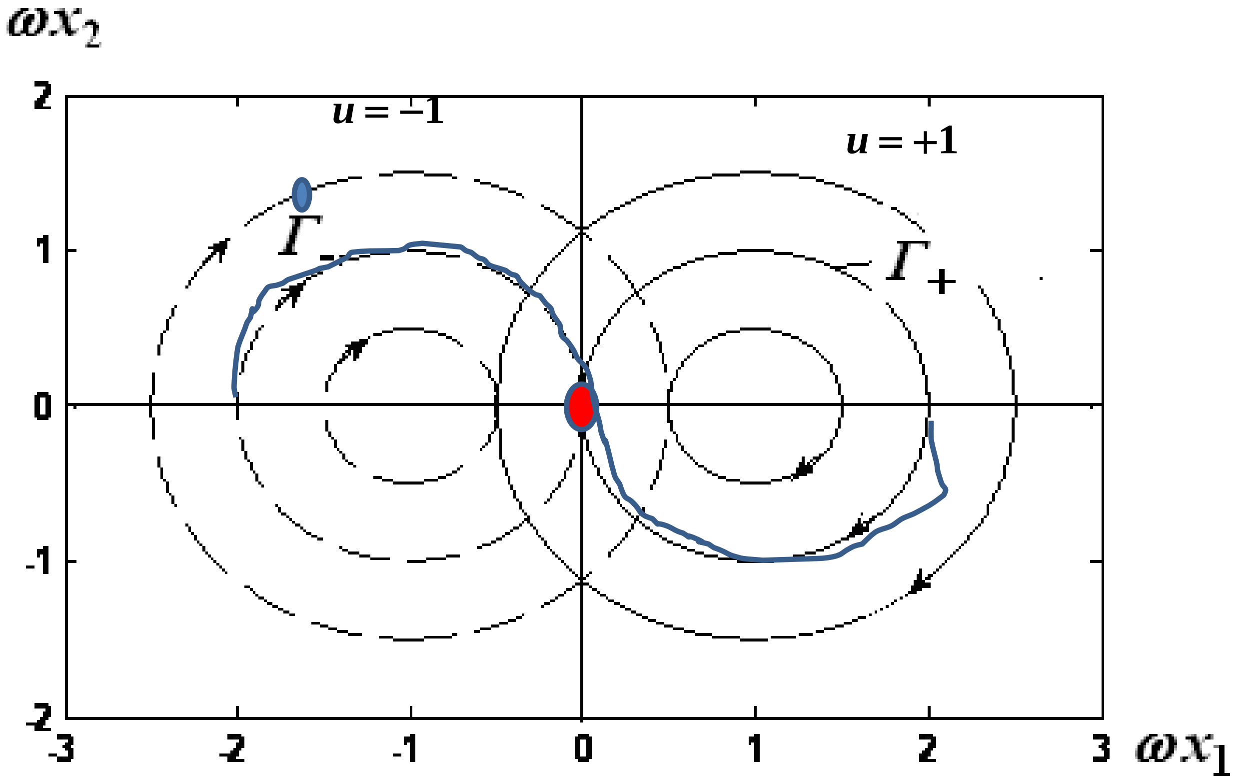
$$\omega x_1(t) = (\omega x_{10} - u)\cos\omega t + \omega x_{20}\sin\omega t + u,$$
$$\omega x_2(t) = -(\omega x_{10} - u)\sin\omega t + \omega x_{20}\cos\omega t.$$

$$[\omega x_1(t) - u]^2 + [\omega x_2]^2 = (\omega x_{10} - u)^2 + (\omega x_{20})^2$$

$x_1, x_2 \longrightarrow \omega x_1, \omega x_2$

$$\omega x_2(t) = \pm\sqrt{(\omega x_{10} - u)^2 + (\omega x_{20})^2 - [\omega x_1(t) - u]^2}$$

```
>> u=1;
omegax10=[1.5   2   2.5];
omegax20=0;
omegax1=-3:0.001:3;
for i=1:3
omegax2=sqrt((omegax10(i)    -    u).^2+    omegax20.^2-(omegax1-
u).^2);
plot(omegax1, omegax2, '-')
hold on
omegax2=-sqrt((omegax10(i)    -    u).^2+    omegax20.^2-(omegax1-
u).^2);
plot(omegax1, omegax2, '-')
hold on
end
>>u=-1;
omegax10=[-1.5   -2   -2.5];
for i=1:3
omegax2=sqrt((omegax10(i)    -    u).^2+    omegax20.^2-(omegax1-
u).^2);
plot(omegax1, omegax2, '--')
hold on
omegax2=-sqrt((omegax10(i)    -    u).^2+    omegax20.^2-(omegax1-
u).^2);
plot(omegax1, omegax2, '--')
hold on
end
```

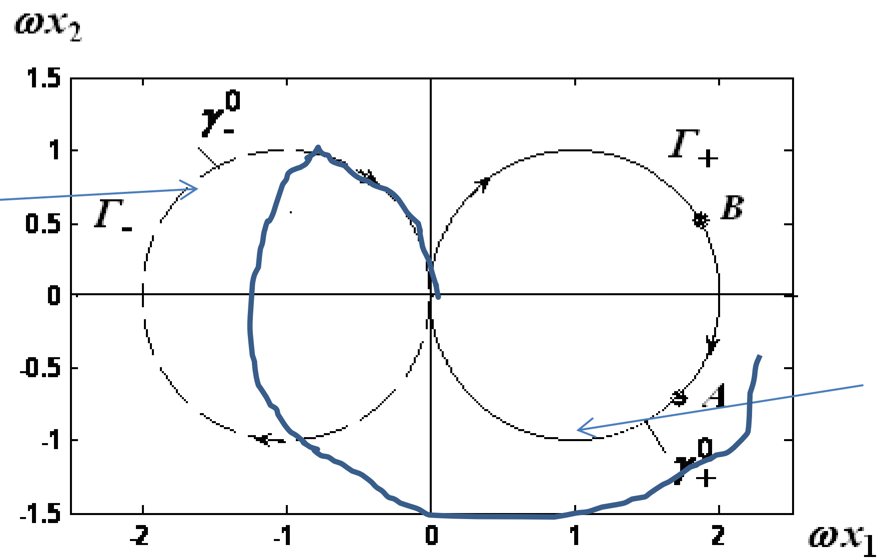


$$\Gamma_+ = \{(\omega x_1, \omega x_2) : (\omega x_1 - 1)^2 + (\omega x_2)^2 = 1\};$$

$$\Gamma_- = \{(\omega x_1, \omega x_2) : (\omega x_1 + 1)^2 + (\omega x_2)^2 = 1\}.$$

Krok 1

$$\omega x_2 = \sqrt{1-(\omega x_1 + 1)^2}$$



$$\omega x_2 = -\sqrt{1-(\omega x_1 - 1)^2}$$

(2, 0) → (0, 0)

$$t = \frac{\pi}{\omega}$$

$$u^* = u^*(\omega x_1, \omega x_2) = +1 \; jeśli \; (\omega x_1, \omega x_2) \in \gamma_+^0;$$

$$u^* = u^*(\omega x_1, \omega x_2) = -1 \; jeśli \; (\omega x_1, \omega x_2) \in \gamma_-^0.$$

$$\gamma_+^0 = \{(\omega x_1, \omega x_2) : (\omega x_1 - 1)^2 + (\omega x_2)^2 = 1, \quad \omega x_2 < 0\}$$

$$\gamma_-^0 = \{(\omega x_1, \omega x_2) : (\omega x_1 + 1)^2 + (\omega x_2)^2 = 1, \quad \omega x_2 > 0\}$$

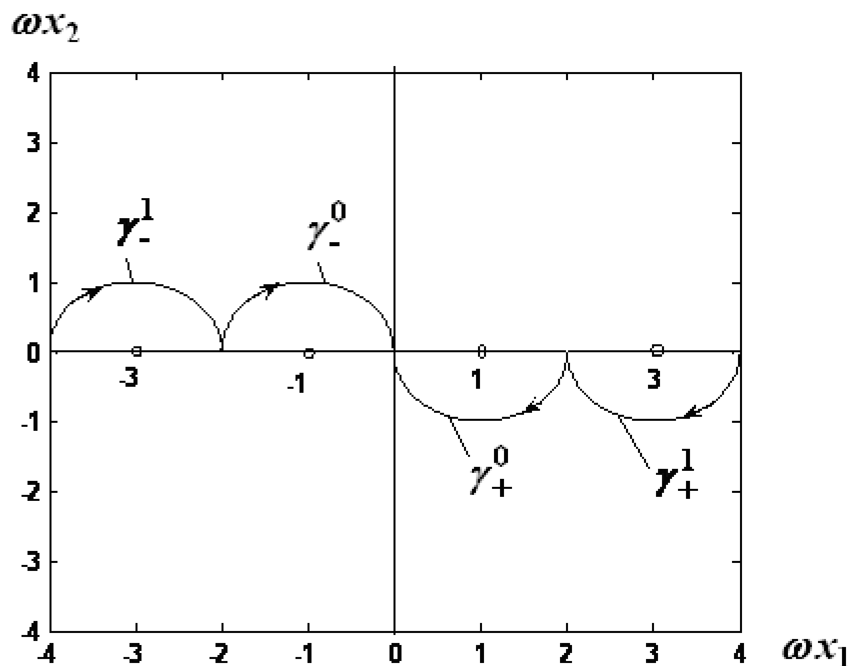
Krok 2

$$\gamma_{+}^{1}=\{(\omega x_1,\omega x_2):(\omega x_1-3)^2+(\omega x_2)^2=1,\quad \omega x_2<0\}\qquad \omega x_2=-\sqrt{1-(\omega x_1-3)^2}$$

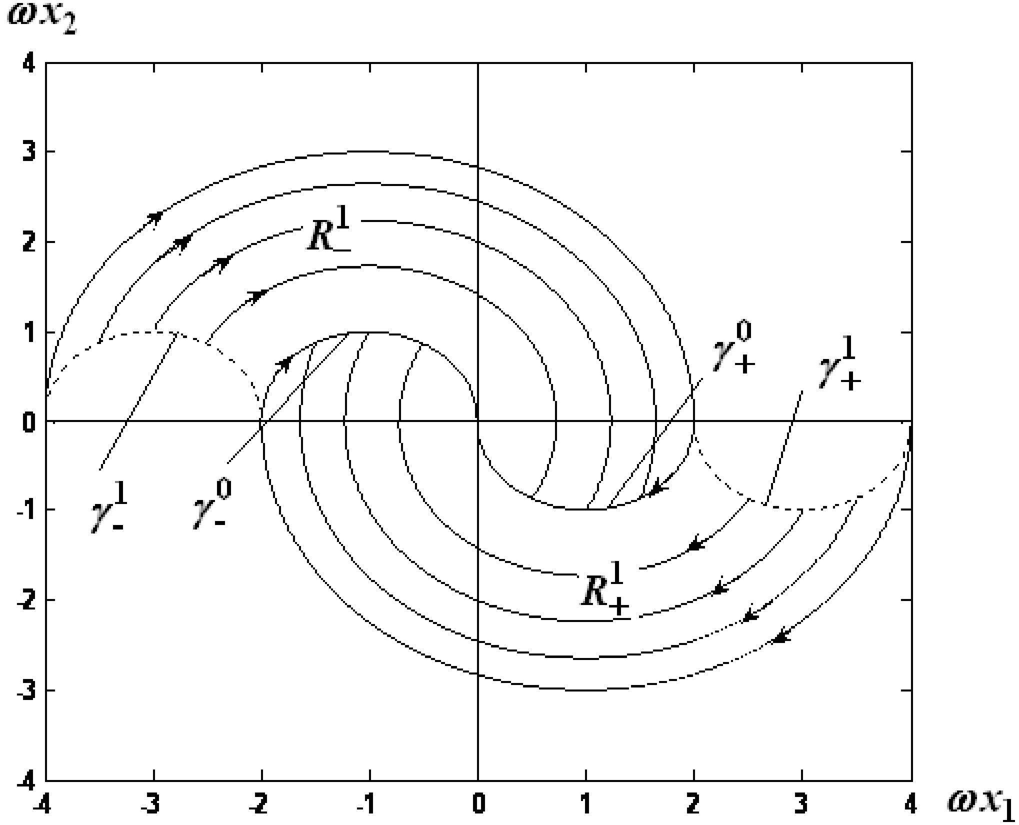
$$\gamma_{-}^{1}=\{(\omega x_1,\omega x_2):(\omega x_1+3)^2+(\omega x_2)^2=1,\quad \omega x_2>0\}\qquad \omega x_2=\sqrt{1-(\omega x_1+3)^2}$$

$$u^{*}=u^{*}(\omega x_1,\omega x_2)=+1\ \textit{jeśli}\ (\omega x_1,\omega x_2)\in R_{+}^{1};$$
$$u^{*}=u^{*}(\omega x_1,\omega x_2)=-1\ \textit{jeśli}\ (\omega x_1,\omega x_2)\in R_{-}^{1}.$$

```
>> %gammaplus0
omegax1=[0:0.001:4];
omegax2=- sqrt(1-(omegax1 - 1).^2);
plot(omegax1, omegax2, '-')
hold on
%gammaminus0
omegax1=[-4:0.001:0];
omegax2= sqrt(1-(omegax1 + 1).^2);
plot(omegax1, omegax2, '-')
hold on
%gammaplus1
omegax1=[0:0.001:4];
omegax2=- sqrt(1-(omegax1 - 3).^2);
plot(omegax1, omegax2, '-')
hold on
%gammaminus1
omegax1=[-4:0.001:0];
omegax2= sqrt(1-(omegax1 + 3).^2);
plot(omegax1, omegax2, '-')
```



Uogólnienie

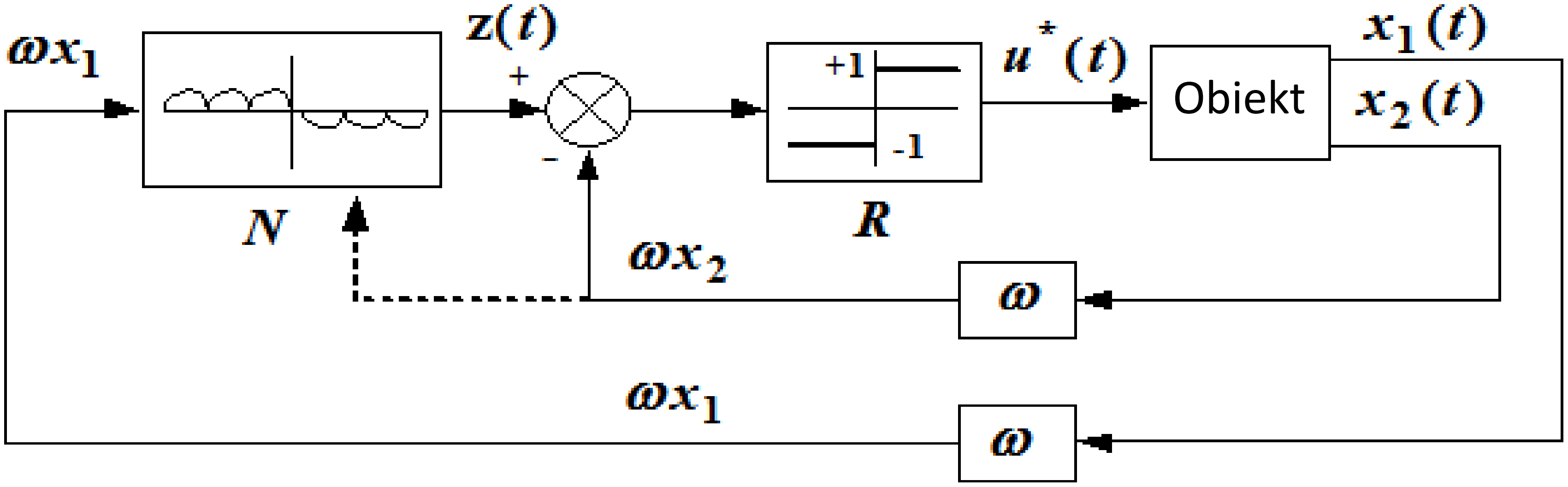


$$\gamma = \left[\bigcup_{k=0}^{\infty} \gamma_+^k\right] \cup \left[\bigcup_{k=0}^{\infty} \gamma_-^k\right] = \gamma_+ \cup \gamma_-$$

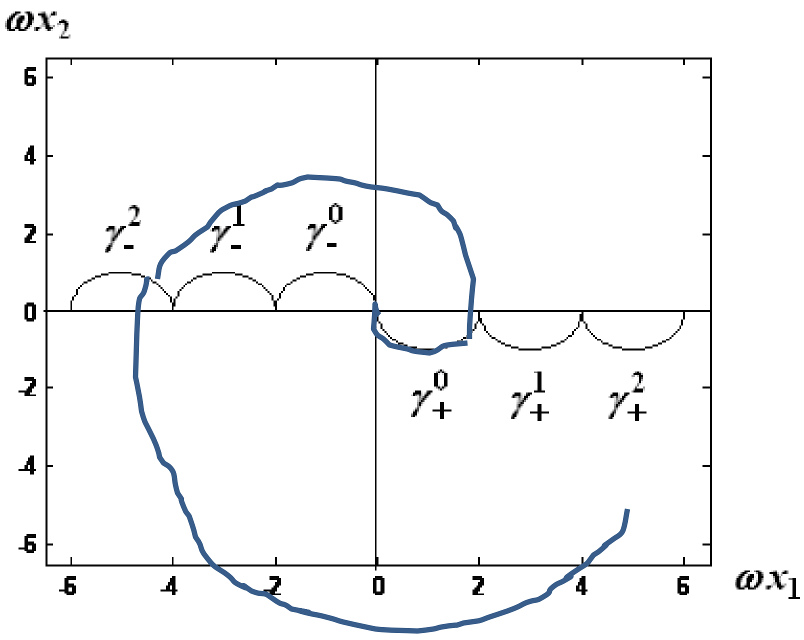
$$\begin{cases} R_- = \bigcup_{k=0}^{\infty} R_-^k, \\ R_+ = \bigcup_{k=0}^{\infty} R_+^k, \end{cases}$$

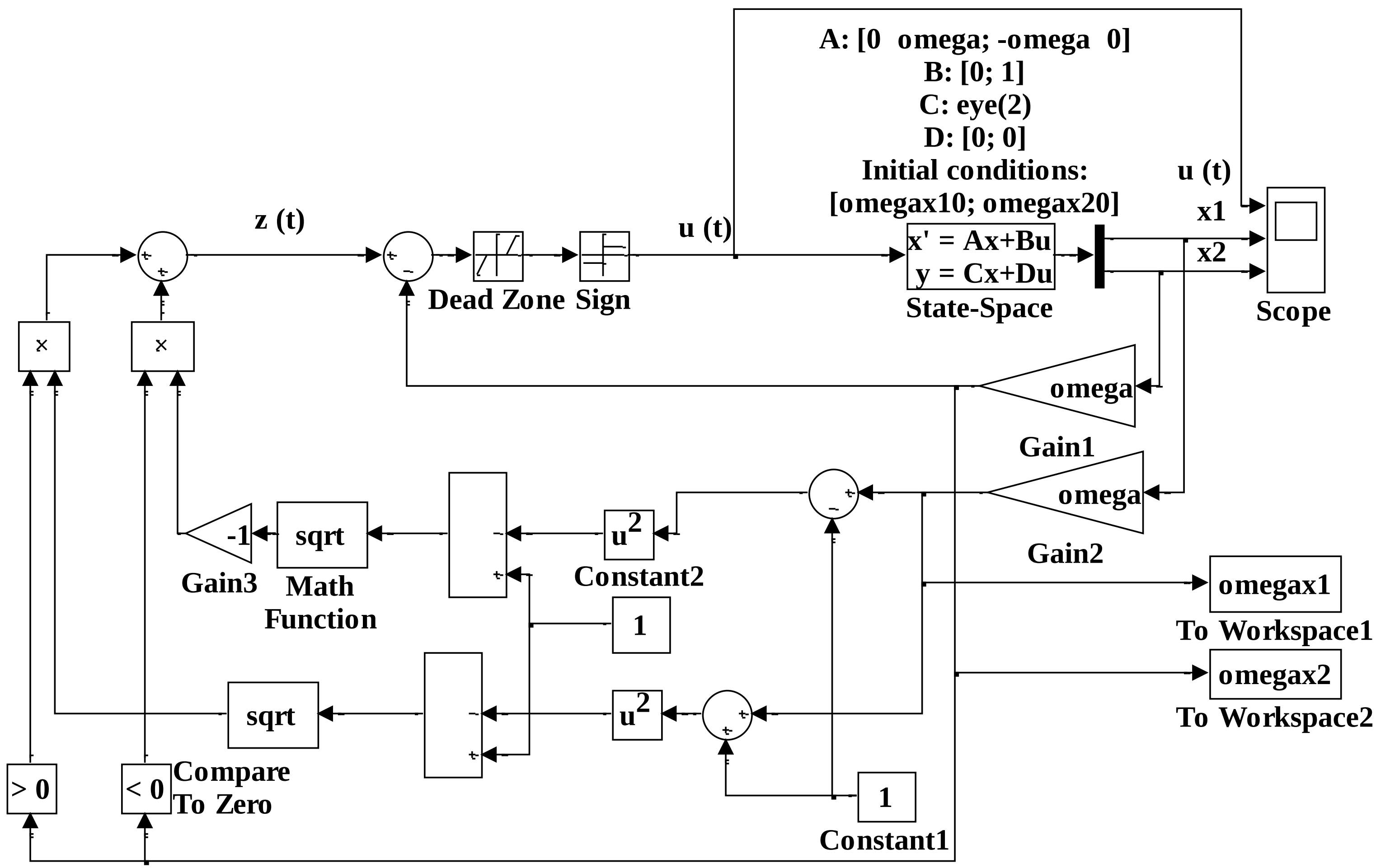
$$\begin{cases} u^* = u^*(\omega x_1, \omega x_2) = +1 \; jeśli \; (\omega x_1, \omega x_2) \in R_+ \cup \gamma_+, \\ u^* = u^*(\omega x_1, \omega x_2) = -1 \; jeśli \; (\omega x_1, \omega x_2) \in R_- \cup \gamma_-. \end{cases}$$

Schemat



```
>>for  k=1:3;
%gammaplusk
omegax1=[0:0.001:2*k];
omegax2=- sqrt(1-(omegax1 - (2*k-1)).^2);
plot(omegax1, omegax2, '-')
hold on
%gammaminusk
omegax1=[-2*k:0.001:0];
omegax2= sqrt(1-(omegax1 + (2*k-1)).^2);
plot(omegax1, omegax2, '-')
end
```

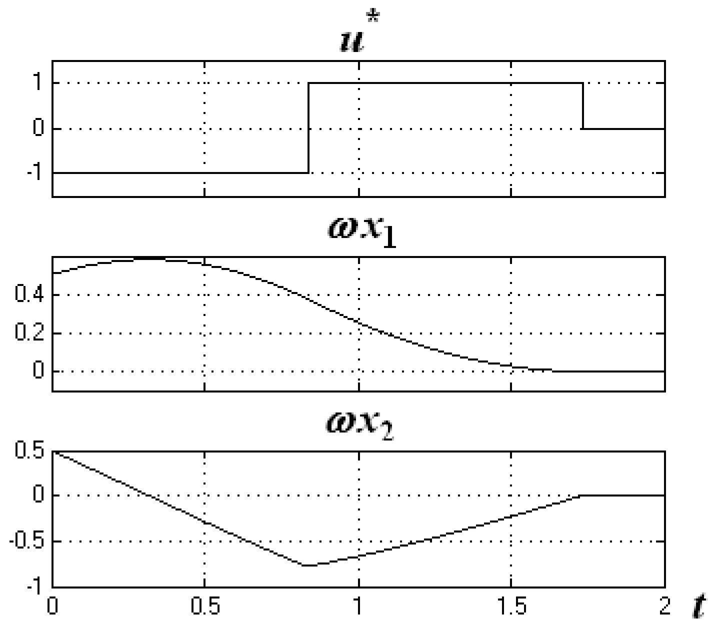


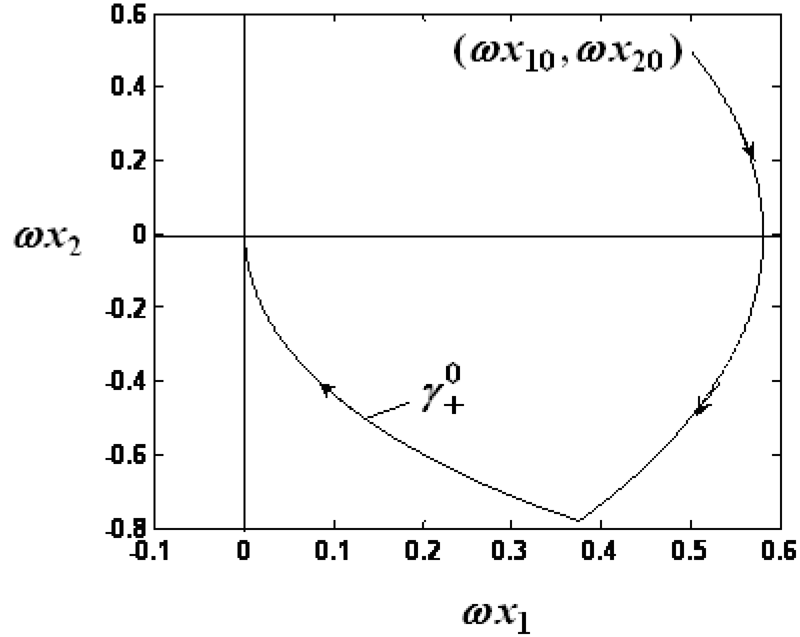


sys8_2order_complex.mdl

Przykład

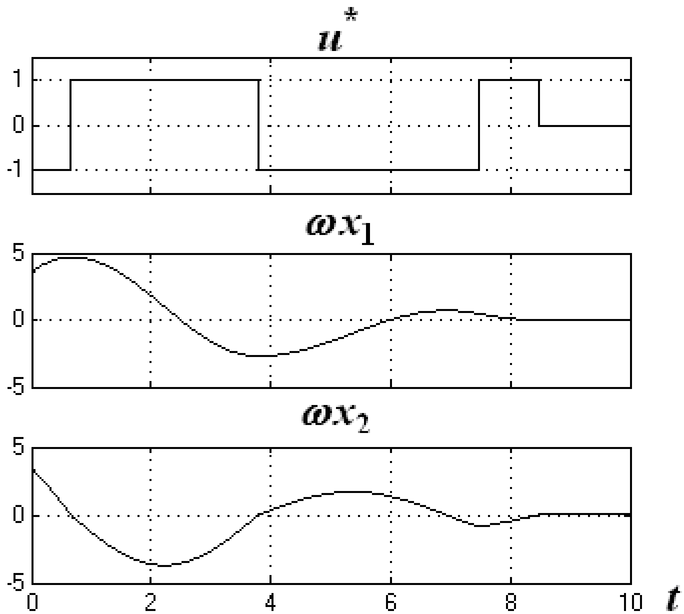
$\omega x_1(0) = 0{,}5 \qquad \omega x_2(0) = 0{,}5$



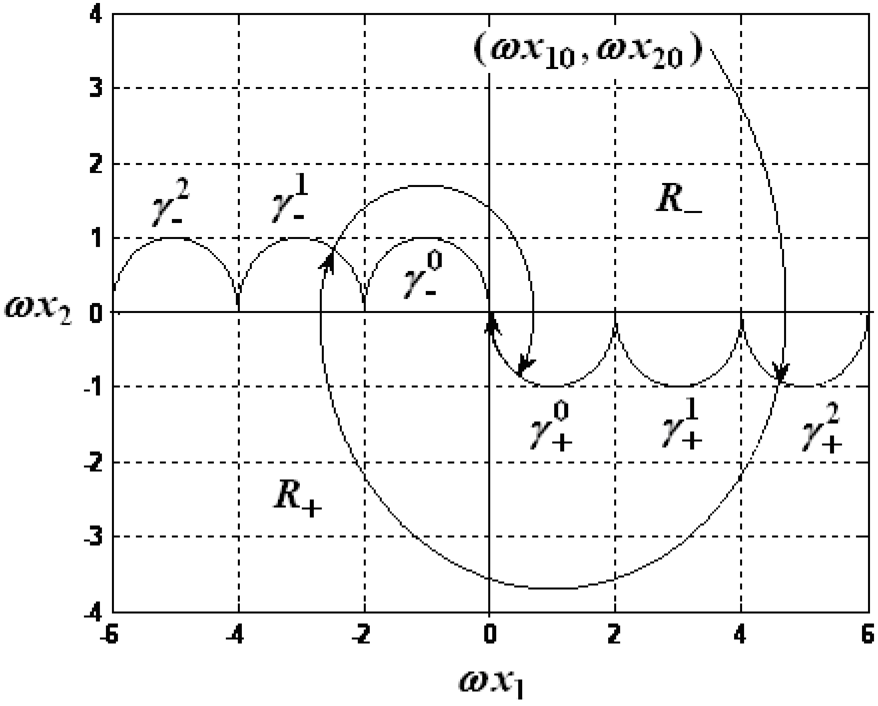


plot(omegax1.Data,omegax2.Data)

Przykład

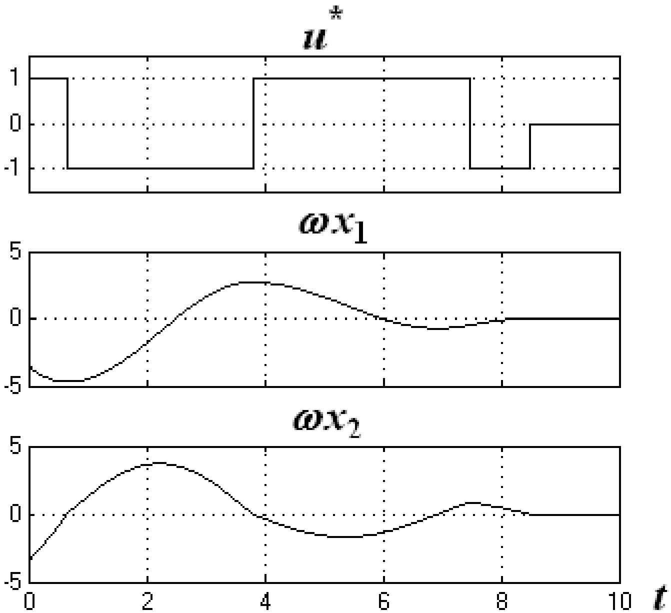


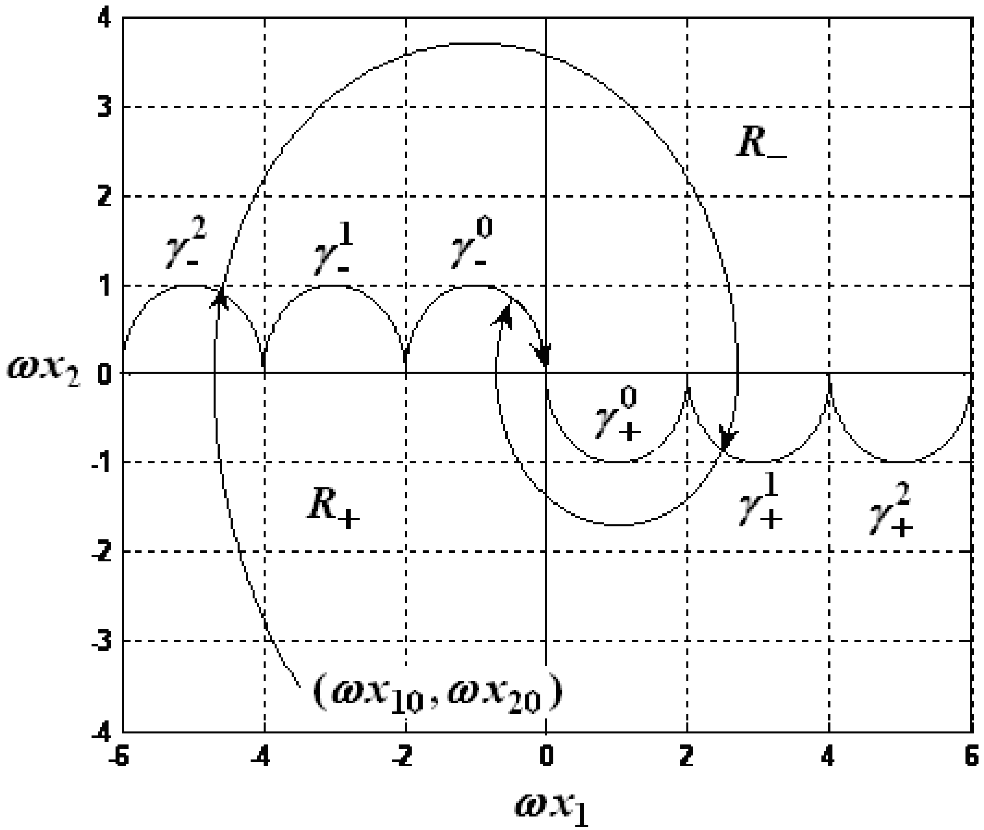
$\omega x_1(0) = 3{,}5$ $\omega x_2(0) = 3{,}5$



Przykład

$\omega x_1(0) = -3{,}5 \qquad \omega x_2(0) = -3{,}5$





Przykład

$\omega x_1(0) = 12$ $\omega x_2(0) = -10$