Teoria i metody optymalizacji

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Optymalizacja układu dynamicznego

OGRANICZENIE

$$\dot{x}(t) = \varphi\,[x(t), u(t), t)], \quad x(t_0) = x_0$$

Opis układu dynamicznego

CEL

CEL ZMODYFIKOWANY

$$J\left(x(t)\right) = \int_{t_0}^{t_f} G\,(x, \dot{x}, u, t)dt \to \min$$

$$J = \int_{t_0}^{t_f} F\,(x, \dot{x}, t)dt \to \min$$

Metoda podstawienia

$$\frac{d^2x(t)}{dt^2} = u(t) \quad (1)$$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \qquad x_1 = x, \quad x_2 = \dot{x}_1 = \dot{x}$$

OGRANICZENIE

CEL

$$W = \int_0^T u^2(t)dt \to \min$$

$$x_1(0) = x_{10}, \quad x_1(T) = x_{1T},$$

$$x_2(0) = x_{20}, \quad x_2(T) = x_{2T},$$

$$t_0 = 0, \quad t_1 = T.$$

Z (1) wynika

$$W = \int_0^T \ddot{x}^2(t)dt.$$

Rozwiązanie

$$F(x)=\left(\frac{d^2x}{dt^2}\right)^2;F_x=\frac{\partial F}{\partial x}=0;\quad F_{\dot{x}}=\frac{\partial F}{\partial \dot{x}}=0;\quad F_{\ddot{x}}=\frac{\partial F}{\partial \ddot{x}}=2\frac{d^2x}{\partial t^2}.$$

$$F_x-\frac{d}{dt}F_{\dot{x}}+\frac{d^2}{dt^2}F_{\ddot{x}}+\ldots+(-1)^n\frac{d^n}{dt^n}F_{x^{(n)}}=0,$$

Rozwiązanie

$$2\frac{d^4x}{dt^4}=0.$$

Całkując dwa razy otrzymamy

$$u(t)=\ddot{x}(t)=\dot{x}_2(t)=C_1t+C_2.$$

Metoda mnożników Lagrange'a

CEL

OGRANICZENIE

$$f(x_1, x_2) \to \min$$
$$g(x_1, x_2) = 0$$

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

Równania algebraiczne

$$\partial L / \partial \mathrm{x} = 0,$$
$$\partial L / \partial \lambda = 0$$

$$d^2 L(\mathrm{x}_0, \lambda) > 0$$

Metoda mnożników Lagrange'a (cd)

CEL

$$J = \int_{t_0}^{t_f} F(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$

OGRANICZENIE

$$g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0; \quad i = 1, 2, \ldots, m$$

$$J_a = \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t)dt$$

$$\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = F(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \sum_i \boldsymbol{\lambda}_i(t) g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

$$\boldsymbol{\lambda}(t) = \left[\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t)\right]'$$

Układ równań Eulera-Lagrange'a

$$J(x_1, \dots, x_n) = \int_{t_0}^{t_1} F\,(x_1, \dots, x_n; \dot{x}_1, \dots, \dot{x}_n, t)dt$$

Układ równań Eulera-Lagrange'a

$$F_{x_i} - \frac{d}{dt} F_{\dot{x}_i} = 0, \quad i = 1, \dots, n.$$

Warunki Legendre'a

$$F_{\dot{x}_1,\dot{x}_1} \geq 0, \quad \begin{vmatrix} F_{\dot{x}_1,\dot{x}_1} & F_{\dot{x}_1,\dot{x}_2} \\ F_{\dot{x}_2,\dot{x}_1} & F_{\dot{x}_2,\dot{x}_2} \end{vmatrix} \geq 0, \dots,$$

$$\begin{vmatrix} F_{\dot{x}_1,\dot{x}_1} & \dots & F_{\dot{x}_1,\dot{x}_n} \\ \dots & \dots & \dots \\ F_{\dot{x}_n,\dot{x}_1} & \dots & F_{\dot{x}_n,\dot{x}_n} \end{vmatrix} \geq 0.$$

Przykład dla funkcji 2-ch argumentów $F_{\dot{x}_1,\dot{x}_1} \geq 0, \quad F_{\dot{x}_1,\dot{x}_1} F_{\dot{x}_2,\dot{x}_2} - F_{\dot{x}_1,\dot{x}_2} F_{\dot{x}_2,\dot{x}_1} \geq 0$

Metoda mnożników Lagrange'a

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}_i}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}_i}\right)_* = 0 \Rightarrow g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{\lambda}}_i} = 0$$

$$\mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), t) = F(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \sum_i \boldsymbol{\lambda}_i(t) g_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

Przykład

CEL

$$J = \int_0^1 \left[x^2(t) + u^2(t) \right] dt$$

$$x(0) = 1; \qquad x(1) = 0$$

OGRANICZENIE

$$\dot{x}(t) = -x(t) + u(t)$$

Metoda podstawienia

$$J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt$$

$$\dot{x}(t) = -x(t) + u(t)$$

$$J = \int_0^1 [x^2(t) + (\dot{x}(t) + x(t))^2] dt$$
$$= \int_0^1 [2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t)] dt.$$

Metoda podstawienia (cd)

$$V = 2x^2(t) + \dot{x}^2(t) + 2x(t)\dot{x}(t);$$

$$\boxed{\left(\frac{\partial V}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right)_* = 0}$$

$$4x^*(t) + 2\dot{x}^*(t) - \frac{d}{dt}(2\dot{x}^*(t) + 2x^*(t)) = 0$$

$$\boxed{\ddot{x}^*(t) - 2x^*(t) = 0}$$

$$x^*(t) = C_1 e^{-\sqrt{2}t} + C_2 e^{\sqrt{2}t}$$

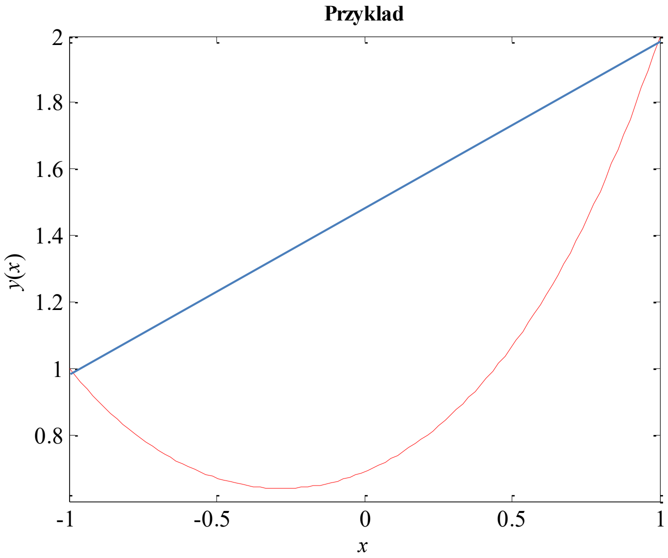
$$x(0) = 1; \qquad x(1) = 0$$

$$C_1 = 1/(1 - e^{-2\sqrt{2}}); \qquad C_2 = 1/(1 - e^{2\sqrt{2}}).$$

$$\begin{aligned} u^*(t) &= \dot{x}^*(t) + x^*(t) \\ &= C_1(1 - \sqrt{2})e^{-\sqrt{2}t} + C_2(1 + \sqrt{2})e^{\sqrt{2}t} \end{aligned}$$

Metoda podstawienia (cd)

Rozwiazanie
Dane wejsciowe:
Funkcja podcalkowa F(x,y,y')=Dy^2 + 2*Dy*y + 2*y^2
Warunki z lewej strony: y(-1)=1
warunki z prawej strony: y(1)=2
Fy=2*Dy + 4*y
Fy'=2*Dy + 2*y
dFy'/dx=2*D2y + 2*Dy
warunek Legendre a:
Fy'y'=2
Rownianie Eulera:
4*y - 2*D2y=0
Rozwiazanie ogolne rowniania Eulera :
y(x)=C2*exp(2^(1/2)*x) + C3/exp(2^(1/2)*x)



Jextr = 9.4496

Jlin = 12.8333

w1ex3

Metoda mnożników Lagrange'a

CEL

$$J = \int_0^1 \left[x^2(t) + u^2(t)\right] dt$$

$$x(0) = 1; \qquad x(1) = 0$$

OGRANICZENIE

$$\dot{x}(t) = -x(t) + u(t)$$

$$g(x(t), \dot{x}(t), u(t)) = \dot{x}(t) + x(t) - u(t) = 0$$

Metoda mnożników Lagrange'a (cd)

$$J = \int_0^1 \left[x^2(t) + u^2(t) + \lambda(t)\left\{\dot{x}(t) + x(t) - u(t)\right\}\right] dt$$
$$= \int_0^1 \mathcal{L}(x(t), \dot{x}(t), u(t), \lambda(t)) dt$$

$$\mathcal{L}(x(t), \dot{x}(t), u(t), \lambda(t)) = x^2(t) + u^2(t) + \lambda(t)\left\{\dot{x}(t) + x(t) - u(t)\right\}$$

Metoda mnożników Lagrange'a (cd)

$$\mathcal{L}(x(t),\dot{x}(t),u(t),\lambda(t)) = x^2(t)+u^2(t)+\lambda(t)\{\dot{x}(t)+x(t)-u(t)\}$$

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_* = 0 \longrightarrow \boxed{2x^*(t)+\lambda^*(t)-\dot{\lambda}^*(t)=0}$$

$$\left(\frac{\partial \mathcal{L}}{\partial u}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right)_* = 0 \longrightarrow \boxed{2u^*(t)-\lambda^*(t)=0}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\lambda}}\right)_* = 0 \longrightarrow \boxed{\dot{x}^*(t)+x^*(t)-u^*(t)=0}$$

$$\lambda^*(t) = 2u^*(t) = 2(\dot{x}^*(t)+x^*(t))$$

$$2x^*(t)+2(\dot{x}^*(t)+x^*(t))-2(\ddot{x}^*(t)+\dot{x}^*(t))=0$$

$$\boxed{\ddot{x}^*(t)-2x^*(t)=0} \longrightarrow \boxed{x^*(t)=C_1e^{-\sqrt{2}t}+C_2e^{\sqrt{2}t}}$$

```
x=dsolve('D2x-2*x=0','x(0)=1,x(1)=0')
```

Przykład

$$J = \frac{1}{2}\int_0^2 u^2 dt$$

$$\ddot{x} = u(t)$$

$$J = \frac{1}{2}\int_0^2 (\ddot{x})^2 dt$$

$$x(t=0)=1, \quad x(t=2)=0$$

$$\dot{x}(t=0)=1, \quad \dot{x}(t=2)=0$$

Metoda podstawienia

$$J = \int_{t_0}^{t_1} F\,[x(t), \dot{x}(t), \ldots, x^{(n)}(t), t]dt.$$

$$F_x - \frac{d}{dt}F_{\dot{x}} + \frac{d^2}{dt^2}F_{\ddot{x}} + \ldots + (-1)^n \frac{d^n}{dt^n}F_{x^{(n)}} = 0,$$

Dostateczne warunki minimum

$$F_{x^{(n)}, x^{(n)}} \geq 0;$$

Przykład

$$\frac{d^2 x(t)}{dt^2} = u(t)$$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \qquad x_1 = x, \quad x_2 = \dot{x}_1 = \dot{x}$$

$$x_1(0) = x_{10}, \quad x_1(T) = x_{1T},$$

$$x_2(0) = x_{20}, \quad x_2(T) = x_{2T},$$

$$t_0 = 0, \quad t_1 = T.$$

$$W = \int_0^T u^2(t)dt \to \min$$

$$W = \int_0^T \ddot{x}^2(t)dt.$$

Rozwiązanie

$$F(x)=\left(\frac{d^2x}{dt^2}\right)^2; F_x=\frac{\partial F}{\partial x}=0; \quad F_{\dot{x}}=\frac{\partial F}{\partial \dot{x}}=0; \quad F_{\ddot{x}}=\frac{\partial F}{\partial \ddot{x}}=2\frac{d^2x}{\partial t^2}.$$

$$F_x-\frac{d}{dt}F_{\dot{x}}+\frac{d^2}{dt^2}F_{\ddot{x}}+\ldots+(-1)^n\frac{d^n}{dt^n}F_{x^{(n)}}=0,$$

Rozwiązanie

$$2\frac{d^4x}{dt^4}=0.$$

Całkując dwa razy otrzymamy

$$u(t)=\ddot{x}(t)=\dot{x}_2(t)=C_1t+C_2.$$

Model w przestrzeni stanów

$$\ddot{\theta} = u(t) \quad J = \frac{1}{2}\int_0^2 (\ddot{\theta})^2 dt \quad \begin{matrix} \theta(t=0) = 1, & \theta(t=2) = 0 \\ \dot{\theta}(t=0) = 1, & \dot{\theta}(t=2) = 0 \end{matrix}$$

$$x_1(t) = \theta(t), \quad \dot{x}_1 = x_2(t), \quad \dot{x}_2 = u(t)$$

$$\dot{x}(t) = Ax(t) + bu(t) \quad x^T = [x_1 x_2], \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b^T = [01]$$

$$L = \int_0^2 \left\{ \frac{1}{2}u^2(t) + \lambda^T(t)[\mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) - \dot{\mathbf{x}}] \right\} dt =$$

$$\int_0^2 \left\{ \frac{1}{2}u^2(t) + \lambda_1(t)\left[x_2(t) - \dot{x}_1\right] + \lambda_2(t)\left[u(t) - \dot{x}_2\right] \right\} dt$$

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x'} = 0 \Rightarrow$$

x1 x2 u

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1(t), \quad u(t) = -\lambda_2(t)$$

Równania Eulera Lagranga'e dla **x1,x2,u**

$$u = 3t - \frac{7}{2} \Rightarrow x_2 = \frac{3}{2}t^2 - \frac{7}{2}t + 1 \Rightarrow x_1 = \frac{1}{2}t^3 - \frac{7}{4}t^2 + t + 1$$

Rozwiązanie przez równanie Eulera-Poissona

$$\ddot{x} = u(t) \quad J = \frac{1}{2}\int_0^2 u^2 dt$$

F

$$L = \int_0^2 \left\{ \frac{1}{2}u^2(t) + \lambda(t)[\ddot{x} - u(t)] \right\} dt$$

$$F_x - \frac{d}{dt}F_{\dot{x}} + \frac{d^2}{dt^2}F_{\ddot{x}} = 0$$

$$F_u - \frac{d}{dt}F_{\dot{u}} + \frac{d^2}{dt^2}F_{\ddot{u}} = 0$$

$$u - \lambda = 0$$

$$u = \lambda$$

$$\ddot{\lambda} = 0$$

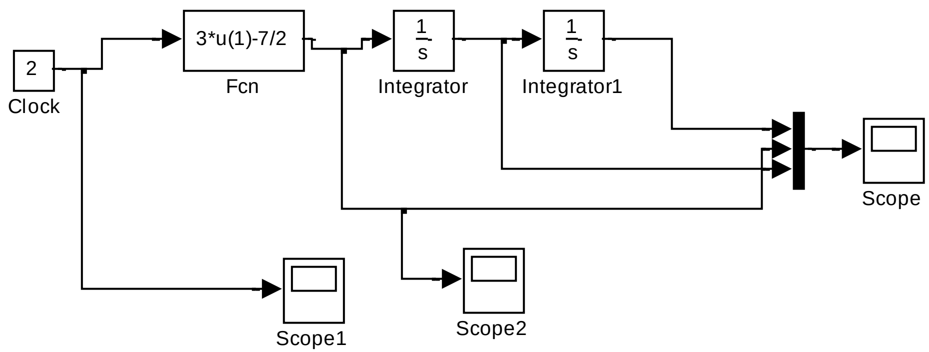
$$\dot{\lambda} = c_1$$

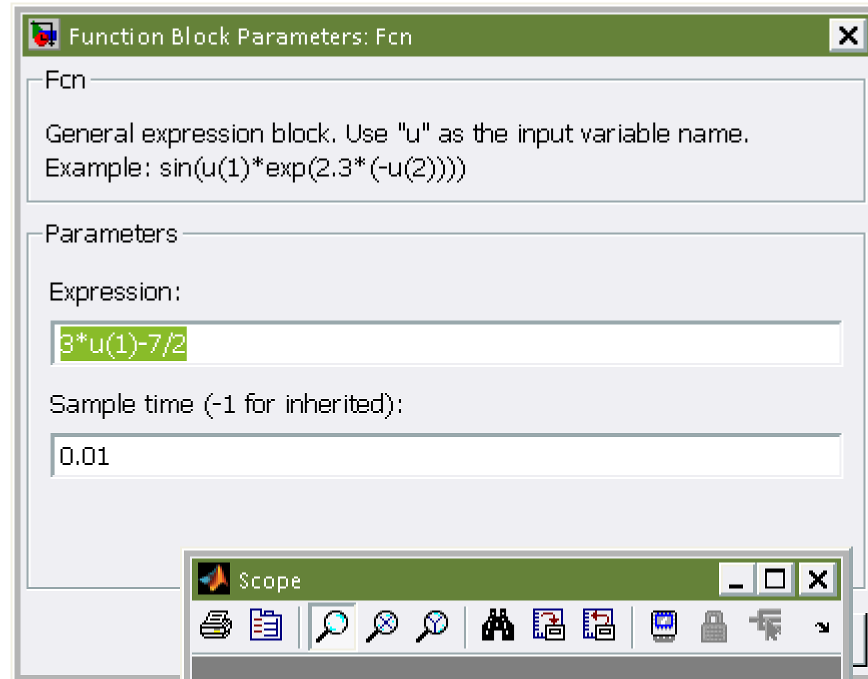
$$\lambda = c_1 t + c_2$$

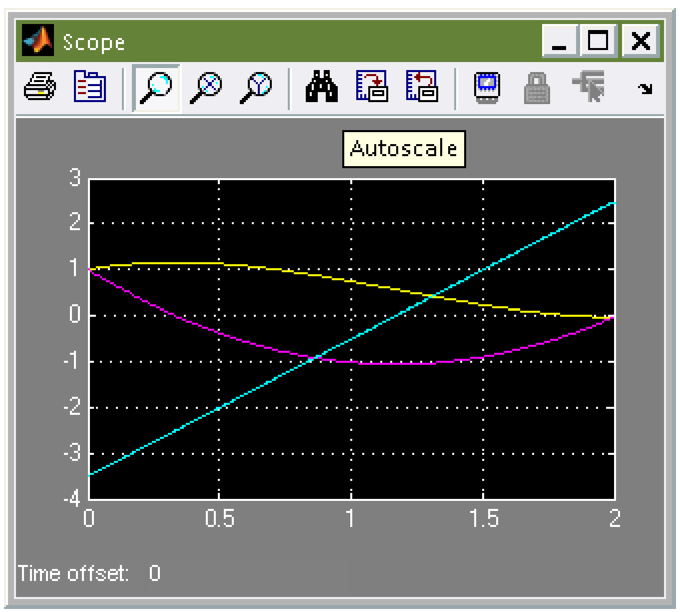
$$u = c_1 t + c_2$$

$$\ddot{x} = u \Rightarrow \dot{x} = c_1 t^2 + c_2 t + c_3 \Rightarrow x = c_1 t^3 + c_2 t^2 + c_3 t + c_4$$

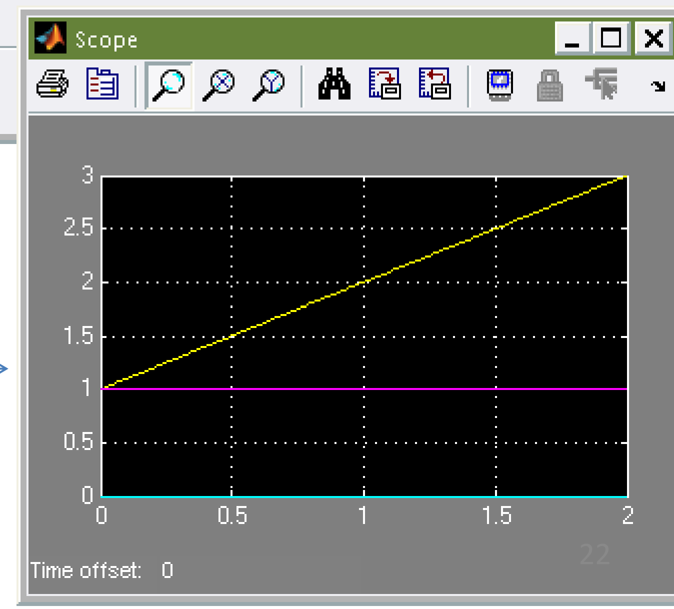
Symulacja (w2p1.mdl)







u=0



Przykład 1. Sterowanie optymalne

$$\dot{x} = u$$

$$x(0)=0, x(2)=1\,, \quad t \in [0,2]$$

$$J = \int_0^2 u^2 dt \rightarrow \min$$

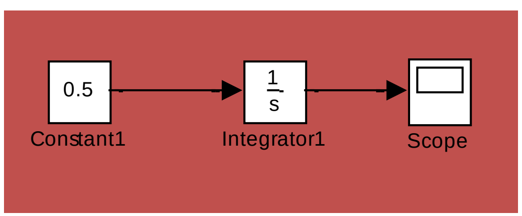
Funkcja podcałkowa zależy wyłącznie od **x’**

$$J = \int_0^2 \dot{x}^2 dt \quad \Rightarrow \quad x(t) = C_1 \cdot t + C_2$$

$$x(0)=0 \Rightarrow C_2 = 0$$

$$u = C_1$$

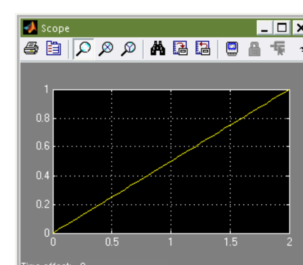
$$x(2)=1 \Rightarrow C_1 = \frac{1}{2} \Rightarrow u = \frac{1}{2}$$



$$J = \int_0^2 0.5^2 dt = 0.5$$

$$u = 0.5$$

W3p1.mdl



Szczególne przypadki równania Eulera

Funkcja podcałkowa zależy wyłącznie od **y'**:

$$J\left(y\right)=\int_a^b F(x,y(x),y'(x))dx$$

$$F=F\left(y'\right)$$

$$F_y-\frac{d}{dx}F_{y'}=0$$

$$\frac{d}{dx}F_{y'}=0\Rightarrow F_{y'}=C\Rightarrow y\left(x\right)=C_1x+C_2$$

$$y\left(x\right)=C_1x+C_2$$

Przykład 1 Sterowanie nieoptymalne

$$\dot{x} = u$$

$$x(0) = 0, x(2) = 1, \quad t \in [0, 2]$$

Nieoptymalne sterowanie

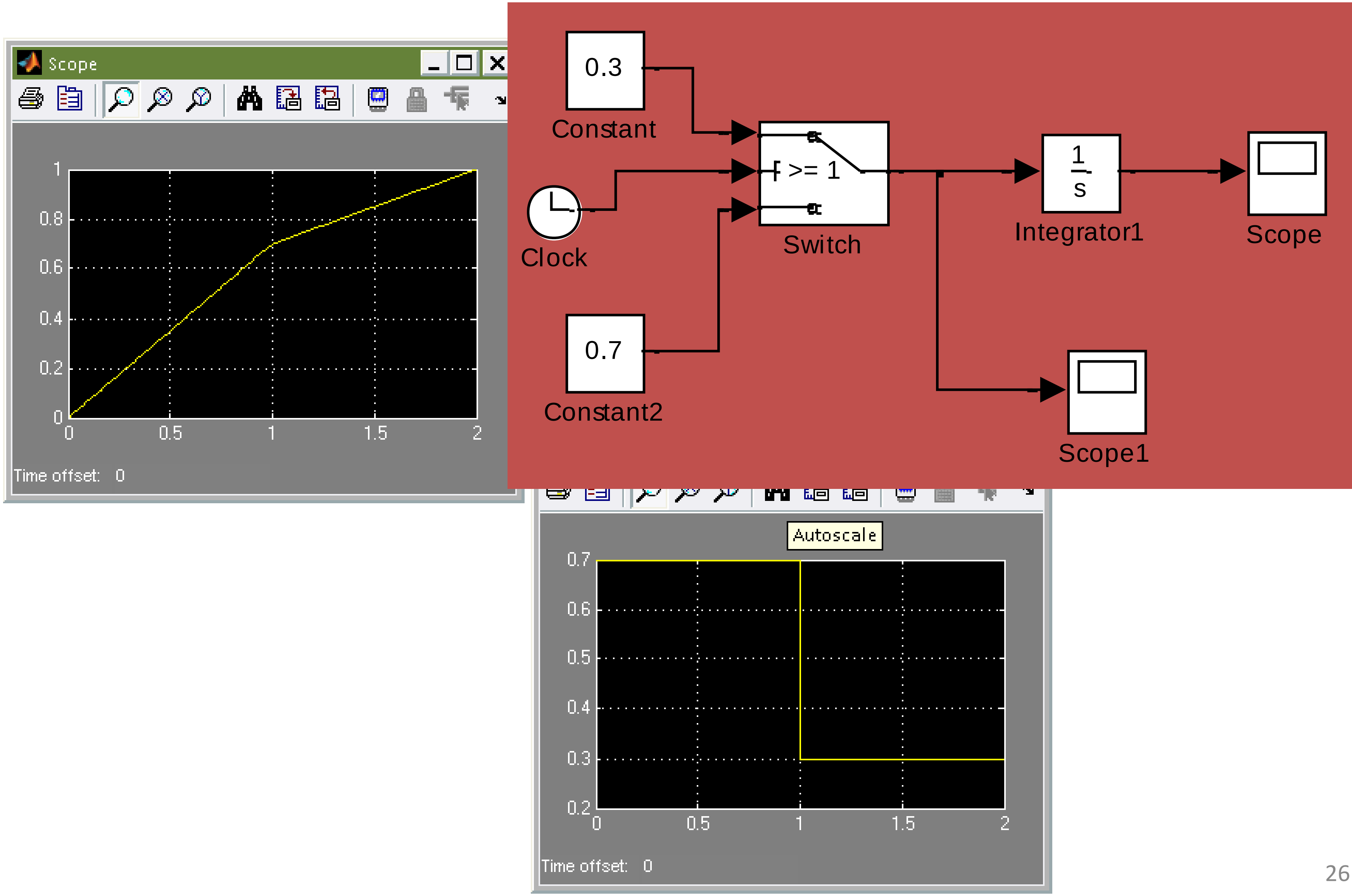
$$x(t) = u_1 \cdot t, \quad t \in [0, 1]$$

$$x(t) = u_2 \cdot t, \quad t \in (1, 2]$$

$$u_1 = 0.7, \ u_2 = 0.3$$

$$J = \int_0^1 0.7^2 dt + \int_1^2 0.3^2 dt = 0.49 + 0.09 = 0.58$$

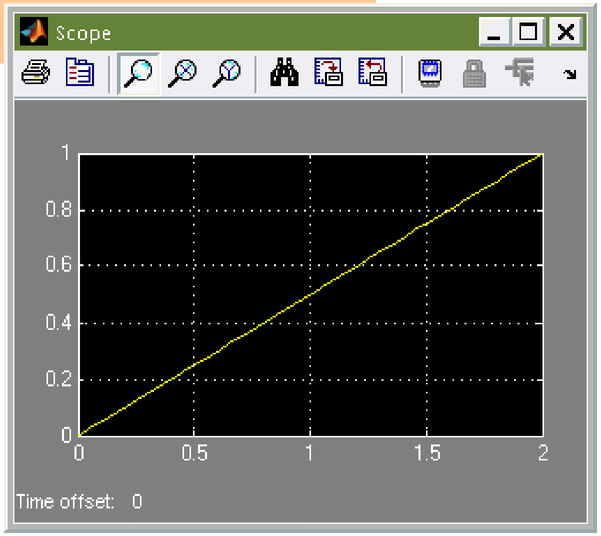
W3p2.mdl

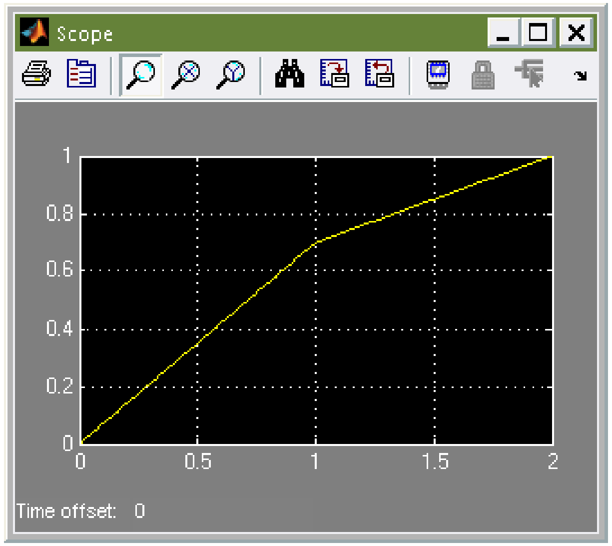


Scope
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
Time offset: 0
0.3
Constant
Clock
>= 1
Switch
1
s
Integrator1
Scope
0.7
Constant2
Scope1
Autoscale
0.7
0.6
0.5
0.4
0.3
0.2
0
0.5
1
1.5
2
Time offset: 0

Przykład 1. Sprawdzanie

$$J = \int_0^2 0.5^2\,dt = 0.5$$





$$J = \int_0^1 0.7^2\,dt + \int_1^2 0.3^2\,dt = 0.49 + 0.09 = 0.58$$

Przykład 1. Metoda podstawienia

$$\dot{x} = u$$

$$x(0) = 0, x(2) = 1, \quad t \in [0,2]$$

$$J = \int_0^2 u^2 dt \to \min$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$J = \int_0^2 u^2 dt = \int_0^2 \dot{x}^2 dt$$

$$F(t, x, \dot{x}) = \dot{x}^2$$

$$-2\frac{d}{dt}\dot{x} = 0$$

$$\ddot{x} = 0 \Rightarrow \dot{x} = C_1 \Rightarrow x(t) = C_1 t + C_2$$

$$C_1 = 0.5, \; C_2 = 0$$

$$u = 0.5$$

Przykład 1. Metoda mnożników Lagrange'a

$$\dot{x} = u$$

$$x(0) = 0, x(2) = 1 , \quad t \in [0, 2]$$

$$J = \int_0^2 u^2 dt \rightarrow \min$$

$$J' = \int_0^2 \left[u^2 + \lambda (\dot{x} - u) \right] dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0$$

$$F(t, x, \dot{x}, u, \dot{u}) = u^2 + \lambda (\dot{x} - u)$$

$$\dot{\lambda} = 0 \Rightarrow \lambda = C$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt} \frac{\partial F}{\partial \dot{u}} = 0$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = \frac{C}{2}$$

$$x(t) = C_1 t + C_2$$

$$C_1 = 0.5, \; C_2 = 0$$

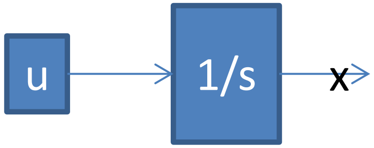
$$u = 0.5$$

Zadania ze swobodnym punktem końcowym

$$x(t_0) = x_0,\ x(t_f) \to \max, \quad t \in [t_0, t_f]$$

$$J = \int_{t_0}^{t_f} F(x(t), \dot{x}(t), t)dt$$

$$g(x(t), \dot{x}(t), t) = 0$$



Przykład 3

dx/dt

$$\dot{x} = u$$

$$x(0) = 0,\ x(2) \to \max, \quad t \in [0, 2]$$

$$J = \int_0^2 u^2 dt \to \min$$

min

$$J' = -x(2) + \int_0^2 u^2 dt = -x(0) + \int_0^2 (-\dot{x})\, dt + \int_0^2 u^2 dt = \int_0^2 \left[-\dot{x} + u^2\right] dt$$

$$J' = \int_0^2 \left[-\dot{x} + \dot{x}^2\right] dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = -1 + 2\dot{x} \qquad \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 2\ddot{x}$$

Przykład 3 (cd)

$$\ddot{x} = 0 \Rightarrow \dot{x} = C_1 \Rightarrow x(t) = C_1 t + C_2$$

$$x(0) = 0, x(2) \to \max$$

$$C_2 = 0, \qquad C_1 = ?$$

$$J' = \int_0^2 \left[-\dot{x} + \dot{x}^2 \right] dt = \int_0^2 \left[-C_1 + C_1^{\,2} \right] dt = 2\left(-C_1 + C_1^{\,2} \right)$$

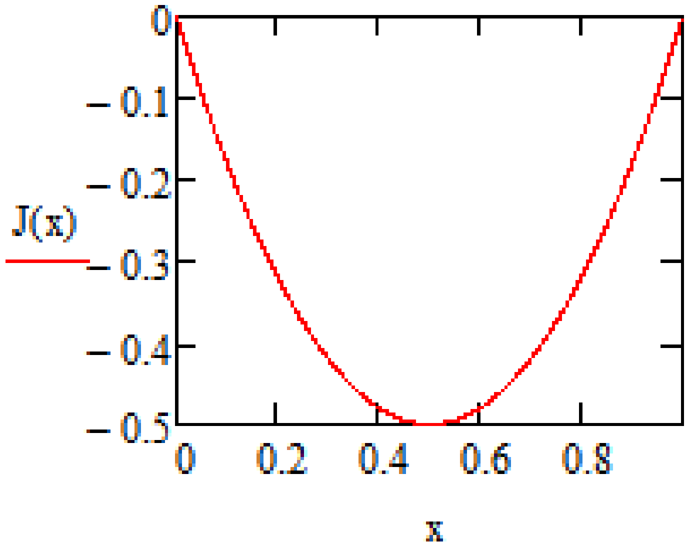
Przykład 3 (cd)

$$J' = \int_0^2 \left[-\dot{x} + \dot{x}^2\right] dt = \int_0^2 \left[-C_1 + C_1^2\right] dt = 2\left(-C_1 + C_1^2\right)$$

$$J(c) := \int_0^2 \left(-c + c^2\right) dt$$

$$\int_0^2 \left(-c + c^2\right) dt \rightarrow 2 \cdot c \cdot (c - 1)$$

$$\frac{d}{dc} J(c) \rightarrow 4 \cdot c - 2$$



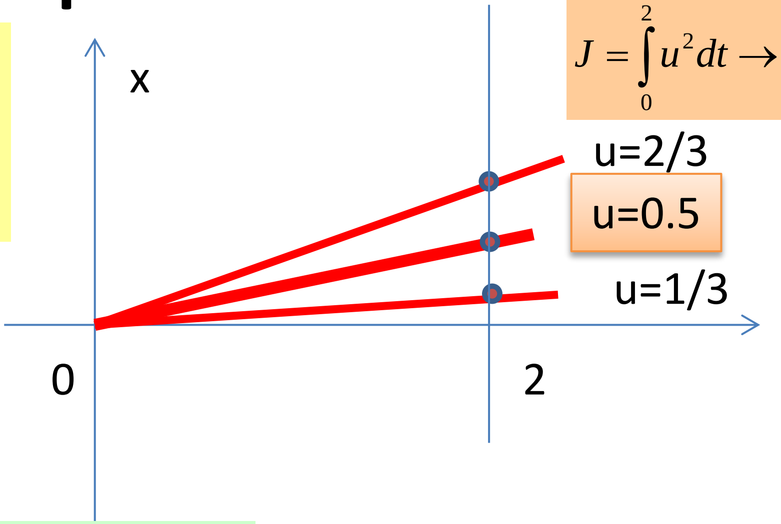
$$C_1 = 0.5$$

Przykład 3. (cd).Sprawdzanie

$$\dot{x} = u$$

$$J' = -x(2) + \int_0^2 u^2 dt$$

$$x(0) = 0, x(2) \to \max$$



$$u = 0.5, \; x(t) = 0.5t$$ $$x(2) = 1$$ $$J' = -0.5$$ min

$$u = \frac{1}{3}, \; x(t) = \frac{1}{3}t$$ $$x(2) = \frac{2}{3}$$ $$J' = -\frac{2}{9} > -0.5$$

$$u = \frac{2}{3}, \; x(t) = \frac{2}{3}t$$ $$x(2) = \frac{4}{3} > 1$$ $$J' = -\frac{3}{9} = -\frac{1}{3} > -0.5$$

Zadania ze swobodnym punktem końcowym i czasem

$$x\left(t_0\right)=x_0,\, x\left(t_f\right)\rightarrow \max, \quad t\in\left[t_0,t_f\right]$$

$$t_f \rightarrow \min$$

$$J=\int_{t_0}^{t_f} F(x(t),\dot{x}(t),t)dt \rightarrow \min$$

$$g(x(t),\dot{x}(t),t)=0$$

Zadania ze swobodnym punktem końcowym i czasem

$$\dot{x} = u \qquad x(0) = 0,\ x(t_f) \to \max, \quad t \in [0, t_f]$$

$$J = \int_0^{t_f} u^2 dt \to \min \qquad t_f \to \min$$

Przykład 4 (Dalszy ciąg)

$$\dot{x} = u$$

$$x(0) = 0, x(t_f) \to \max, \quad t \in [0, t_f]$$

$$t_f \to \min$$

$$J = \int_0^{t_f} u^2 dt \to \min$$

$$J' = -x(t_f) + \frac{1}{2} t_f^{\,2} + \int_0^{t_f} u^2 dt =$$

$$= -x(0) + \int_0^{t_f} (-\dot{x}) dt + \int_0^{t_f} t dt + \int_0^{t_f} u^2 dt = \int_0^{t_f} \left[-\dot{x} + u^2 + t \right] dt$$

Przykład 4 (Dalszy ciąg)

$$J' = -x(t_f) + \frac{1}{2}t_f^{\,2} + \int_0^{t_f} u^2 dt =$$

$$= -x(0) + \int_0^{t_f} (-\dot{x})\,dt + \int_0^{t_f} t\,dt + \int_0^{t_f} u^2 dt = \int_0^{t_f} \left[-\dot{x} + u^2 + t\right] dt$$

$$J' = \int_0^{t_f} \left[-\dot{x} + \dot{x}^2 + t\right] dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = -1 + 2\dot{x} \qquad \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 2\ddot{x}$$

Przykład 4 (Dalszy ciąg)

$$\ddot{x} = 0 \Rightarrow \dot{x} = C_1 \Rightarrow x(t) = C_1 t + C_2$$

$$x(0) = 0, x(t_f) \to \max$$

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_* = 0$$

$$J' = \int_0^{t_f} \left[-\dot{x} + \dot{x}^2 + t\right] dt = \int_0^{t_f} \left[-C_1 + C_1{}^2 + t\right] dt = t_f\left(-C_1 + C_1{}^2\right) + \frac{1}{2} t_f^2$$

Przykład 4 (Dalszy ciąg)

$$J' = \int_0^{t_f} \left[-\dot{x} + \dot{x}^2 + t \right] dt = \int_0^{t_f} \left[-C_1 + C_1^{\,2} + t \right] dt = t_f \left(-C_1 + C_1^{\,2} \right) + \frac{1}{2} t_f^2$$

$$\frac{\partial J'}{\partial C_1} = t_f \left(2C_1 - 1 \right) = 0$$

$$\frac{\partial J'}{\partial t_f} = C_1^{\,2} - C_1 + t_f = 0$$

$$C_1 = 0.5$$

$$t_f = 0.25$$

$$\dot{x} = u$$

$$x(0) = 0, x\left(t_f\right) \to \max, \quad t \in \left[0, t_f\right]$$

$$u = 0.5, \; x\left(0.25\right) = 0.125$$

Sprawdzanie

$$J = -x(t_f) + \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt$$

$$J(C1, tf) := -C1 \cdot tf + \frac{1}{2}tf^2 + \int_0^{tf} C1^2\, dt$$

$J(0.5, 0.25) = -0.031$

$J(0.5, 0.3) = -0.03$

$J(0.5, 0.15) = -0.026$

$J(0.6, 0.25) = -0.029$

$J(0.4, 0.25) = -0.029$

$J(0.4, 0.33) = -0.025$

Zadania ze swobodnym czasem

$$\dot{x} = u$$

$$x(0) = 0, x(t_f) = 1, \quad t \in [0, t_f]$$

$$J = \int_0^{t_f} u^2 dt \rightarrow \min$$

$$t_f \rightarrow \min$$

Rozwiązanie

$$x(0)=0, x\left(t_f\right)=1, \quad t\in\left[0,t_f\right]$$

$$\dot{x}=u$$

$$J=\int_0^{t_f} u^2 dt \rightarrow \min$$

$$t_f \rightarrow \min$$

$$J=\frac{1}{2}t_f^2+\int_0^{t_f} u^2 dt \rightarrow \min$$

$$J'=\frac{1}{2}t_f^2+\int_0^{t_f} u^2 dt$$

$$J'=\int_0^{t_f} t dt+\int_0^{t_f} u^2 dt$$

$$J'=\int_0^{t_f}\left(t+\dot{x}^2\right)dt$$

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}=0$$

$$2\dot{x}'=2\ddot{x}=0$$

$$\dot{x}=c_1$$

$$x(t)=c_1 t+c_2$$

$$x(0)=0\Rightarrow c_2=0$$

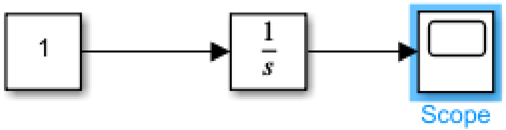
$$x\left(t_f\right)=c_1\cdot t_f=1\Rightarrow t_f=\frac{1}{c_1}$$

$$J'=\int_0^{\frac{1}{c_1}}\left(t+c_1^2\right)dt=\frac{1}{2}t^2\Big|_0^{\frac{1}{c_1}}+c_1^2\cdot t\Big|_0^{\frac{1}{c_1}}=\frac{1}{2}\frac{1}{c_1^2}+c_1\Rightarrow \min$$

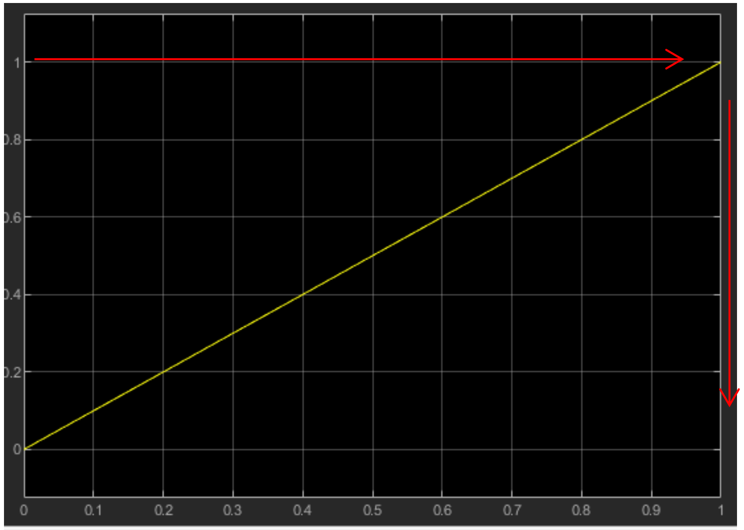
$$\frac{dJ'}{dc_1}=0\Rightarrow 1-\frac{1}{c_1^3}=0\Rightarrow c_1=1$$

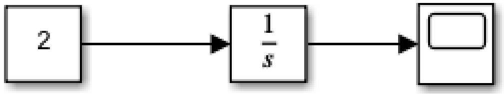
$$x(t)=t$$
$$x(0)=0$$
$$x(1)=1$$
$$t_f=1$$

Ex4.slx

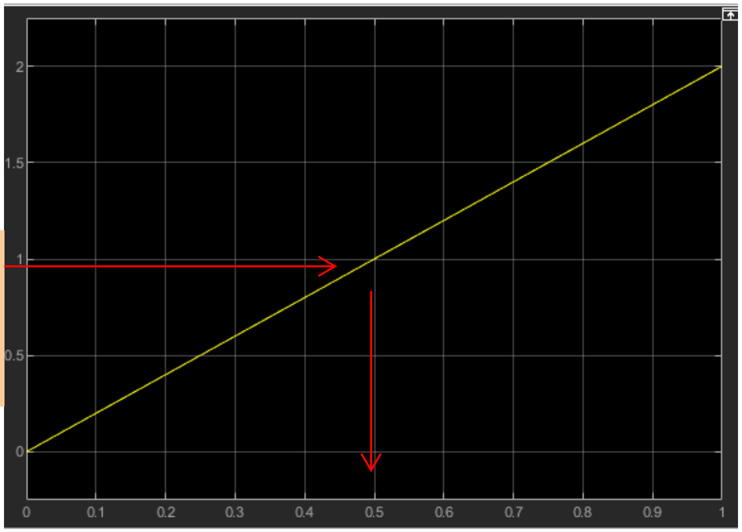


$$J = \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt = \frac{1}{2} + \int_0^1 1dt = 1.5$$

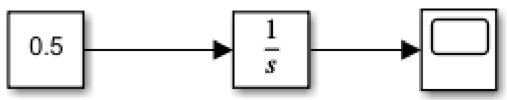




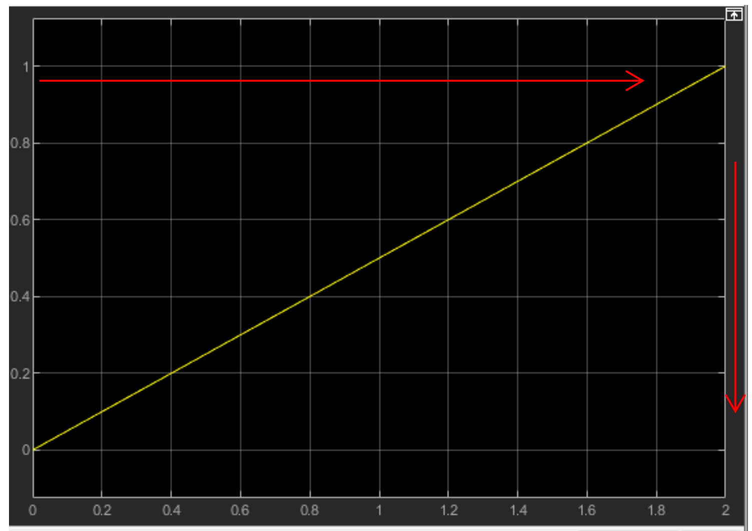
$$J = \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt = \frac{1}{2}0.5^2 + \int_0^{0.5} 2^2 dt = 2.125$$

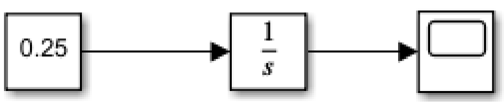


Ex4.slx

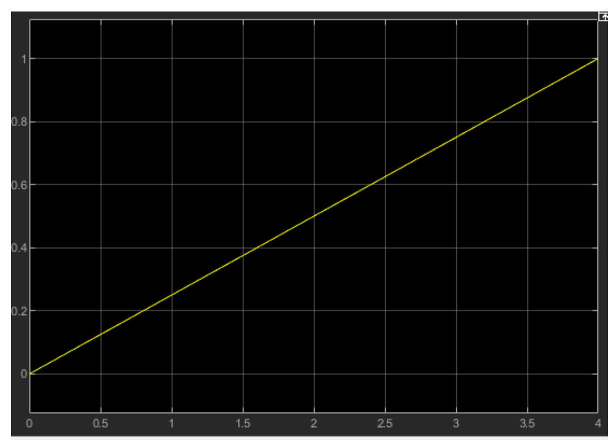


$$J = \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt = 2 + \int_0^2 0.5^2 dt = 0.25 \cdot 2 = 3$$





$$J = \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt = 8 + \int_0^4 0.25^2 dt = 8.25$$



$$J = \frac{1}{2}t_f^2 + \int_0^{t_f} u^2 dt = 50 + \int_0^{10} 0.1^2 dt = 50.1$$

Zagadnienie Bolzy

01-06

Wprowadzamy funkcjonał kosztu (cost functional) (lub funkcjonał wypłaty (payoff functional))

$$J(u(t)) = S\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} V(x(t),u(t),t)dt$$

Lambda

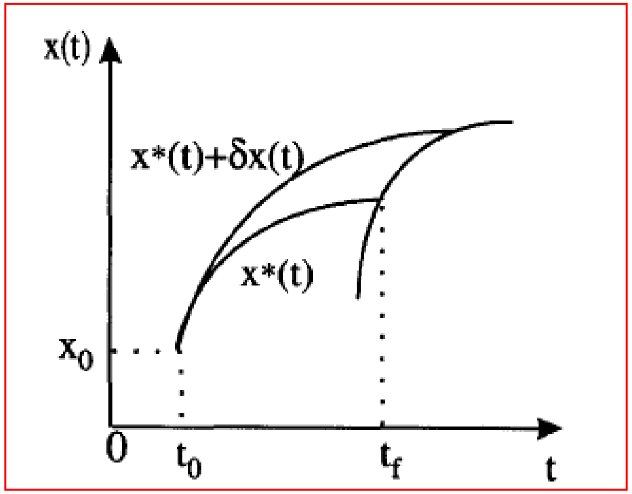
Rozpatrujemy więc zagadnienie:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

z danymi początkowymi

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$\mathbf{x}(t_f)$ i/lub t_f są swobodne



Oskar Bolza
1857 – 1942

Zagadnienie Bolzy (cd.)

$$\Rightarrow \quad J(u(t)) = S\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} V(x(t),u(t),t)dt$$

$$\int_{t_0}^{t_f} \frac{dS(x(t),t)}{dt}dt = S(x(t),t)\Big|_{t_0}^{t_f} = S\left(x\left(t_f\right),t_f\right) - \boxed{S\left(x\left(t_0\right),t_0\right)}$$

$$\Rightarrow \quad J_2(u(t)) = \int_{t_0}^{t_f}\left[V(x(t),u(t),t) + \frac{dS}{dt}\right]dt =$$

$$\int_{t_0}^{t_f} V(x(t),u(t),t)dt + S\left(x\left(t_f\right),t_f\right) - S\left(x\left(t_0\right),t_0\right)$$

$$\frac{d[S(x(t),t)]}{dt} = \left(\frac{\partial S}{\partial x}\right)' \dot{x}(t) + \frac{\partial S}{\partial t}$$

Nowe zagadnienie

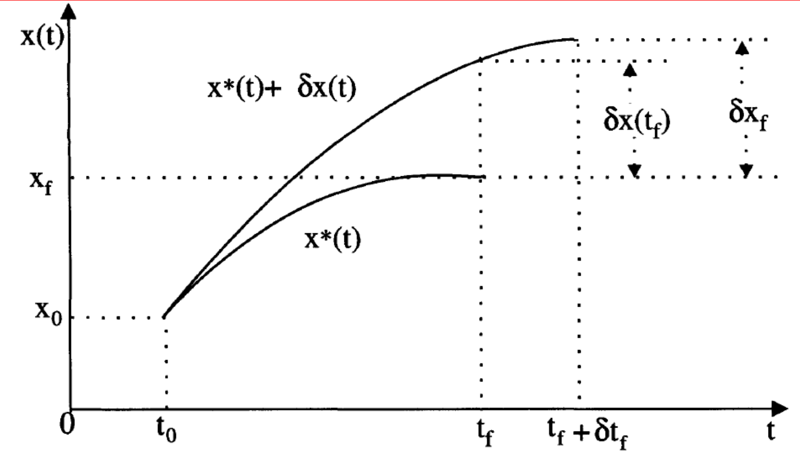
$$J(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \left[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \frac{dS(\mathbf{x}^*(t), t)}{dt} \right] dt$$

$$\dot{\mathbf{x}}^*(\mathbf{t}) = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t).$$

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \delta\mathbf{x}(t)$$

* - optymalne

$$\mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t)$$



$$\frac{d[S(\mathbf{x}(t), t)]}{dt} = \left(\frac{\partial S}{\partial \mathbf{x}}\right)' \dot{\mathbf{x}}(t) + \frac{\partial S}{\partial t}$$

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} [V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* + \boldsymbol{\lambda}'(t) \{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\}] dt$$

Zagadnienie Bolzy (cd.)

Funkcjonał

$$J_a(\mathbf{u}^*(t)) = \int_{t_0}^{t_f} \Big[V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_*$$

$$+\boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\Big]dt$$

$$\mathbf{u}(t) = \mathbf{u}^*(t) + \delta\mathbf{u}(t)$$

$$\mathbf{x}(t) = \mathbf{x}^*(t) + \delta\mathbf{x}(t)$$

$$J_a(\mathbf{u}(t)) = \int_{t_0}^{t_f+\delta t_f} \Big[V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)$$

$$+ \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_*$$

$$+\boldsymbol{\lambda}'(t)\Big[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)$$

$$- \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\Big]\Big]dt.$$

Zagadnienie Bolzy (cd.)

Lagranżjan

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^\delta &= \mathcal{L}^\delta(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' [\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)] + \left(\frac{\partial S}{\partial t}\right)_* \\ &\quad + \boldsymbol{\lambda}'(t)\left[\mathbf{f}(\mathbf{x}^*(t) + \delta\mathbf{x}(t), \mathbf{u}^*(t) + \delta\mathbf{u}(t), t)\right. \\ &\quad \left. - \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\right].\end{aligned}$$

Zagadnienie Bolzy (cd.)

Funkcjonał przez Lagranżjan

$$J_a\left(u^*(t)\right) = \int_{t_0}^{t_f} L\left(x^*(t),\dot{x}^*(t),u^*(t),l(t),t\right)dt = \int_{t_0}^{t_f} Ldt$$

$$J_a(u(t)) = \int_{t_0}^{t_f+\delta t_f} L^\delta dt = \int_{t_0}^{t_f} L^\delta dt + \int_{t_f}^{t_f+\delta t_f} L^\delta dt$$

Zagadnienie Bolzy (cd.)

$$\int_{t_f}^{t_f+\delta t_f} \mathcal{L}^\delta dt = \mathcal{L}^\delta\Big|_{t_f} \delta t_f$$

$$\approx \left\{ \mathcal{L} + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) \right.$$

$$\left. + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t) \right\}\Bigg|_{t_f} \delta t_f$$

$$\approx \mathcal{L}|_{t_f} \delta t_f .$$

Zagadnienie Bolzy (cd.)

$$
\begin{aligned}
\Delta J &= J_a(\mathbf{u}(t)) - J_a(\mathbf{u}^*(t)) \\
&= \int_{t_0}^{t_f} (\mathcal{L}^\delta - \mathcal{L})dt + \mathcal{L}|_{t_f}\,\delta t_f
\end{aligned}
$$

$$
\begin{aligned}
\delta J = \int_{t_0}^{t_f} &\left\{ \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_*' \delta \mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta \dot{\mathbf{x}}(t) + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_*' \delta \mathbf{u}(t) \right\} dt \\
&+ \mathcal{L}|_{t_f}\,\delta t_f .
\end{aligned}
$$

Zagadnienie Bolzy (cd.)

$$\int udv = uv - \int vdu$$

$$\int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta\dot{\mathbf{x}}(t)dt = \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \frac{d}{dt}\left(\delta\mathbf{x}(t)\right)dt$$

$$= \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \delta\mathbf{x}(t)\right]\Bigg|_{t_0}^{t_f}$$

$$- \int_{t_0}^{t_f} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*\right]' \delta\mathbf{x}(t)dt.$$

$$\boxed{\delta\mathbf{x}(t_0) = 0}$$

Zagadnienie Bolzy (cd.)

$$\delta J = \int_{t_0}^{t_f} \left[\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_* \right]' \delta \mathbf{x}(t) dt$$

$$+ \int_{t_0}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} \right)_*' \delta \mathbf{u}(t) dt$$

$$+ \mathcal{L}|_{t_f} \delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_*' \delta \mathbf{x}(t) \right] \Bigg|_{t_f} .$$

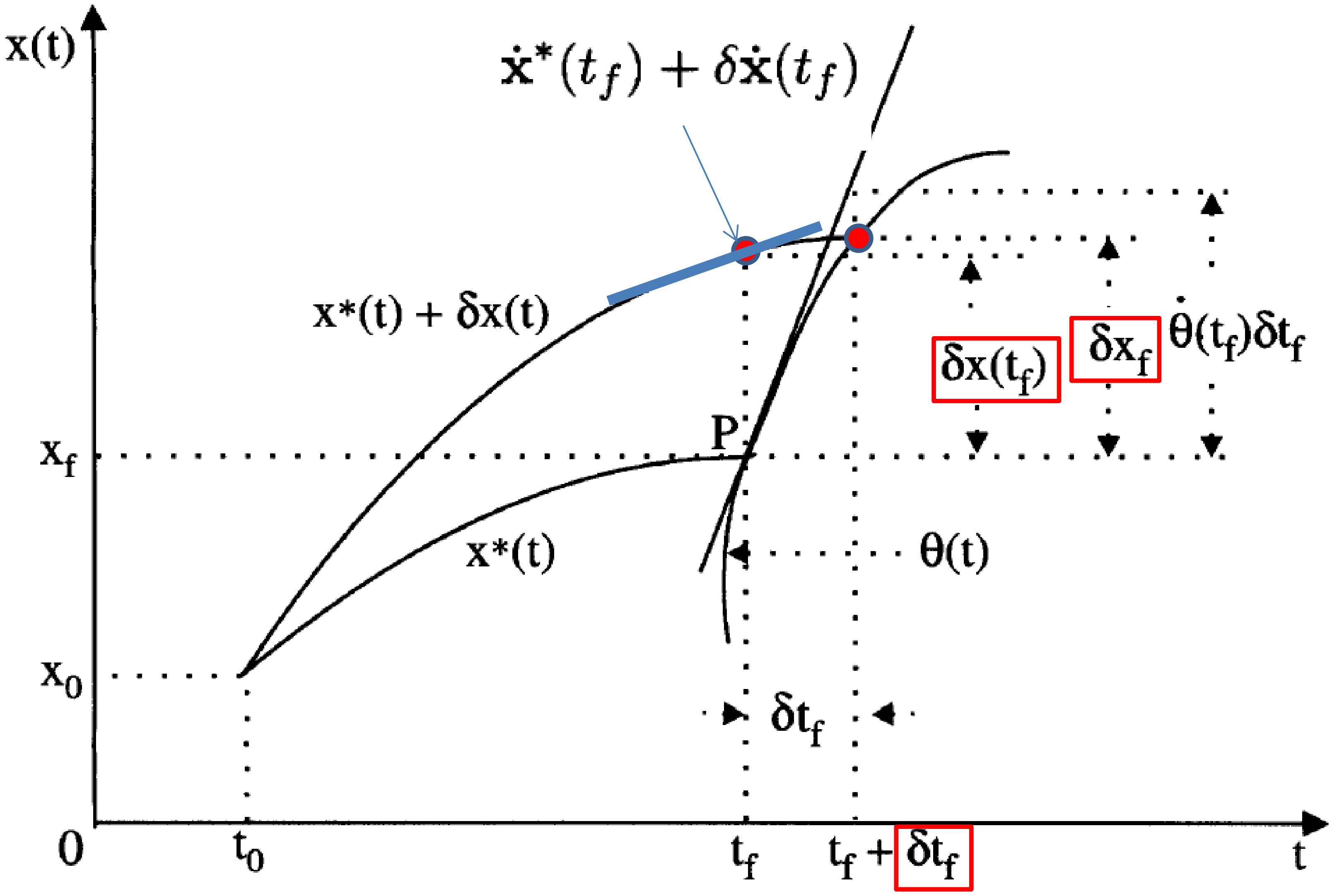
$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)_* - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} \right)_* = 0.$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}} \right)_* = 0.$$

$$\mathcal{L}^*|_{t_f} \delta t_f + \left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} \right)_*' \delta \mathbf{x}(t) \right] \Bigg|_{t_f} = 0$$

Zagadnienie Bolzy (cd.)



Zagadnienie Bolzy (cd.)

$$\delta\mathbf{x}(t_f) \longrightarrow \delta\mathbf{x}_f$$

$$\dot{\mathbf{x}}^*(t_f) + \delta\dot{\mathbf{x}}(t_f) \approx \frac{\delta\mathbf{x}_f - \delta\mathbf{x}(t_f)}{\delta t_f}$$

$$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) + \{\dot{\mathbf{x}}^*(t) + \delta\dot{\mathbf{x}}(t)\}\,\delta t_f$$

$$\delta\dot{\mathbf{x}}(t)\quad \delta t_f \longrightarrow 0$$

$$\delta\mathbf{x}(t_f) = \delta\mathbf{x}_f - \dot{\mathbf{x}}^*(t_f)\delta t_f.$$

Warunek transwersalności

$$\left[\mathcal{L}^* - \left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}\right)_*^{\prime}\dot{\mathbf{x}}(t)\right]\Bigg|_{t_f}\delta t_f + \left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}\right)_*^{\prime}\Bigg|_{t_f}\delta\mathbf{x}_f = 0$$

Funkcja Lagrange'a

$$\mathcal{L} = \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)$$
$$= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t}$$
$$+\boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

Warunek transwersalności

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

Przykład 3. Metoda 2.

$$\dot{x} = u$$

$$x(0) = 0,\ x(2) \to \max, \quad t \in [0,2]$$

$$J = \int_0^2 u^2 dt \to \min$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$J' = -x(2) + \int_0^2 u^2 dt$$

$$S(x(t),t) = -x(t)$$

$$\frac{\partial}{\partial x} S(x(t),t) = -1 \qquad \frac{\partial}{\partial t} S(x(t),t) = 0$$

$$\mathcal{L} = \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)$$
$$= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t}$$
$$+ \boldsymbol{\lambda}'(t)\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\}$$

$$\frac{d}{dt} S(x(t),t) = \frac{\partial}{\partial x} S(x(t),t)\frac{dx}{dt} + \frac{\partial}{\partial t} S(x(t),t)\frac{dt}{dt}$$

$$J'' = \int_0^2 \left[-\dot{x} + u^2 + \lambda(\dot{x} - u) \right] dt$$

L

0

(-1+lambda(t))

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

$$\dot{\lambda} = 0 \Rightarrow \lambda(t) = C$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = \frac{C}{2}$$

$$x = C_1 t + C_2$$

Przykład 3 (Metoda 2. Dalszy ciąg)

warunek transwersalności

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

$$\left(\lambda(t)-1\right)\Big|_{t=2} = 0 \Rightarrow \lambda(2) = 1$$

$$C = 1$$

$$\dot{\lambda} = 0 \Rightarrow \lambda = C$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = 0.5$$

$$\dot{x} = u \Rightarrow \dot{x} = C_1 \Rightarrow x(t) = C_1 t + C_2$$

$$x(0) = 0, x(2) \rightarrow \max$$

$$C_2 = 0, \quad C_1 = 0.5$$

$$x(t) = 0.5t$$

Zadanie (przykład 4)

$$\dot{x} = u$$

$$x(0) = 0,\ x(t_f) \to \max, \quad t \in [0, t_f]$$

$$J = \int_0^{t_f} u^2 dt \to \min$$

$$t_f \to \min$$

Rozważmy problem na podstawie zagadnienia Bolzy

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\}\end{aligned}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

Rozwiązanie

$$J = -x(t_f) + \frac{1}{2}t_f^2 + \int_0^{t_f} \left[u^2\right] dt$$

$$S(x(t),t) = -x(t) + \frac{1}{2}t^2$$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\end{aligned}$$

$$L = u^2 - \dot{x} + t + \lambda\{\dot{x} - u\}$$

$$\left(\frac{\partial L}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)_* = 0 \rightarrow \dot{\lambda} = 0 \rightarrow \lambda = C$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0 \rightarrow 2u - \lambda = 0$$

$$\left(\lambda(t) - 1\right)\Big|_{t_f} = 0 \rightarrow \lambda(t_f) = 1 \rightarrow u = 0.5$$

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'\Bigg|_*\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

$$u^2 - \dot{x} + t + \lambda\{\dot{x} - u\} - (-1 + \lambda)\dot{x} =$$

$$\left(u^2 + t - \lambda u\right)\Big|_{t_f} = 0.25 + t_f - 0.5 = 0$$

$$\rightarrow t_f = 0.25$$

$$\dot{x} = u \rightarrow x(t) = 0.5t + C \qquad x(0) = 0 \qquad C = 0$$

Przykład 5. Metoda 2.

$$\dot{x} = u$$

$$x(0) = 0, \quad x(t_f) \to 4, \quad t \in [0, t_f]$$

$$J = \int_0^{t_f} u^2 dt \to \min$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$$S(x(t), t) = \frac{1}{2} t^2$$

$$\mathcal{L} = \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} + \boldsymbol{\lambda}'(t)\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\}$$

$$J = \frac{1}{2} t_f^2 + \int_0^{t_f} [u^2] dt$$

$$\frac{\partial}{\partial x} S(x(t), t) = 0 \qquad \frac{\partial}{\partial t} S(x(t), t) = t$$

$$\frac{d}{dt} S(x(t), t) = \frac{\partial}{\partial x} S(x(t), t) \frac{dx}{dt} + \frac{\partial}{\partial t} S(x(t), t) \frac{dt}{dt}$$

$$J'' = \int_0^2 \left[t + u^2 + \lambda (\dot{x} - u) \right] dt$$

L

0 (-1+lambda(t))

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

$$\dot{\lambda} = 0 \Rightarrow \lambda(t) = C$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = \frac{C}{2}$$

$$x = C_1 t + C_2$$

Rozwiązanie

$$J = \frac{1}{2}t_f^2 + \int_0^{t_f}\left[u^2\right]dt$$

$$S\left(x(t),t\right) = \frac{1}{2}t^2$$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\end{aligned}$$

$$L = u^2 + t + \lambda\{\dot{x} - u\}$$

$$\left(\frac{\partial L}{\partial x}\right)_* - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)_* = 0 \;\Rightarrow\; \dot{\lambda} = 0 \;\Rightarrow\; \lambda = C$$

$$\frac{C_1^2}{1} + t_f - \frac{2C_1^2}{1} = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0 \;\Rightarrow\; 2u - \lambda = 0$$

$$4 = C_1 t_f$$

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)' \dot{\mathbf{x}}(t)\right]_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_*\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

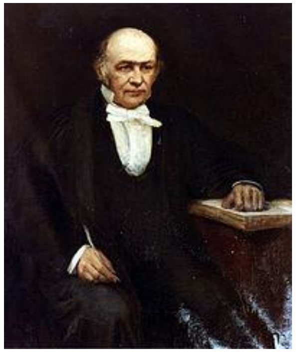
$$u^2 + t + \lambda\{\dot{x} - u\} - \lambda\dot{x} =$$

$$\left(u^2 + t - \lambda u\right)\Big|_{t_f} = \frac{C^2}{4} + t_f - \frac{C^2}{2} = 0$$

$$\Rightarrow\; t_f = 2, \ldots$$

$$\dot{x} = u \;\Rightarrow\; x(t) = C_1 t + C_2 \qquad x(0) = 0 \qquad C_2 = 0 \qquad x = {*}{*}{*}t$$

Hamiltonian lub funkcja Pontriagina



William Rowan Hamilton

$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*'}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\mathcal{H}^* = \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t)$$

$$\begin{aligned}\mathcal{L}^* &= \mathcal{L}^*(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &= \mathcal{H}^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}^*(t), t) \\ &\quad + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \left(\frac{\partial S}{\partial t}\right)_* - \boldsymbol{\lambda}^{*'}(t)\dot{\mathbf{x}}^*(t)\end{aligned}$$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \\ &= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t} \\ &\quad + \boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}\end{aligned}$$

Równoważność warunków Lagrange'a a Pontriagina

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0}$$

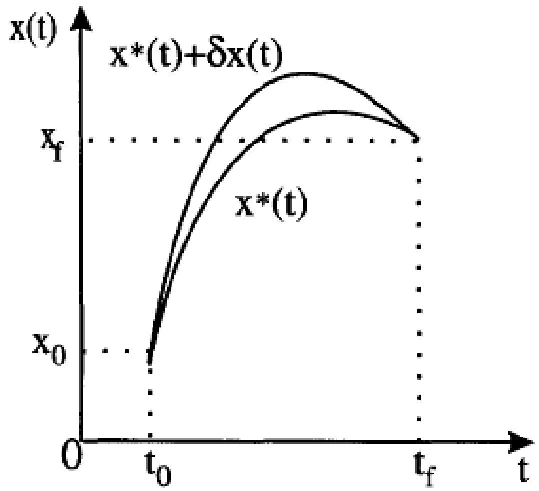
$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}^*}{\partial \dot{\mathbf{x}}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)_* = 0 \longrightarrow \boxed{\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).}$$

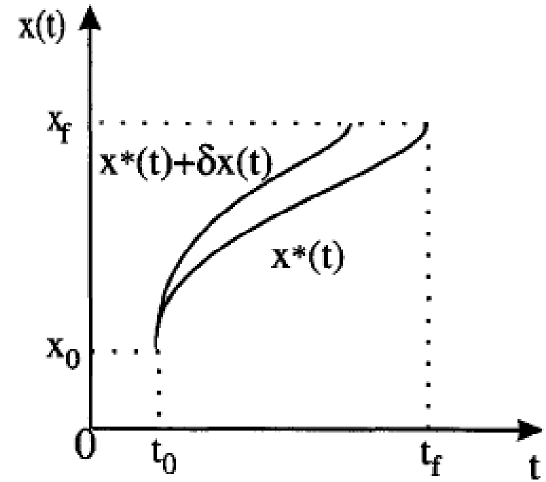
Warunek transwersalności

$$\boxed{\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.}$$

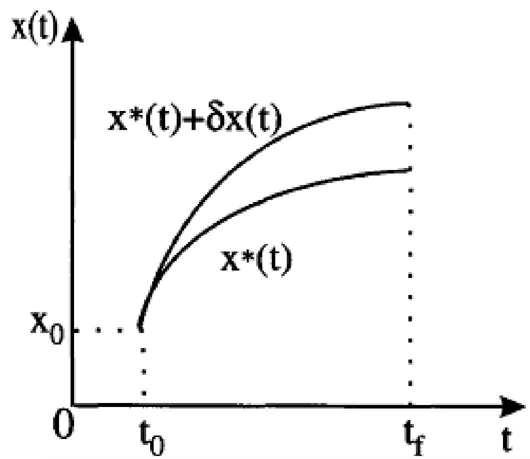
Typy zagadnień optymalizacji



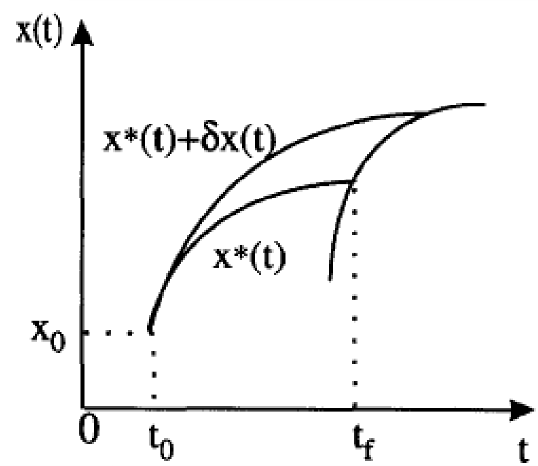
Końce i czas ustalone



Końce ustalone i czas ruchomy



Koniec ruchomy i czas ustalony

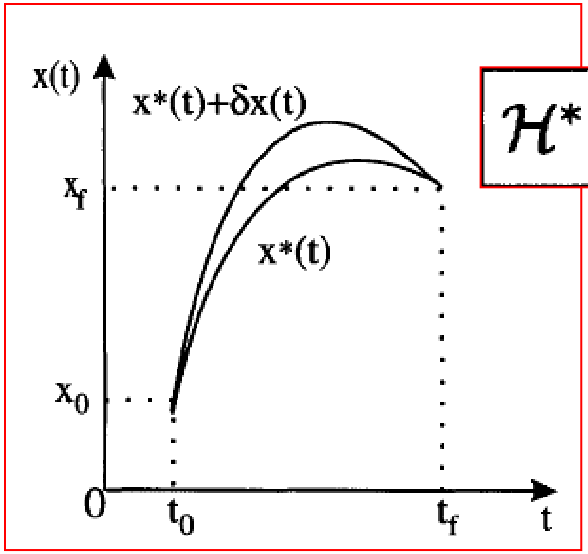


Koniec i czas są ruchome

Końce i czas są ustalone

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J = \int_{t_0}^{t_f} V(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)dt$$



$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*\prime}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).$$

Warunek transwersalności

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.$$

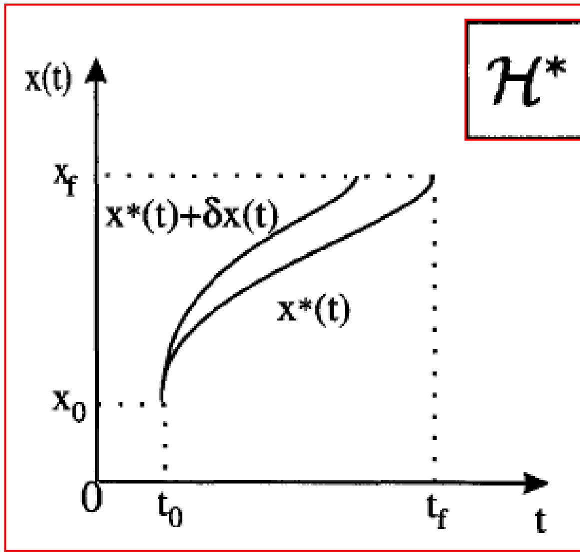
0

0

Końce są ustalone i czas jest ruchomy

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$



$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*'}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).$$

Warunek transwersalności

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0.$$

0

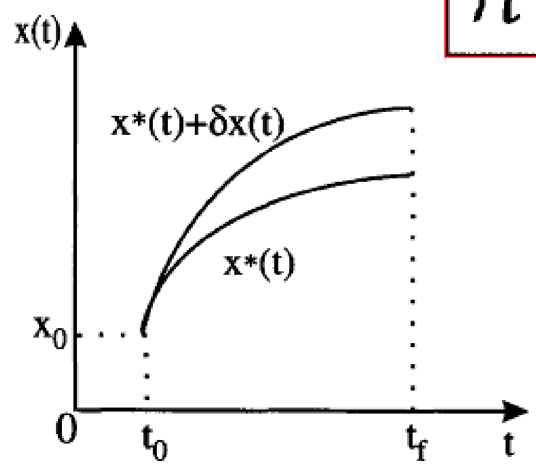
$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0.$$

Koniec jest ruchomy i czas jest ustalony

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*\prime}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$



$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).$$

Warunek transwersalności

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}^{\prime} \delta \mathbf{x}_f = 0.$$

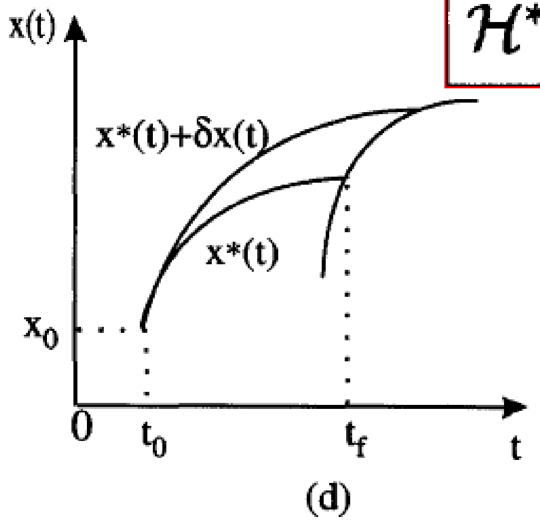
0

$$\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}.$$

Koniec i czas ruchome (I)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$



$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*\prime}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).$$

Warunek transwersalności

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}^{\prime} \delta \mathbf{x}_f = 0.$$

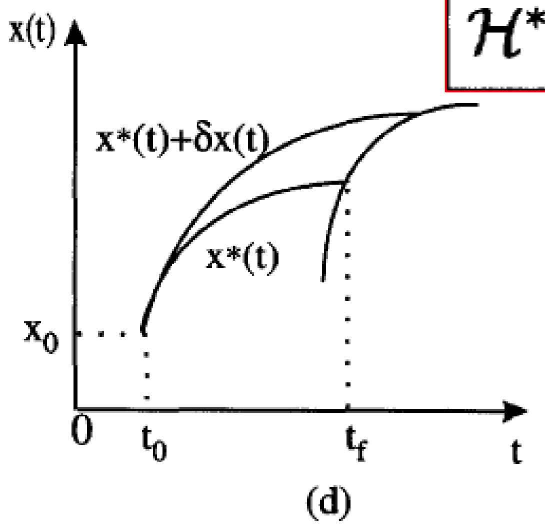
$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{*t_f} = 0$$

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_{*t_f} = 0$$

Koniec i czas są ruchome (II)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$



$$\mathcal{H}^* = V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \boldsymbol{\lambda}^{*'}(t)\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_* = -\dot{\boldsymbol{\lambda}}^*(t)$$

$$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_* = \dot{\mathbf{x}}^*(t).$$

Warunek transwersalności

$$\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$$

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f}' \delta \mathbf{x}_f = 0.$$

$$\delta \mathbf{x}_f \approx \dot{\boldsymbol{\theta}}(t_f)\delta t_f$$

$$\left[\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_* + \left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}^*(t)\right)_*' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0.$$

Warunek dostateczny

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$\delta^2 J = \int_{t_0}^{t_f} \left[\frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} (\delta \mathbf{x}(t))^2 + \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} (\delta \mathbf{u}(t))^2 + 2 \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u} \partial \mathbf{x}} (\delta \mathbf{u}(t) \delta \mathbf{x}(t)) \right]_* dt$$

$$= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x}^2} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} \\ \frac{\partial^2 \mathcal{H}}{\partial \mathbf{x} \partial \mathbf{u}} & \frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \end{bmatrix}_* \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt$$

$$= \int_{t_0}^{t_f} \left[\delta \mathbf{x}'(t) \;\; \delta \mathbf{u}'(t) \right] \Pi \begin{bmatrix} \delta \mathbf{x}(t) \\ \delta \mathbf{u}(t) \end{bmatrix} dt.$$

f(x+dx)=f(x)+**f'(x)+**f''(x)

f(x+dx)=f(x)**+**f''(x)

$$\left(\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} \right)_* > 0$$

Algorytm

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

$$\mathbf{x}(t_f) = \mathbf{x}_f$$

ruchome lub ustalone

1

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = V(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

2

$$\left(\frac{\partial \mathcal{H}}{\partial \mathbf{u}}\right)_* = 0$$

$$\mathbf{u}^*(t) = \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t)$$

$$\mathcal{H}^*(\mathbf{x}^*(t), \mathbf{h}(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t), \boldsymbol{\lambda}^*(t), t) = \mathcal{H}^*(\mathbf{x}^*(t), \boldsymbol{\lambda}^*(t), t)$$

3

$$\dot{\mathbf{x}}^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}\right)_*$$

$$\dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right)_*$$

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0$$

Algorytm (kontynuacja)

4

$$\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]'_{t_f} \delta \mathbf{x}_f = 0$$

$\delta t_f = 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f$
$\delta t_f \neq 0, \delta \mathbf{x}_f = 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \mathbf{x}_f, \left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0$
$\delta t_f = 0, \delta \mathbf{x}_f \neq 0$	$\mathbf{x}(t_0) = \mathbf{x}_0, \boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}$
$\delta \mathbf{x}_f = \boldsymbol{\theta}(t_f)\delta t_f$	$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t} + \left\{\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right\}' \dot{\boldsymbol{\theta}}(t)\right]_{t_f} = 0$
$\delta t_f \neq 0$ $\delta \mathbf{x}_f \neq 0$	$\delta \mathbf{x}(t_0) = \mathbf{x}_0$ $\left[\mathcal{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} = 0, \left[\left(\frac{\partial S}{\partial \mathbf{x}}\right)_* - \boldsymbol{\lambda}^*(t)\right]_{t_f} = 0$

Przykład 1

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$$\mathbf{x}(0) = [1 \;\; 2]'; \qquad \mathbf{x}(2) = [1 \;\; 0]'.$$

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

S=0

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = V(u(t)) = \frac{1}{2}u^2(t)$$
$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = [f_1, \; f_2]',$$

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t).\end{aligned}$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t)$$

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t).\end{aligned}$$

Przykład 1 (dalszy ciąg)

$$\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) = \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t)$$
$$= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t).$$

$$\dot{x}_1^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)_* = x_2^*(t)$$
$$\dot{x}_2^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)_* = -\lambda_2^*(t)$$
$$\dot{\lambda}_1^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)_* = 0$$
$$\dot{\lambda}_2^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)_* = -\lambda_1^*(t).$$

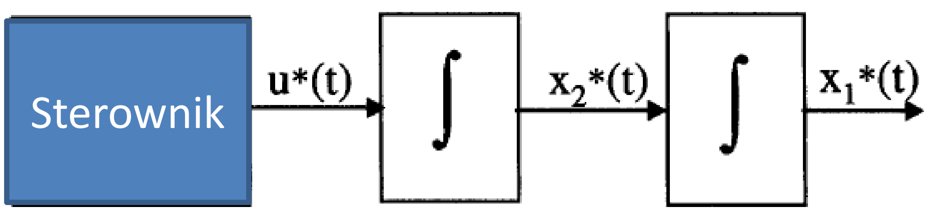
$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$
$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$
$$\lambda_1^*(t) = C_3$$
$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$$\mathbf{x}(0) = [1 \quad 2]'; \qquad \mathbf{x}(2) = [1 \quad 0]'.$$

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = 3, \quad C_4 = 4.$$

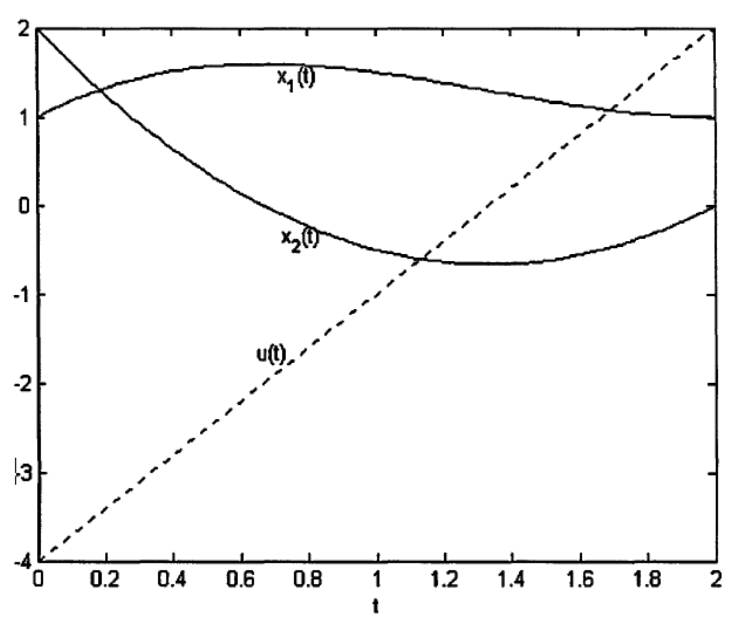
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$



Realizacja w MatLabie

S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1, ...
x1(0)=1,x2(0)=2,x1(2)=1,x2(2)=0‘)

S.x1	1+2*t-2*t^2+1/2*t^3
S.x2	2-4*t+3/2*t^2
S.lambda1	3
S.lambda2	4-3*t




```
Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
```

Przykład 2

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$\mathbf{x}(0) = [1 \quad 2]'; \qquad x_1(2) = 0; \qquad x_2(2)$ dowolny

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

S=0

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = V(u(t)) = \frac{1}{2}u^2(t)$$
$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = [f_1, \ f_2]',$$

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t).\end{aligned}$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t)$$

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t).\end{aligned}$$

Przykład 2 (Kontynuacja)

$$\dot{x}_1^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_1}\right)_* = x_2^*(t)$$

$$\dot{x}_2^*(t) = +\left(\frac{\partial \mathcal{H}}{\partial \lambda_2}\right)_* = -\lambda_2^*(t)$$

$$\dot{\lambda}_1^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_1}\right)_* = 0$$

$$\dot{\lambda}_2^*(t) = -\left(\frac{\partial \mathcal{H}}{\partial x_2}\right)_* = -\lambda_1^*(t).$$

$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$

$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$

$$\lambda_1^*(t) = C_3$$

$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$\mathbf{x}(0) = [1 \;\; 2]'; \quad x_1(2) = 0; \quad x_2(2)$ dowolny

$x_2(2)$ ruchome, δx_{2f}

$$\lambda_2(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{*t_f}$$

ponieważ S=0

$$\lambda_2(t_f) = 0$$

$$x_1(0) = 1; \quad x_2(0) = 2; \quad x_1(2) = 0; \quad \lambda_2(2) = 0.$$

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 15/8; \quad C_4 = 15/4.$$

Realizacja w MatLabie

S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1, x1(0)=1,x2(0)=2,x1(2)=0,lambda2(2)=0')

S.x1

ans= 5/16*t^3+2*t+1-15/8*t^2

S.x2

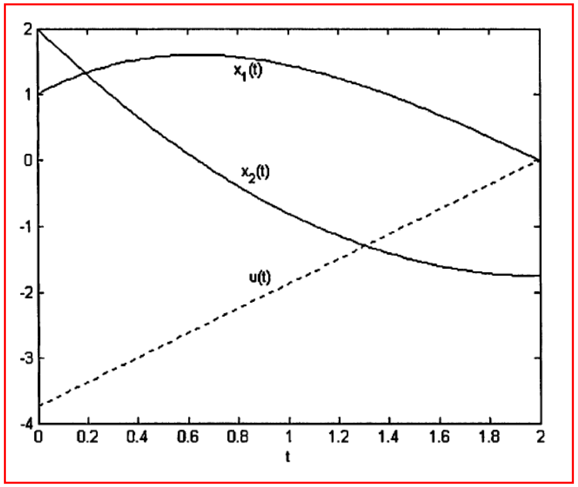
ans= 15/16*t^2+2-15/4*t

S.lambda1

ans=15/8

S.lambda2

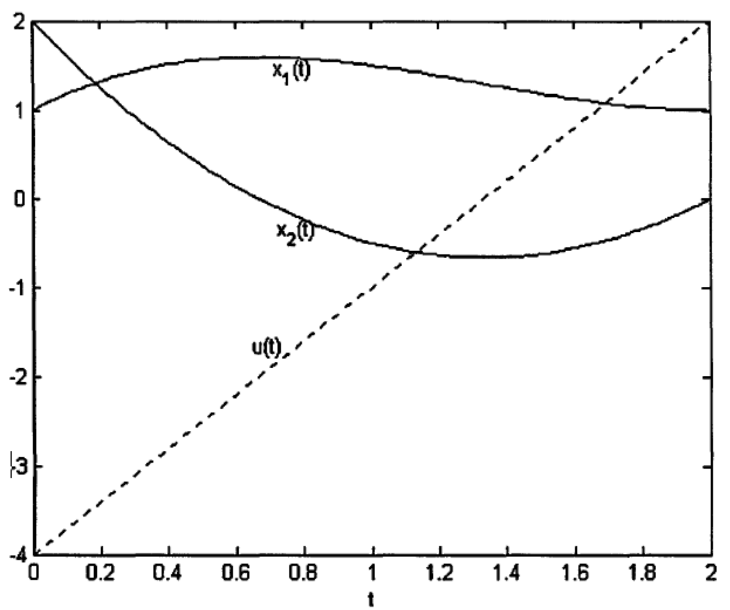
ans=-15/8*t+15/4



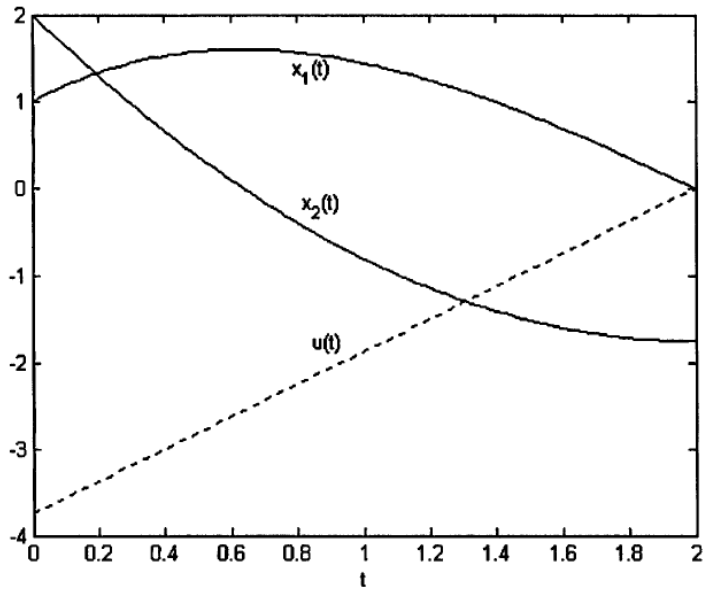

```
%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
```

Przykład 1 vs przykład 2

$$\mathbf{x}(0) = [1 \;\; 2]'; \qquad \mathbf{x}(2) = [1 \;\; 0]'.$$



$$\mathbf{x}(0) = [1 \;\; 2]'; \qquad x_1(2) = 0; \qquad x_2(2)$$ dowolny



Przykład 3

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$\mathbf{x}(0) = [1 \;\; 2]'; \;\; x_1(t_f) = 3; \qquad x_2(t_f)$ dowolny

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

S=0

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = V(u(t)) = \frac{1}{2}u^2(t)$$
$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = [f_1, \; f_2]',$$

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t). \end{aligned}$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t)$$

$$\begin{aligned} \mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t). \end{aligned}$$

Przykład 3 (cd)

$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$
$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$
$$\lambda_1^*(t) = C_3$$
$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$\mathbf{x}(0) = [1 \quad 2]'; \quad x_1(t_f) = 3; \qquad x_2(t_f)$ dowolny

$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{t_f} = 0 \longrightarrow \lambda_1(t_f)x_2(t_f) - 0.5\lambda_2^2(t_f) = 0$$

$$\lambda_2(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{*t_f} = 0$$

ponieważ S=0

$$x_1(0) = 1; \quad x_2(0) = 2; \quad x_1(t_f) = 3;$$
$$\lambda_2(t_f) = 0; \quad \lambda_1(t_f)x_2(t_f) - 0.5\lambda_2^2(t_f) = 0.$$

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 4/9; \quad C_4 = 4/3; \quad t_f = 3$$

Przykład 3 (cd)

$c1 := 1 \qquad c2 := 2$

$c3 := 4 \quad c4 := 6 \quad tf := 7$

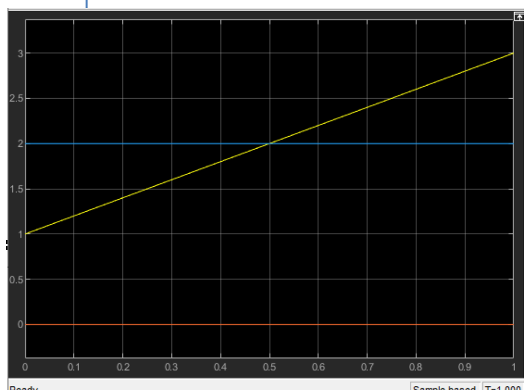
given

$c3 \cdot \frac{tf^3}{6} - c4 \cdot \frac{tf^2}{2} + c2 \cdot tf + c1$

$-c3 \cdot tf + c4 = 0$

$c3 \cdot \left(c3 \cdot \frac{tf^2}{2} - c4 \cdot tf + c2 \right) = 0$

$$\text{find}(c3, c4, tf) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$c1 := 1 \qquad c2 := 2$

$c3 := 4 \quad c4 := 6 \quad tf := 7$

given

$c3 \cdot \frac{tf^3}{6} - c4 \cdot \frac{tf^2}{2} + c2 \cdot tf + c1 = 3$

$-c3 \cdot tf + c4 = 0$

$\left(c3 \cdot \frac{tf^2}{2} - c4 \cdot tf + c2 \right) = 0$

$$\text{find}(c3, c4, tf) = \begin{pmatrix} 0.444 \\ 1.333 \\ 3 \end{pmatrix}$$

$c3 \cdot \left(c3 \cdot \frac{tf^2}{2} - c4 \cdot tf + c2 \right) = 0$

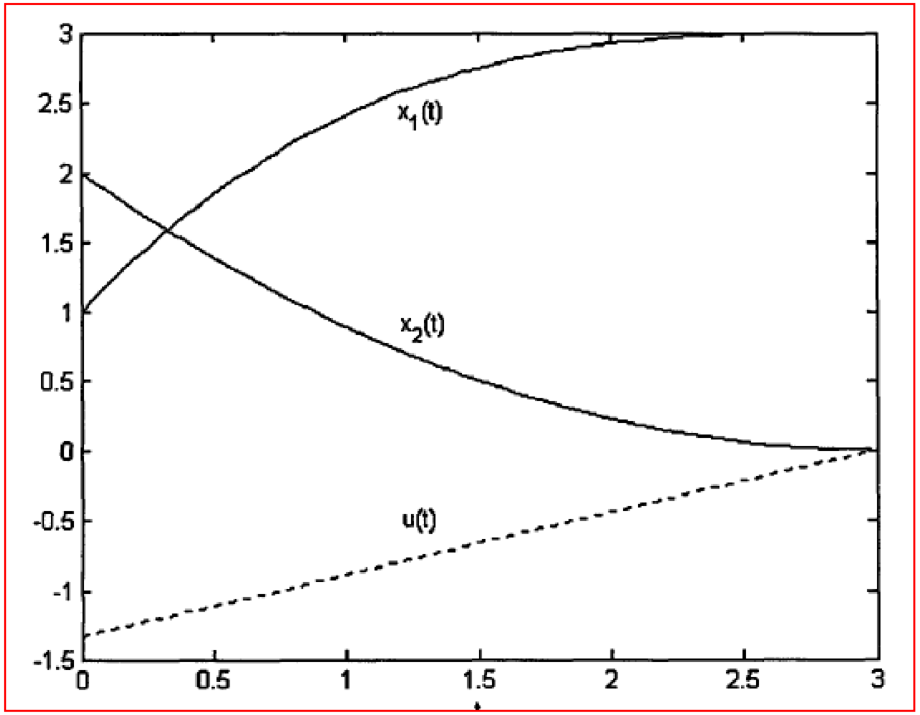
$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

Realizacja w MatLabie

```
clear all
S=dsolve('Dx1=x2,Dx2=-lam2,Dlam1=0,Dlam2=-lam1,x1(0)=1,
    x2(0)=2,x1(tf)=3,lam2(tf)=0')
t='tf';
eq1=subs(S.x1)-'x1tf';
eq2=subs(S.x2)-'x2tf';
eq3=S.lam1-'lam1tf';
eq4=subs(S.lam2)-'lam2tf';
eq5='lam1tf*x2tf-0.5*lam2tf^2';
S2=solve(eq1,eq2,eq3,eq4,eq5,'tf,x1tf,x2tf,lam1tf,
    lam2tf','lam1tf<>0')
%% lam1tf<>0 means lam1tf is not equal to 0;
%% This is a condition derived from eq5.
%% Otherwise, without this condition in the above
%% SOLVE routine, we get two values for tf (1 and 3 in this case)
%%
tf=S2.tf
x1tf=S2.x1tf;
x2tf=S2.x2tf;
clear t
x1=subs(S.x1)
x2=subs(S.x2)
lam1=subs(S.lam1)
```

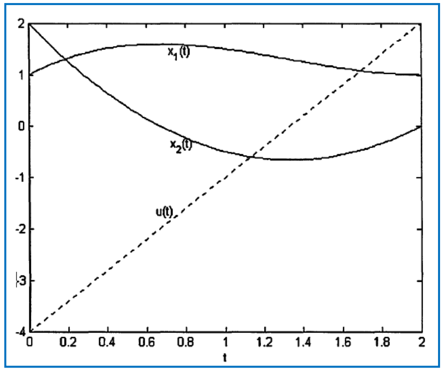
Realizacja w MatLabie

```
lam2=subs(S.lam2)
%% Convert the symbolic values to
%% numerical values as shown below.
j=1;
tf=double(subs(S2.tf))
%% coverts tf from symbolic to numerical
for tp=0:0.05:tf
t=sym(tp);
%% coverts tp from numerical to symbolic
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lam2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end
plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
```

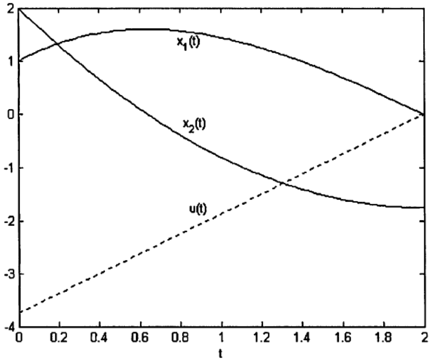


Przykład 1 vs przykład 2,3

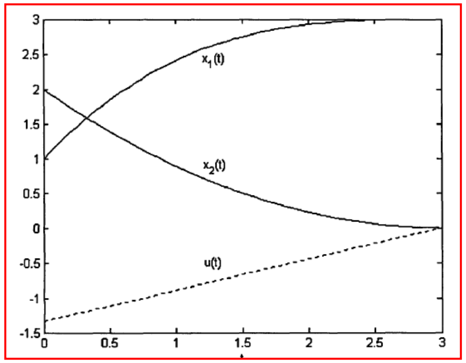
$\mathbf{x}(0) = [1 \quad 2]'; \quad \mathbf{x}(2) = [1 \quad 0]'.$



$\mathbf{x}(0) = [1 \quad 2]'; \quad x_1(2) = 0; \quad x_2(2)$ dowolny



$\mathbf{x}(0) = [1 \quad 2]'; \; x_1(t_f) = 3; \quad x_2(t_f)$ dowolny



Przykład 4

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$\mathbf{x}(0) = [1 \;\; 2]'; \; \mathbf{x}(2) =$ pożądany (wskazany)

$$J = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2}\int_0^2 u^2 dt$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$S(\mathbf{x}(t_f)) = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2$$

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = V(u(t)) = \frac{1}{2}u^2(t)$$
$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = [f_1, \; f_2]',$$

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t).\end{aligned}$$

S≠0

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t)$$

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t).\end{aligned}$$

Przykład 4 (cd)

$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$
$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$
$$\lambda_1^*(t) = C_3$$
$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$\mathbf{x}(0) = [1 \;\; 2]'$; $\mathbf{x}(2) =$ pożądany (wskazany)

δt_f → 0

$$\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}$$

ponieważ S ≠ 0

$$\lambda_1^*(t_f) = \left(\frac{\partial S}{\partial x_1}\right)_{t_f} \longrightarrow \lambda_1^*(2) = x_1(2) - 4$$
$$\lambda_2^*(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{t_f} \longrightarrow \lambda_2^*(2) = x_2(2) - 2.$$

$$\mathbf{x}(0) = [1 \;\; 2]$$

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = \frac{3}{7}, \quad C_4 = \frac{4}{7}$$

Realizacja w MatLabie

```
S=dsolve('Dx1=x2,Dx2=-lambda2,Dlambda1=0,Dlambda2=-lambda1,
x1(0)=1,x2(0)=2,lambda1(2)=x12-4,lambda2(2)=x22-2')
t='2';
S2=solve(subs(S.x1)-'x12',subs(S.x2)-'x22','x12,x22');
%% solves for x1(t=2) and x2(t=2)
x12=S2.x12;
x22=S2.x22;
clear t
```

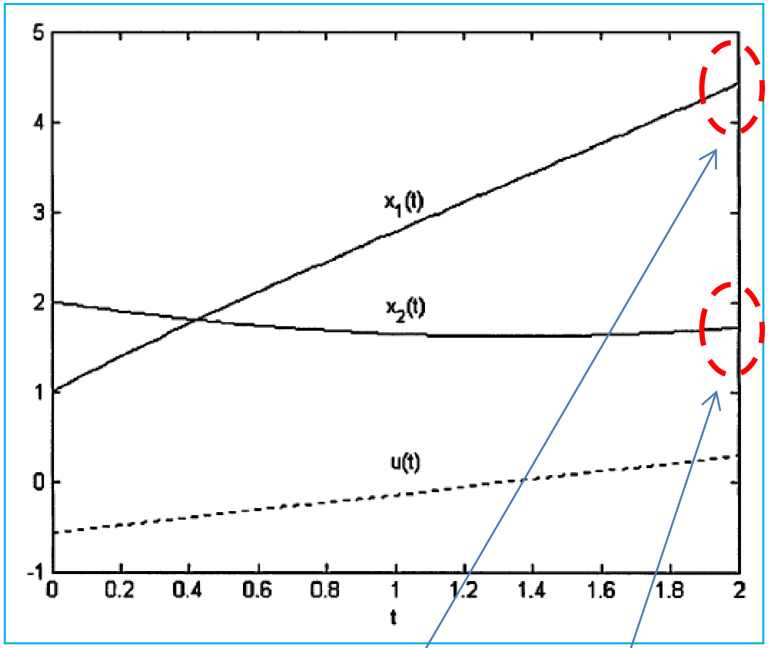
```
S =

lambda1: [1x1 sym]
lambda2: [1x1 sym]
     x1: [1x1 sym]
     x2: [1x1 sym]
```

```
x1=subs(S.x1)

x1 =

1-2/7*t^2+1/14*t^3+2*t


x2=subs(S.x2)

x2 =

-4/7*t+3/14*t^2+2
```

```
lambda1=subs(S.lambda1)

lambda1 =

3/7


lambda2=subs(S.lambda2)

lambda2 =

4/7-3/7*t
```

Realizacja w MatLabie

```
%% Plot command is used for which we need to
%% convert the symbolic values to numerical values.
j=1;
for tp=0:.02:2
t=sym(tp);
x1p(j)=double(subs(S.x1));
%% subs substitutes S.x1 to x1p
x2p(j)=double(subs(S.x2));
%% double converts symbolic to numeric
up(j)=-double(subs(S.lambda2));
%% optimal control u = -lambda_2
t1(j)=tp;
j=j+1;
end

plot(t1,x1p,'k',t1,x2p,'k',t1,up,'k:')
xlabel('t')
gtext('x_1(t)')
gtext('x_2(t)')
gtext('u(t)')
```



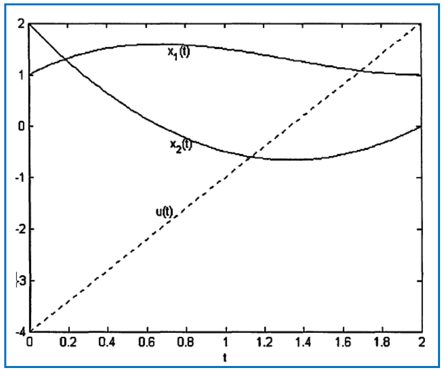
$$J = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2}\int_0^2 u^2 dt$$

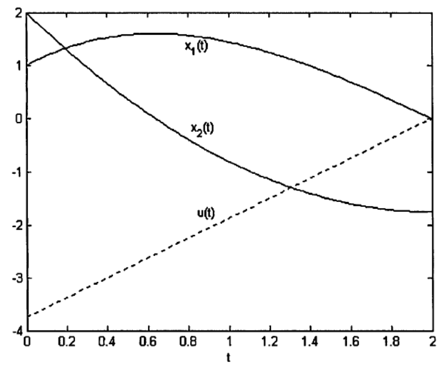
Przykład 1 vs przykład 2,3,4

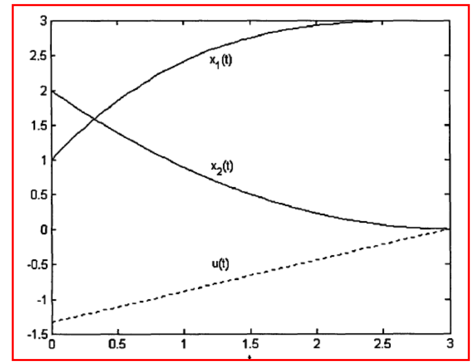
$\mathbf{x}(0) = [1 \;\; 2]'; \quad \mathbf{x}(2) = [1 \;\; 0]'.$

$\mathbf{x}(0) = [1 \;\; 2]'; \quad x_1(2) = 0; \quad x_2(2)$ dowolny

$\mathbf{x}(0) = [1 \;\; 2]'; \;\; x_1(t_f) = 3; \quad x_2(t_f)$ dowolny

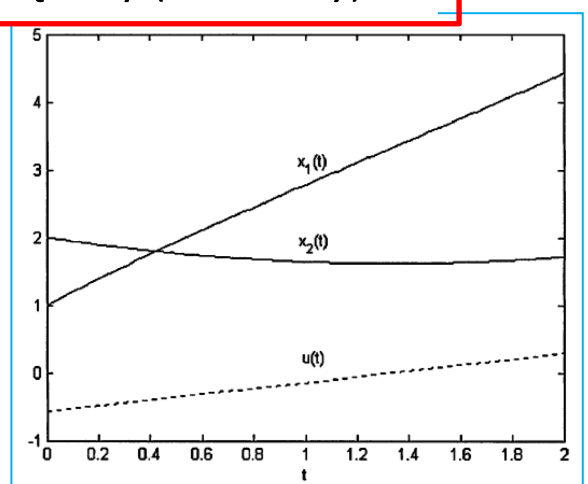






$\mathbf{x}(0) = [1 \;\; 2]'; \;\; x_1(t_f)$ pożądany (wskazany)

$$\frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2$$



Przykład 5

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

$\mathbf{x}(0) = [1 \;\; 2]'$; $\mathbf{x}(t_f) =$ pożądany (wskazany)

$$J = \frac{1}{2}[x_1(t_f) - 4]^2 + \frac{1}{2}[x_2(t_f) - 2]^2 + \frac{1}{2}\int_0^{t_f} u^2 dt$$

$$J = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

$$S(\mathbf{x}(t_f)) = \frac{1}{2}[x_1(t_f - 4]^2 + \frac{1}{2}[x_2(t_f - 2]^2$$

$$V(\mathbf{x}(t), \mathbf{u}(t), t) = V(u(t)) = \frac{1}{2}u^2(t)$$
$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = [f_1, \; f_2]',$$

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)) \\ &= V(u(t)) + \boldsymbol{\lambda}'(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ &= \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)u(t).\end{aligned}$$

S≠0

$$\frac{\partial\mathcal{H}}{\partial u} = 0 \longrightarrow u^*(t) + \lambda_2^*(t) = 0 \longrightarrow u^*(t) = -\lambda_2^*(t)$$

$$\begin{aligned}\mathcal{H}^*(x_1^*(t), x_2^*(t), \lambda_1^*(t), \lambda_2^*(t)) &= \frac{1}{2}\lambda_2^{*^2}(t) + \lambda_1^*(t)x_2^*(t) - \lambda_2^{*^2}(t) \\ &= \lambda_1^*(t)x_2^*(t) - \frac{1}{2}\lambda_2^{*^2}(t).\end{aligned}$$

Przykład 5 (cd)

$$x_1^*(t) = \frac{C_3}{6}t^3 - \frac{C_4}{2}t^2 + C_2 t + C_1$$
$$x_2^*(t) = \frac{C_3}{2}t^2 - C_4 t + C_2$$
$$\lambda_1^*(t) = C_3$$
$$\lambda_2^*(t) = -C_3 t + C_4.$$

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

$\mathbf{x}(0) = [1 \;\; 2]'$; $\mathbf{x}(t_f)$ = pożądany (wskazany)

ponieważ S ≠ 0

$$\left(\mathcal{H} + \frac{\partial S}{\partial t}\right)_{t_f} = 0 \longrightarrow \lambda_1(t_f)x_2(t_f) - 0.5\lambda_2^2(t_f) = 0$$

$$\boldsymbol{\lambda}^*(t_f) = \left(\frac{\partial S}{\partial \mathbf{x}}\right)_{*t_f}$$

$$\lambda_1^*(t_f) = \left(\frac{\partial S}{\partial x_1}\right)_{t_f} \longrightarrow \lambda_1^*(t_f) = x_1(t_f) - 4$$
$$\lambda_2^*(t_f) = \left(\frac{\partial S}{\partial x_2}\right)_{t_f} \longrightarrow \lambda_2^*(t_f) = x_2(t_f) - 2.$$

$$\mathbf{x}(0) = [1 \;\; 2]$$

$$C_1 = 1, \quad C_2 = 2,$$

Przykład 5 (cd)

$c1 := 1 \qquad c2 := 2$

$c3 := 4 \quad c4 := 6 \quad tf := 7$

given

$$c3 \cdot \frac{tf^3}{6} - c4 \cdot \frac{tf^2}{2} + c2 \cdot tf + c1 - 4 = c3$$

$$-c3 \cdot tf + c4 = 0$$

$$\left(c3 \cdot \frac{tf^2}{2} - c4 \cdot tf + c2 - 2\right) = -c3 \cdot tf + c4$$

$$\text{find}(c3, c4, tf) = \begin{pmatrix} -2.733 \times 10^{-10} \\ -3.139 \times 10^{-10} \\ 1.5 \end{pmatrix}$$

$c1 := 1 \qquad c2 := 2$

$c3 := 4 \quad c4 := 6 \quad tf := 7$

given

$$c3 \cdot \frac{tf^3}{6} - c4 \cdot \frac{tf^2}{2} + c2 \cdot tf + c1 - 14 = c3$$

$$-c3 \cdot tf + c4 = 0$$

$$\left(c3 \cdot \frac{tf^2}{2} - c4 \cdot tf + c2 - 5\right) = -c3 \cdot tf + c4$$

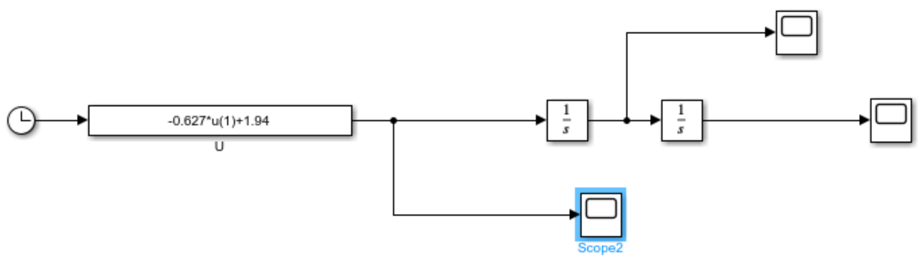
$$\text{find}(c3, c4, tf) = \begin{pmatrix} -0.627 \\ -1.94 \\ 3.093 \end{pmatrix}$$

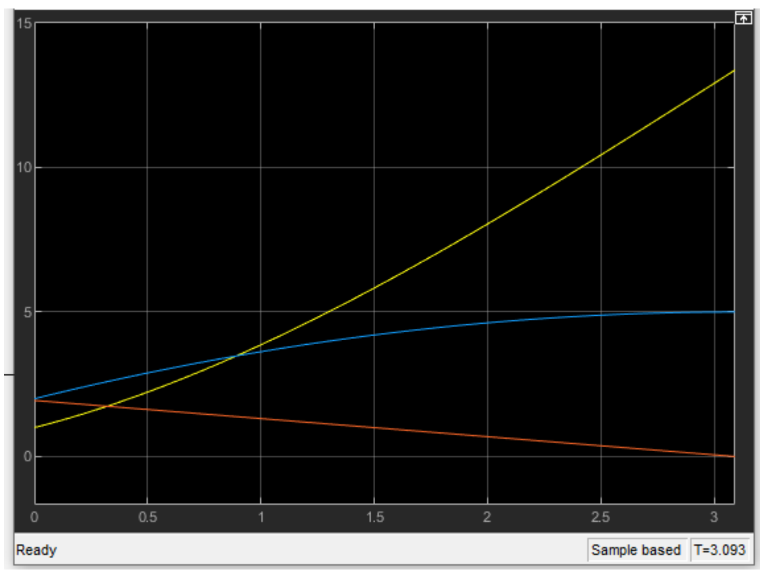
x1(tf)=14

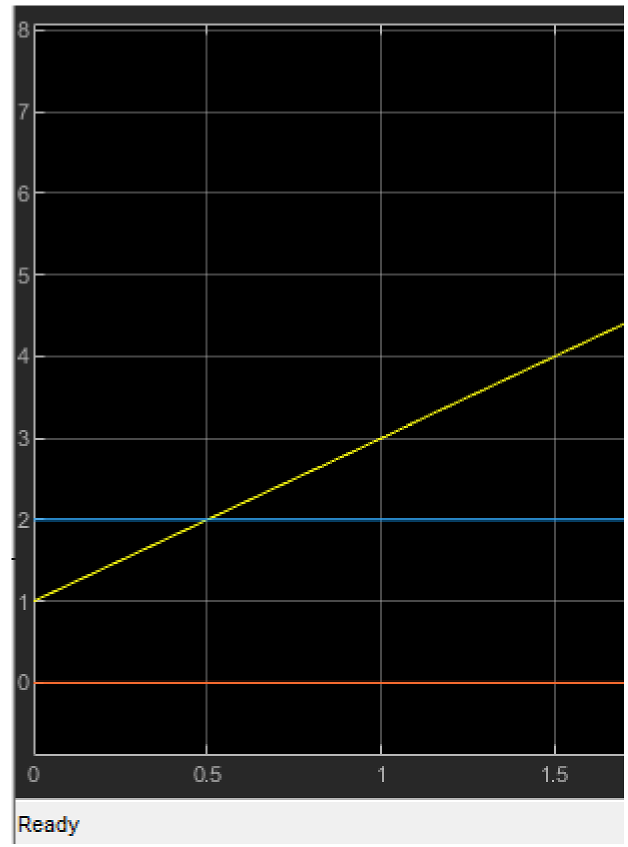
x2(tf)= 5

$$u^*(t) = -\lambda_2^*(t) = C_3 t - C_4$$

Przykład 5 (cd)







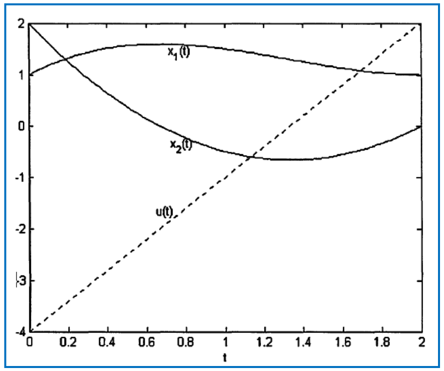
bellman5

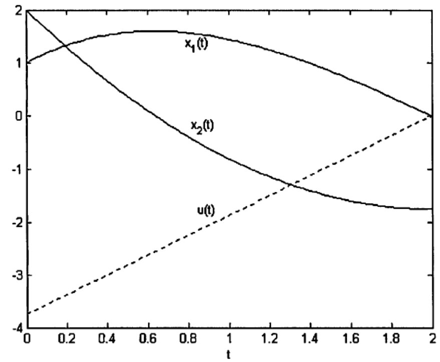
Przykład 1 vs przykład 2,3,4,5

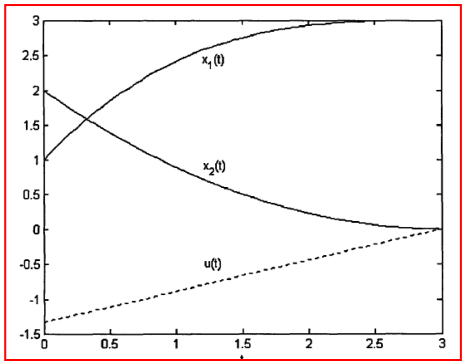
$\mathbf{x}(0) = [1 \ \ 2]'; \quad \mathbf{x}(2) = [1 \ \ 0]'.$

$\mathbf{x}(0) = [1 \ \ 2]'; \quad x_1(2) = 0; \quad x_2(2)$ dowolny

$\mathbf{x}(0) = [1 \ \ 2]'; \ x_1(t_f) = 3; \quad x_2(t_f)$ dowolny





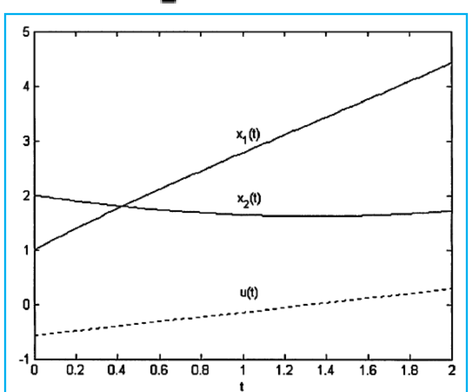


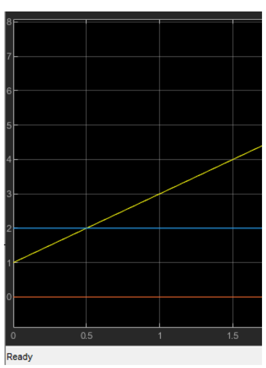
$\mathbf{x}(0) = [1 \ \ 2]'; \ \mathbf{x}(2) =$ dowolny

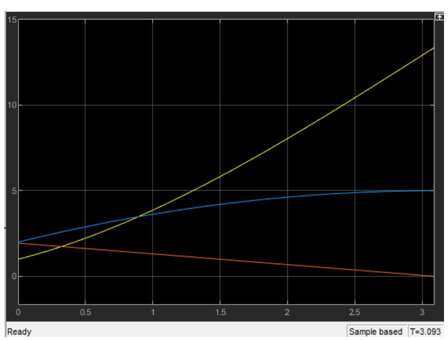
$\mathbf{x}(0) = [1 \ \ 2]'; \ x_1(t_f)$ pożądany (wskazany)

$$\frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2$$

$$\frac{1}{2}[x_1(t_f) - 4]^2 + \frac{1}{2}[x_2(t_f) - 2]^2$$







x1(tf)=14

x2(tf)= 5

Ograniczenia - nierówności

$$J = \int_{t_0}^{t_f} \varphi(\mathbf{x}, \dot{\mathbf{x}}, t)\,dt$$

$$\Lambda(x, \dot{x}, t) = 0$$

$$\Gamma_{\min} \leqslant \Gamma(\mathbf{x}, \dot{\mathbf{x}}, t) \leqslant \Gamma_{\max}$$

Ekstrema warunkowe na przedziale domkniętym

$$f(x) \to \underset{x \in X}{\text{extr}} \qquad x_0 \in X \qquad X \subset R^n$$

Tw. Weierstrassa
Każda funkcja ciągła **na przedziale domkniętym** ma wartość najmniejszą i największą.

Aby znaleźć największą i najmniejszą wartość funkcji f (x) w obszarze X:
- znajdź punkty krytyczne **wewnątrz** obszaru X, oblicz w nich wartości funkcji f (x);
- znajdź największą i najmniejszą wartość funkcji f (x) **na granicy** obszaru X;
- porównaj znalezione wartości i wybierz spośród nich największą i najmniejszą.

Warunki Kuhna-Tuckera

Warunki konieczne istnienia ekstremum.

$$\min_{x \in R^n} F(x)$$
$$c_i(x) \leqslant 0, i = 1 \ldots m \qquad (1)$$
$$F(x), c_i(x) \in C^1$$

funkcja jest różniczkowalna, funkcja pochodna jest ciągła

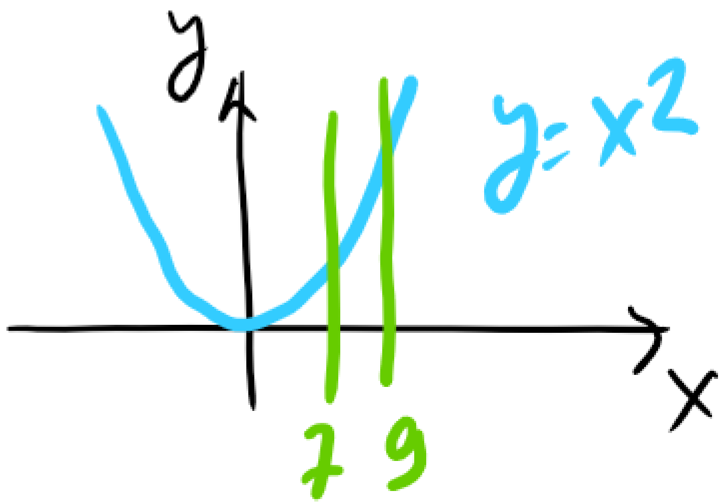
Każdy punkt spełniający ograniczenia nazywany jest **punktem dopuszczalnym**. Celem Optymalizacji z ograniczeniami jest znalezienie punktu dopuszczalnego, w którym minimalizowana funkcja osiąga **przynajmniej lokalnie** najmniejszą możliwą wartość.

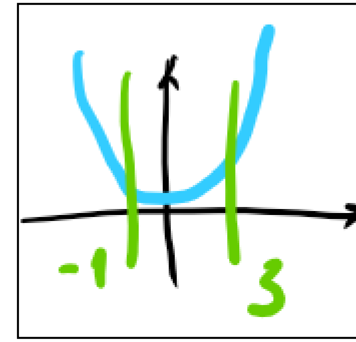
Tw. Kuhna-Truckera

Jeśli w punkcie xo funkcja F(xo) osiąga minimum lokalne, to w punkcie tym istnieją mnożniki lambda spełniające warunki

$$\nabla_x F(x^o) + \sum_i \lambda_i^o \nabla c_i(x^o) = 0$$
$$c_i(x^o) \leqslant 0$$
$$\lambda_i^o c_i(x^o) = 0$$
$$\lambda_i^o > 0$$

Optymalizacja funkcji z ograniczeniami nierównościami





$$\min y(x)$$

$$x \in D_i$$

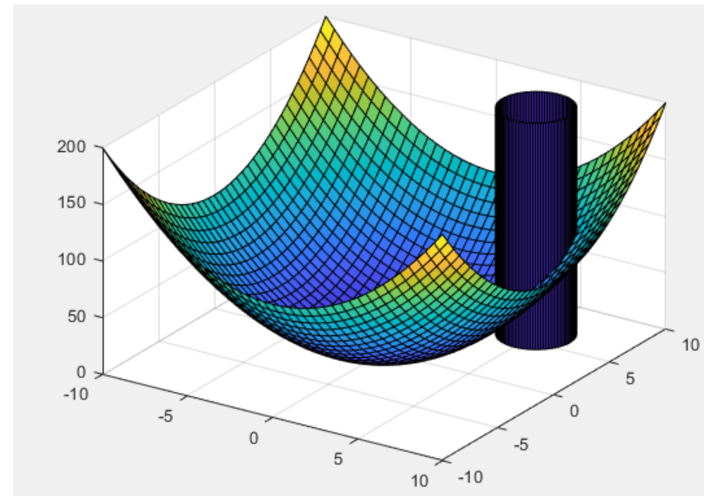
$$D_1: (x-7)(9-x)=0$$

$$D_2: (x-7)(9-x)\geq 0$$

$$D_3: (x-7)(9-x)<0$$

$$D_4: x \in R$$

```
X,Y] = meshgrid(-10:0.5:10,-10:0.5:10);
Z = X.^2+Y.^2;
surf(X,Y,Z);
hold on;
[X Y Z]=cylinder(2,100);
surf(X+5,Y+6,Z.*200);
```



$\min y(x_1,x_2) = x_1^2 + x_2^2$

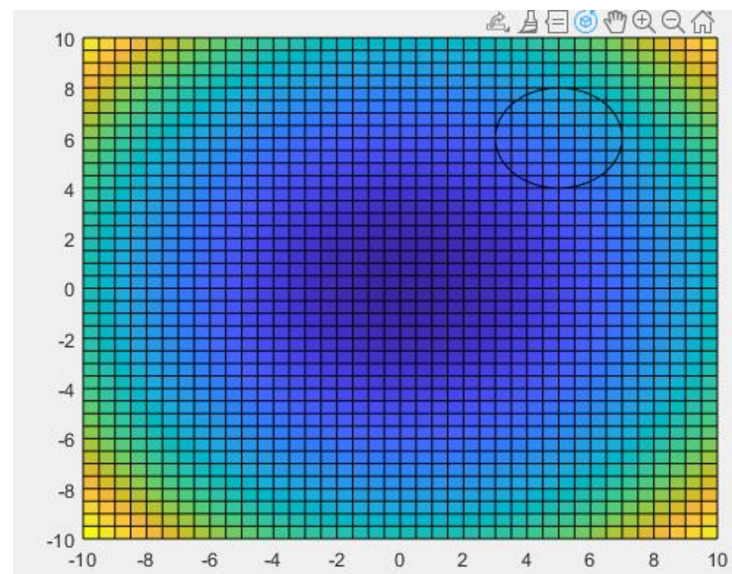
$(x_1,x_2) \in D_i$

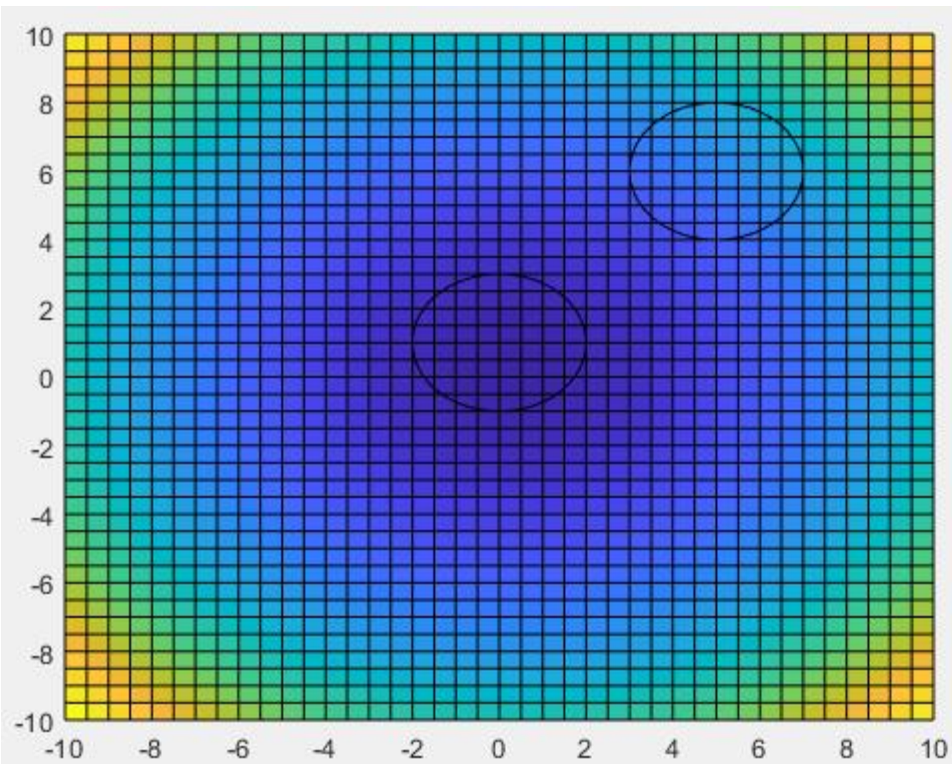
$D_1 = r^2 = (x_1-a)^2+(x_2-b)^2, \forall z$

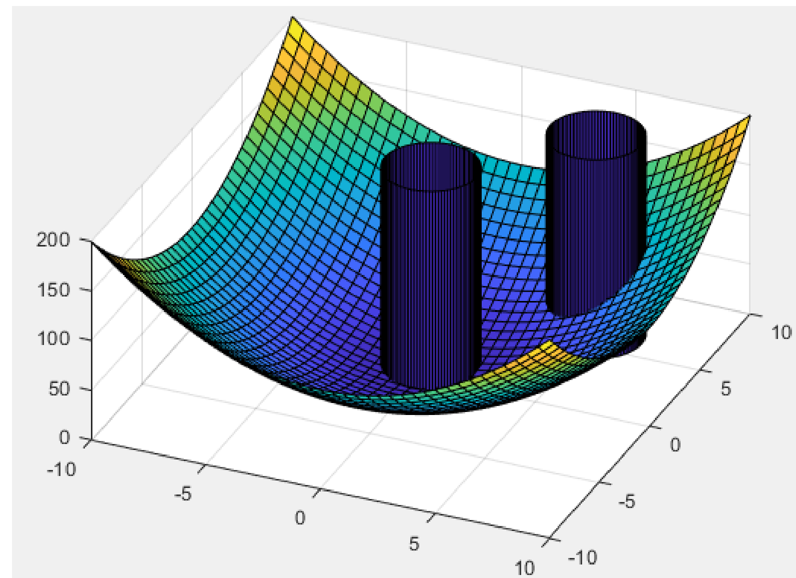
$D_2 : r^2 \leq (x_1-a)^2+(x_2-b)^2, \forall z$

$D_3 : r^2 > (x_1-a)^2+(x_2-b)^2 \; \forall z$

$D_4 : x_1, x_2 \in \mathbb{R}$




10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10


200
150
100
50
0
-10
-5
0
5
10
-10
-5
0
5
10

Przykład.

$$\dot{x} = u$$

$$x(0) = 0, \; x(2) \to \max, \quad t \in [0, 2]$$

$$J = \int_0^2 u^2 dt \to \min$$

$$J(\mathbf{u}(t)) = S(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$$S(x(t), t) = -x(t)$$

$$J' = -x(2) + \int_0^2 u^2 dt$$

$$\frac{\partial}{\partial x} S(x(t), t) = -1 \qquad \frac{\partial}{\partial t} S(x(t), t) = 0$$

$$\mathcal{L} = \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)$$
$$= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t}$$
$$+ \boldsymbol{\lambda}'(t) \{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\}$$

$$\frac{d}{dt} S(x(t), t) = \frac{\partial}{\partial x} S(x(t), t) \frac{dx}{dt} + \frac{\partial}{\partial t} S(x(t), t) \frac{dt}{dt}$$

$$J'' = \int_0^2 \left[-\dot{x} + u^2 + \lambda(\dot{x} - u) \right] dt$$

L

0

(-1+lambda(t))

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

$$\dot{\lambda} = 0 \Rightarrow \lambda(t) = C$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = \frac{C}{2}$$

$$x = C_1 t + C_2$$

Przykład (cd)

warunek transwersalności

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_* \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)'_*\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

$$\left(\lambda(t)-1\right)\Big|_{t=2} = 0 \Rightarrow \lambda(2) = 1$$

$$\dot{\lambda} = 0 \Rightarrow \lambda = C$$

$$C = 1$$

$$2u - \lambda = 0 \Rightarrow u = C_1 = 0.5$$

$$\dot{x} = u \Rightarrow \dot{x} = C_1 \Rightarrow x(t) = C_1 t + C_2$$

$$x(0) = 0, x(2) \to \max$$

$$C_2 = 0, \quad C_1 = 0.5$$

$$x(t) = 0.5t$$

Porównywanie

$$\dot{x} = u$$

$$x(t) = ut$$

$$J = \int_0^2 u^2 dt \to \min$$

$$J' = -x(2) + \int_0^2 u^2 dt$$

X(2)	U	J	J’
0	0	0	0
0.2000	0.1000	0.0200	-0.1800
0.4000	0.2000	0.0800	-0.3200
0.6000	0.3000	0.1800	-0.4200
0.8000	0.4000	0.3200	-0.4800
1	0.5000	0.5000	-0.5000
1.2000	0.6000	0.7200	-0.4800
1.4000	0.7000	0.9800	-0.4200
1.6000	0.8000	1.2800	-0.3200
1.8000	0.9000	1.6200	-0.1800
2	1	2	0

```
Out=[];
for u=0:0.1:1
   Out=[Out; 2*u,u,2*u*u,2*u*u-2*u];
end;
```

Zagadnienie 1

$$\dot{x} = u$$

$$x(0) = 0, x(2) \rightarrow \max, \quad t \in [0,2]$$

$$-1 \leq u \leq 1$$

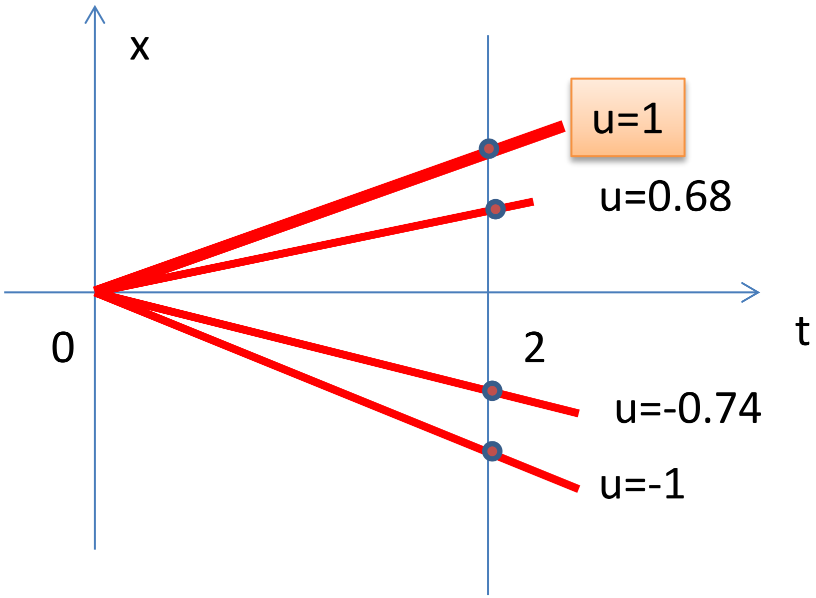
~~$$J = \int_0^2 u^2 dt \rightarrow \min$$~~

Rozwiązanie bez ograniczeń

$$x(t) = u \cdot t$$

Jest zupełnie oczywiste, że

$$u = 1$$



Zagadnienie 1

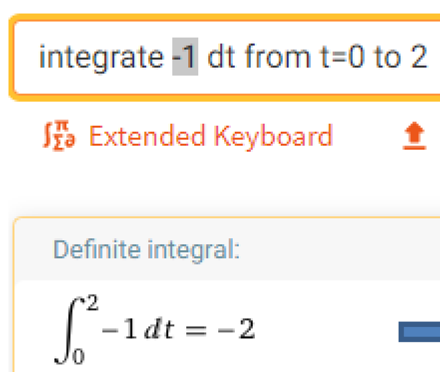
$$\dot{x} = u$$

$$x(0)=0,\ x(2)\to \max, \quad t\in[0,2]$$

$$J=\int_0^2 u^2 dt \to \min$$

$$-1 \le u \le 1$$

$$J=-x(2)=-x(0)+\int_0^2(-\dot{x})\,dt=\int_0^2[-\dot{x}]\,dt=\int_0^2[-u(t)]\,dt\to \min$$



→ U=1

Ograniczenia - nierówności

$$J = \int_{t_0}^{t_f} \varphi(\mathbf{x}, \dot{\mathbf{x}}, t)\,dt$$

$$\Lambda(x, \dot{x}, t) = 0$$

$$\Gamma_{\min} \leqslant \Gamma(\mathbf{x}, \dot{\mathbf{x}}, t) \leqslant \Gamma_{\max}$$

$$-1 \leq u \leq 1$$

Metoda zamiany zmiennych

$$\left(\Gamma_{\max i} - \Gamma_i\right)\left(\Gamma_i - \Gamma_{\min i}\right) = \gamma_i^2; i = 1, 2, \ldots$$

Metoda przez funkcjonał Lagrange'a

$$\mathcal{L} = \mathcal{L}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t)$$
$$= V(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left(\frac{\partial S}{\partial \mathbf{x}}\right)_*' \dot{\mathbf{x}}^*(t) + \frac{\partial S}{\partial t}$$
$$+\boldsymbol{\lambda}'(t)\left\{\mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right\}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)_* - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_* = 0$$

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}}\right)_* = 0$$

Warunek transwersalności

$$\left[\mathcal{L}^* - \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*' \dot{\mathbf{x}}(t)\right]\Bigg|_{t_f} \delta t_f + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}\right)_*'\Bigg|_{t_f} \delta \mathbf{x}_f = 0$$

Zagadnienie 1(cd)

$$-1 \le u \le 1 \Leftrightarrow \left(u_{\max} - u\right)\left(u - u_{\min}\right) = \alpha^2 \ge 0$$

$$u_{\min} = -1, \quad u_{\max} = 1$$

$$J' = \int_0^2 \Big[\underbrace{-\dot{x} + \lambda_1\left(\dot{x} - u\right) + \lambda_2\left(\left(u_{\max} - u\right)\left(u - u_{\min}\right) - \alpha^2\right)}_{F}\Big]dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \qquad \dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = C$$

$$\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}} = 0 \qquad -\lambda_1 + \lambda_2\left(2u\right) = 0$$

$$\frac{\partial F}{\partial \alpha} - \frac{d}{dt}\frac{\partial F}{\partial \dot{\alpha}} = 0 \qquad \lambda_2 \alpha = 0$$

warunek transwersalności (L)

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=2} = 0$$

$$\left.\left(\lambda_1\left(t\right) - 1\right)\right|_{t=2} = 0 \Rightarrow \lambda_1\left(2\right) = 1$$

$$\Rightarrow C = 1 \Rightarrow \lambda_1 = 1$$

Zagadnienie 1 (cd)

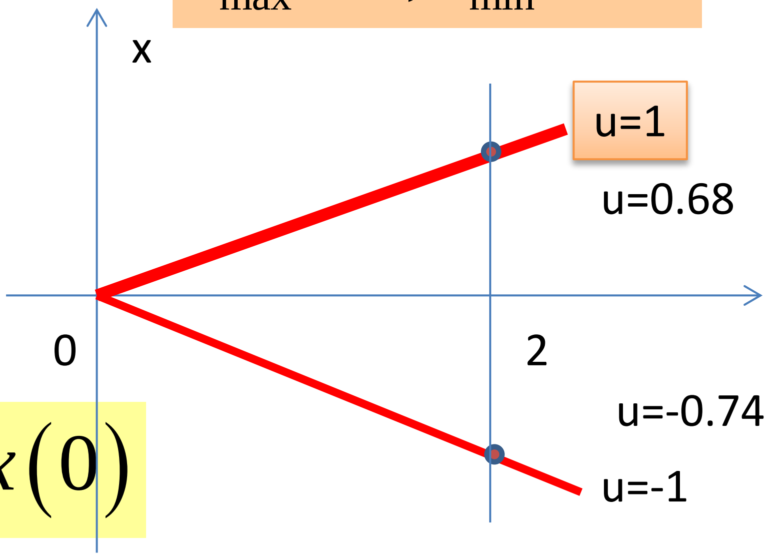
$$\begin{cases} \lambda_2 (2u) = 1 \\ \lambda_2 \alpha = 0 \xrightarrow{\times\alpha} \lambda_2 \alpha^2 = 0 \Rightarrow \lambda_2 (u_{\max} - u)(u - u_{\min}) = 0 \end{cases}$$

$$\lambda_2(t) \neq 0 \Rightarrow u = u_{\max} \quad \text{lub} \quad u = u_{\min}$$

$$u_{\max} = 1, u_{\min} = -1$$

$$x(0) = 0, x(2) \to \max, \quad t \in [0, 2]$$

?



$$x(t) = u \cdot t, u = -1, \Rightarrow x(2) = -2 < x(0)$$

$$x(t) = u \cdot t, u = 1, \Rightarrow x(2) = 2 > x(0)$$

$$u = 1$$

Zagadnienie 2

$$\ddot{x} = u(t)$$

$$\dot{x}_1 = x_2(t),$$
$$\dot{x}_2 = u(t)$$

$x_1(t_0) = x_0$

$x_2(t_0) = v_0$

$x_1(t_f)$ → maksimum

$x_2(t_f) = v_f$

$$u_{\min} \le u \le u_{\max}$$

Rozwiązanie

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u \end{cases}$$

$\dot{x}_1 = x_2(t)$, $\dot{x}_2 = u(t)$ $x_1(t_0) = x_0$ $x_2(t_0) = v_0$

$x_1(t_f)$ → maksimum

$u_{\min} \le u \le u_{\max}$

$x_2(t_f) = v_f$

$$J = -x_1(t_f) = -x_1(t_0) + \int_{t_0}^{t_f} (-\dot{x}_1)\, dt$$

$\dot{x}_1 = x_2(t)$, $x_1(t_0) = x_0$, $x_1(t_f)$ — maksimum

$\dot{x}_2 = u(t)$, $x_2(t_0) = v_0$, $x_2(t_f) = v_f$

→ $(u - u_{\min})(u_{\max} - u) - \alpha^2 = 0.$

$$J = \int_{t_0}^{t_f} \varphi(\mathbf{x}, \dot{\mathbf{x}}, t)\, dt \qquad \Lambda(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$$

→

$$J' = \int_{t_0}^{t_f} [\varphi(\mathbf{x}, \dot{\mathbf{x}}, t) + \lambda^T(t)\, \Lambda(\mathbf{x}, \dot{\mathbf{x}}, t)]\, dt$$

$$J' = -x_1(t_0) + \int_{t_0}^{t_f} \left\{ -\dot{x}_1 + \lambda_1[x_2 - \dot{x}_1] + \lambda_2[u - \dot{x}_2] + \lambda_3\left[(u - u_{\min})^{\cdot}(u_{\max} - u) - \alpha^2\right] \right\} dt$$

Rozwiązanie (kont.)

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\mathbf{x}}} - \frac{\partial \Phi}{\partial \mathbf{x}} = 0 \quad \mathbf{x}^T = \left[x_1, x_2, u\right]$$

$$\Phi = \lambda_1\left[x_2 - \dot{x}_1\right] + \lambda_2\left[u - x_2\right] + \lambda_3\left[\left(u - u_{\min}\right)\left(u_{\max} - u\right) - \alpha^2\right] - \dot{x}_1$$

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1$$

$$0 = -\lambda_2 + \lambda_2\left[2u - u_{\max} - u_{\min}\right], \quad 0 = \alpha\lambda_3$$

warunek transwersalności (L)

$$\left.\frac{\partial \Phi}{\partial \dot{x}_1}\right|_{t=t_f} = 0 = -1 - \lambda_1\left(t_f\right)$$

Rozwiązanie (kont.)

$$\dot{x}_1 = x_2(t), \quad x_1\left(t_0\right) = x_0$$

$$\dot{x}_2 = x(t), \quad x_2\left(t_0\right) = v_0$$

$$\dot{\lambda}_1 = 0, \quad \lambda_1\left(t_f\right) = -1$$

$$\dot{\lambda}_2 = -\lambda_1(t), \quad x_2\left(t_f\right) = v_f$$

$$\alpha(t)\lambda_3(t) = 0$$

$$\lambda_2(t) = \lambda_3(t)\left[2u(t) - u_{max} - u_{\min}\right]$$

$$\alpha^2(t) = \left[u(t) - u_{\min}\right]\left[u_{\max} - u(t)\right]$$

Rozwiązanie (kont.)

$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = C, \lambda_1\left(t_f\right) = -1 \Rightarrow \lambda_1 = -1$$

$$\dot{\lambda}_2 = -\lambda_1$$

$$\lambda_2\left(t\right) = t + C_1$$

$$\begin{cases} \lambda_3\left(2u\right) = t + C_1 \\ \lambda_3 \alpha = 0 \xrightarrow{\times\alpha} \lambda_3 \alpha^2 = 0 \Rightarrow \lambda_3\left(u_{\max} - u\right)\left(u - u_{\min}\right) = 0 \end{cases}$$

$$\lambda_3\left(t\right) \neq 0 \Rightarrow u = u_{\max} \quad \text{lub} \quad u = u_{\min}$$

Rozwiązanie (kont.)

$u_{\max} = 1, u_{\min} = -1$

$t_0 = 0, t_f = 2$

$x_1(0) = 0, \quad x_1(2) = \rightarrow \max$

$x_2(0) = 0, \quad x_2(2) = 0,$

$u = 1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = 1 \end{cases}$$

$$\begin{cases} x_1(t) = \dfrac{t^2}{2} \Rightarrow x_1 = \dfrac{x_2^2}{2} \\ x_2(t) = t \end{cases}$$

$u = -1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -1 \end{cases}$$

$$\begin{cases} x_1(t) = -\dfrac{t^2}{2} \Rightarrow x_1 = -\dfrac{x_2^2}{2} \\ x_2(t) = -t \end{cases}$$

Trzeba znaleźć warunek

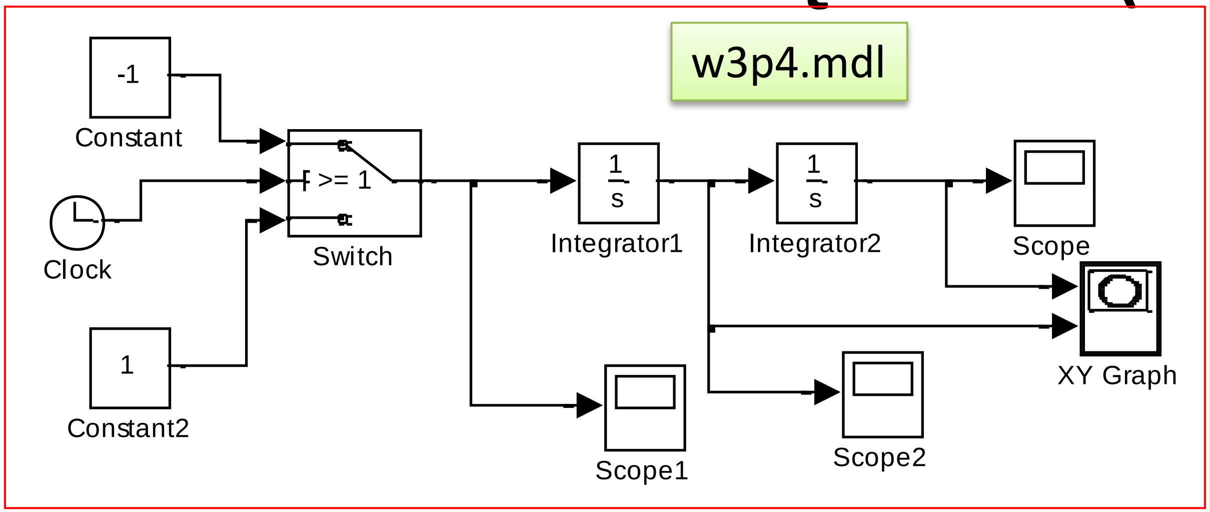
$t^* \in [0, 2]$

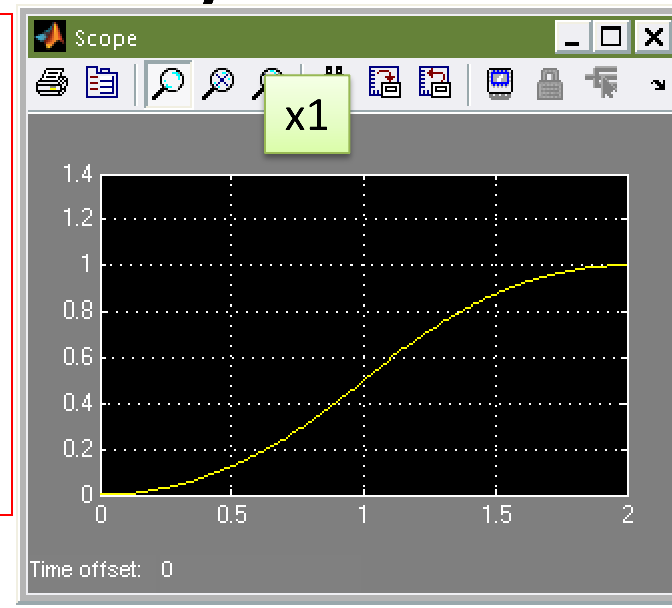
$u = 1, \; jeśli \;\; t \le t^*$

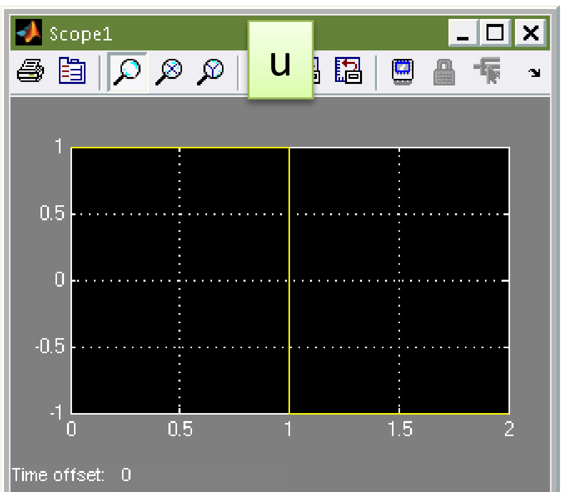
$u = -1, \; jeśli \;\; t > t^*$

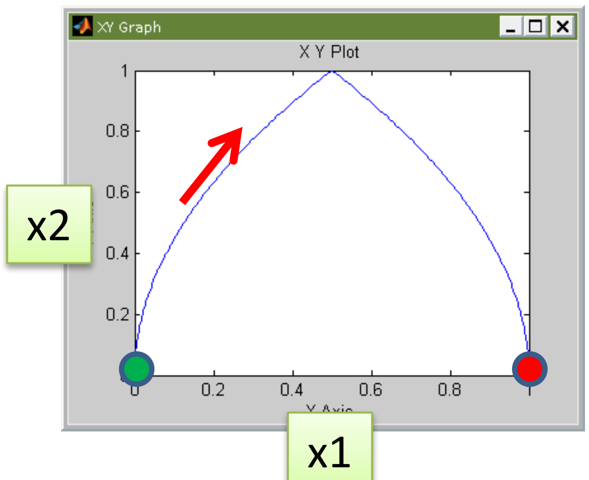
$t^* = 1$

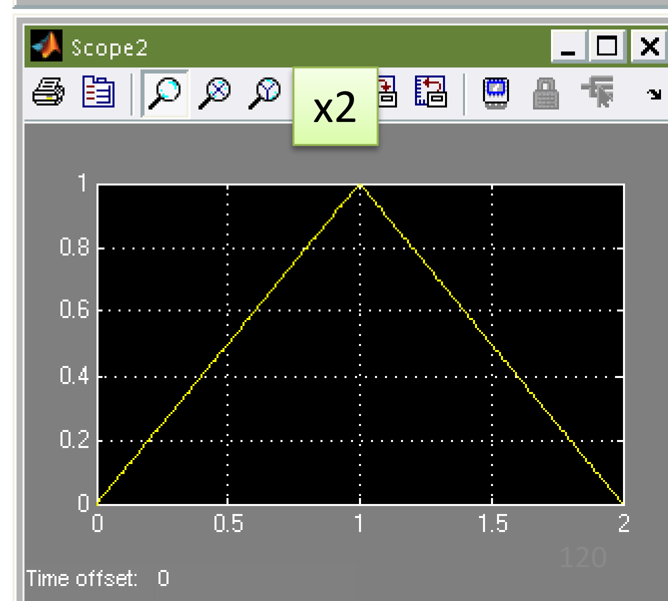
Rozwiązanie (kont.)











Rozwiązanie (kont.)

$u_{\max} = 1, u_{\min} = -1$

$t_0 = 0, t_f = 2$

$x_1(0) = A, \quad x_1(2) = \rightarrow \max$

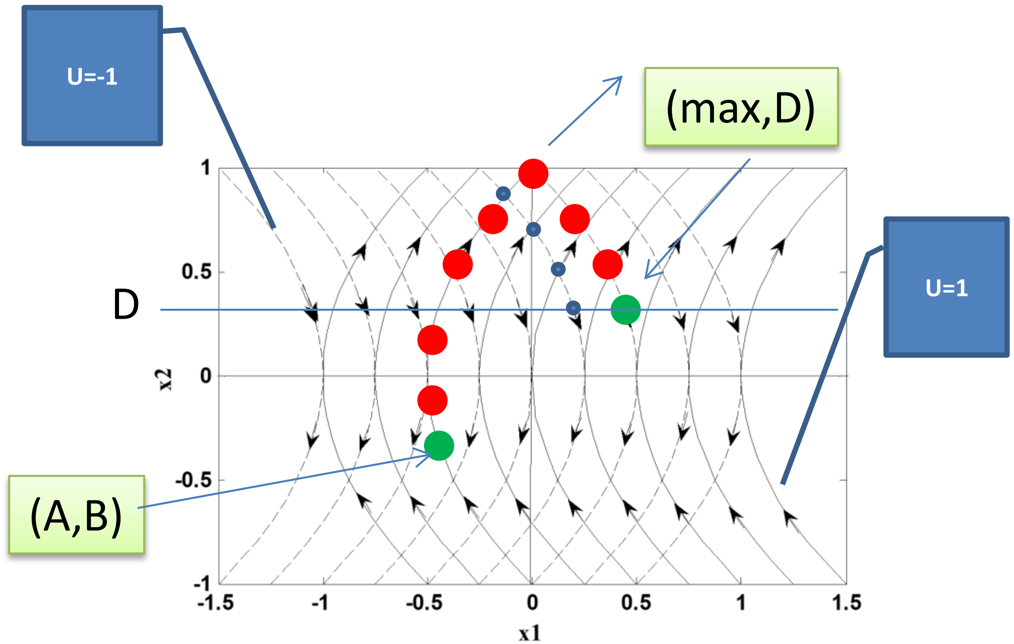
$x_2(0) = B, \quad x_2(2) = D,$

$u = 1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = 1 \end{cases}$$

$u = -1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -1 \end{cases}$$



Trzeba znaleźć warunek

$t^* \in [0, 2]$

$u = 1, \; jeśli \;\; t \le t^*$

$u = -1, \; jeśli \;\; t > t^*$

$t^* = ?$

$$x_1 = \frac{x_2^2}{2} + C$$

$$x_1 = -\frac{x_2^2}{2} + C$$

Rozwiązanie (kont.)

$u = 1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = 1 \end{cases}$$

$$x_1(0) = A, \quad x_1(2) = \to \max$$
$$x_2(0) = B, \quad x_2(2) = D$$

$$\begin{cases} x_1(t) = \dfrac{t^2}{2} + C_1 t + C_2 \\ x_2(t) = t + C_1 \end{cases} \Rightarrow C_1 = B, C_2 = A$$

$$\begin{cases} x_1(t^*) = \dfrac{t^{*2}}{2} + Bt^* + A \\ x_2(t^*) = t^* + B \end{cases} \Rightarrow C_1 = B, C_2 = A$$

Rozwiązanie(kont.)

$u = -1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -1 \end{cases}$$

$x_1(t^*),\quad x_1(2) \Rightarrow \max$

$x_2(t^*),\quad x_2(2) = D$

$$\begin{cases} x_1(t) = -\frac{t^2}{2} + C_3 t + C_4 \\ x_2(t) = -t + C_3 \end{cases} \Rightarrow C_3 = D + 2$$

$$x_1(t^*) = -\frac{t^{*2}}{2} + (D+2)t^* + C_4$$

$$x_1(t^*) = \frac{t^{*2}}{2} + Bt^* + A$$

$$C_4 = t^{*2} + (B - D - 2)t^* + A$$

Rozwiązanie(kont.)

$$u = 1: \quad x_2(t^*) = t^* + B$$

$$u = -1: \quad x_2(t^*) = -t^* + D + 2$$

$$t^* = \frac{(D + 2 - B)}{2}$$

$$C_4 = t^{*2} + (B - D - 2)t^* + A$$

$$x_1(0) = A = 0, \quad x_1(2) = \longrightarrow \max$$

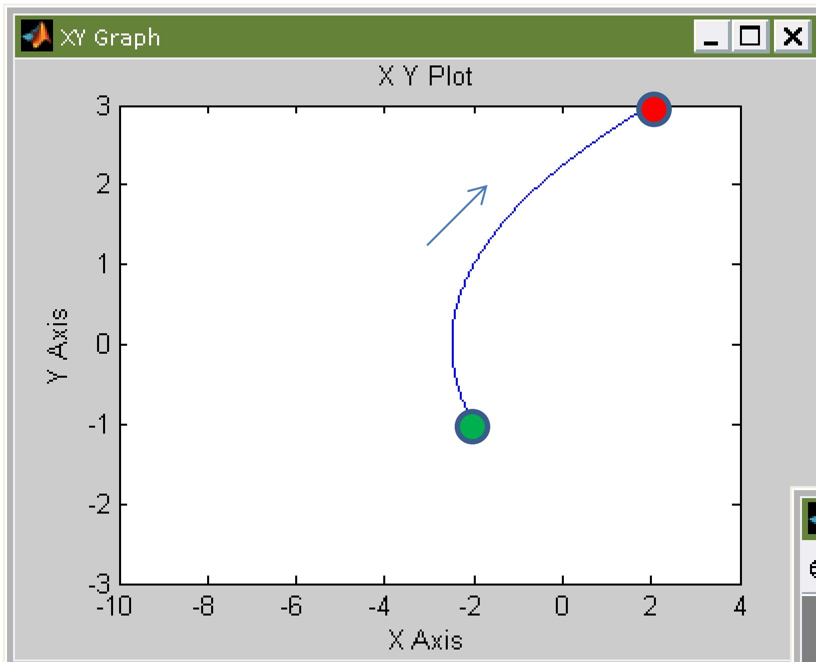
$$x_2(0) = B = 0, \quad x_2(2) = D = 0$$

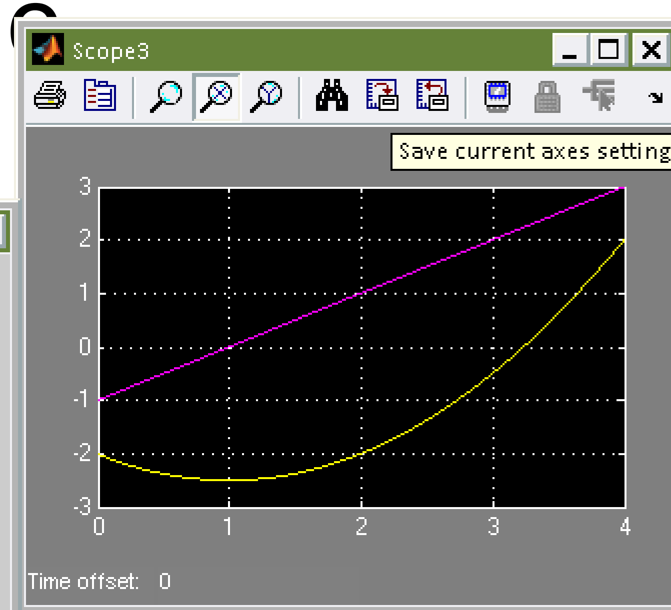
t*=1, C4=-1

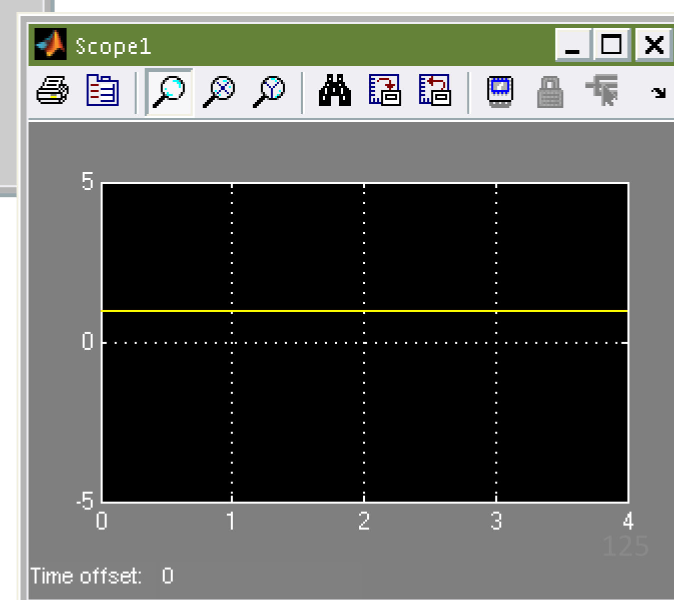
Rozwiązanie

X1(0)=-2
X2(0)=-1
vX1(tf)-max
X2(tf)=3
Tf-?

tf=4
X1(tf)=2







Rozwiązanie (kont.)

$u = 1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = 1 \end{cases}$$

$$x_1(0) = A, \quad x_1(t_f) = \rightarrow \max$$
$$x_2(0) = B, \quad x_2(t_f) = D$$

$$\begin{cases} x_1(t) = \dfrac{t^2}{2} + C_1 t + C_2 \\ x_2(t) = t + C_1 \end{cases} \Rightarrow C_1 = B, C_2 = A$$

$$x_1(t) = \frac{\left(x_2(t) - B\right)^2}{2} + B\left(x_2(t) - B\right) + A$$

Rozwiązanie(kont.)

$$x_1(t_f)=\frac{\left(x_2(t_f)-B\right)^2}{2}+B\left(x_2(t_f)-B\right)+A$$

$$x_1(t_f)\rightarrow \max$$

$$x_2(t_f)=D$$

$$A=-2, B=-1, D=3$$

$$x_1(t_f)=\frac{\left(D-B\right)^2}{2}+B\left(D-B\right)+A=2$$

$$x_1(t_f)=\frac{t_f^{\;2}}{2}+Bt_f+A$$

$$2=\frac{t_f^{\;2}}{2}-t_f-2$$

```
>> s=solve('x^2/2-x-4=0')
 s =
  4
 -2
```

$$t_f=4$$

$$x\left(t_f\right)=2$$

Obliczenie momentu przełączenia

A,B → E,D

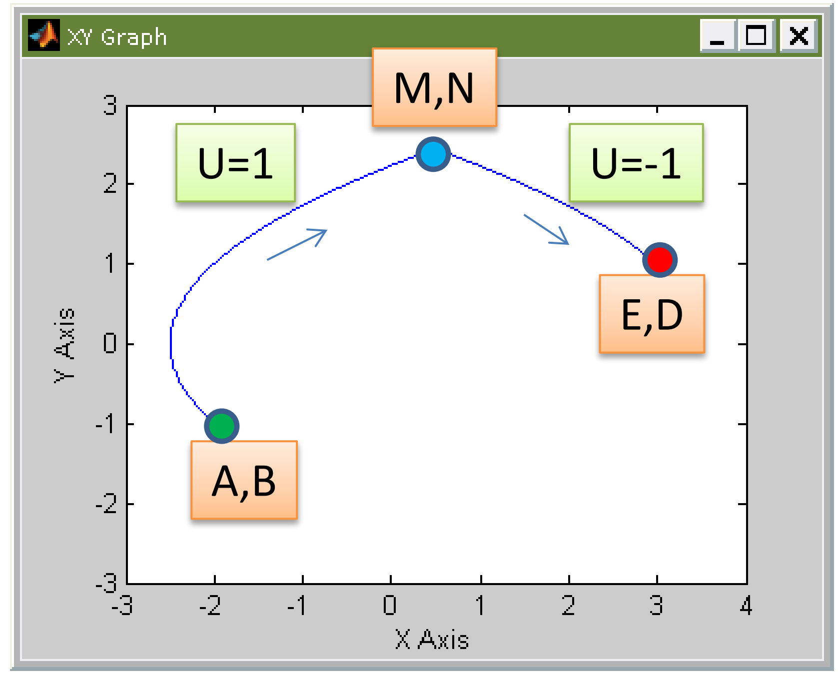
Tmn-?
Tf-?

$u = 1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = 1 \end{cases}$$

$u = -1$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -1 \end{cases}$$



$$x_1(0) = A, \quad x_1(t_f) = E$$
$$x_2(0) = B, \quad x_2(t_f) = D$$

Obliczenie momentu przełączenia

$$x_1(t_{MN}) = M$$

$$x_2(t_{MN}) = N$$

$$u = 1 \quad \begin{cases} x_1(t) = \dfrac{t^2}{2} + C_1 t + C_2 \\ x_2(t) = t + C_1 \end{cases} \Rightarrow C_1 = B, C_2 = A$$

$$t_{MN} = N - B$$

$$x_1(t) = \frac{\left(x_2(t) - B\right)^2}{2} + B\left(x_2(t) - B\right) + A$$

$$t_{MN} : M = \frac{\left(N - B\right)^2}{2} + B\left(N - B\right) + A$$

$$M = \frac{\left(t_{MN}\right)^2}{2} + B\left(t_{MN}\right) + A$$

Obliczenie momentu przełączenia

$$x_1(t_f) = E$$
$$x_2(t_f) = D$$

$$u = -1$$

$$\begin{cases} x_1(t) = -\dfrac{t^2}{2} + C_3 t + C_4 \\ x_2(t) = -t + C_3 \end{cases}$$

$$x_1(t_{NM}) = M$$
$$x_2(t_{NM}) = N$$

$$x_2(t_{MN}) = -t_{MN} + C_3 \Rightarrow C_3 = N + t_{MN}$$

$$x_2(t_f) = -t_f + C_3 \Rightarrow C_3 = D + t_f$$

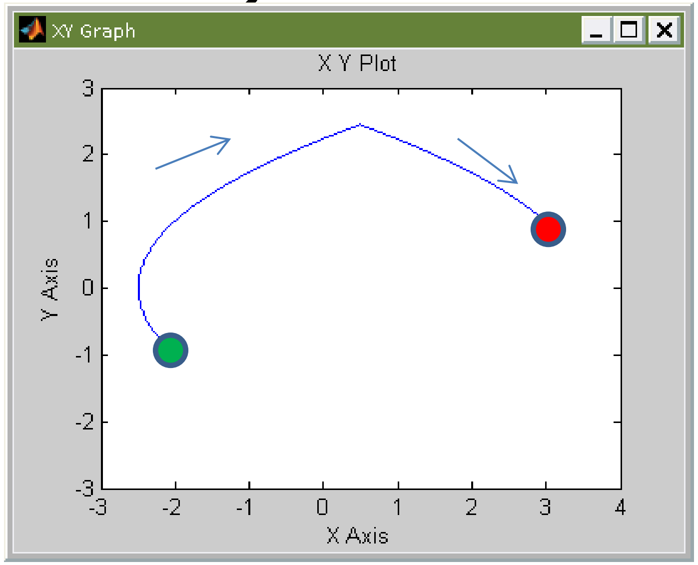
$$N = D + t_f - t_{MN}$$

$$x_1(t_f) = -\frac{t_f^{\,2}}{2} + \left(D + t_f\right)t_f + C_4 \Rightarrow C_4 = E + \frac{t_f^{\,2}}{2} - \left(D + t_f\right)t_f$$

$$x_1(t_{TM}) : \frac{t_{MN}^{\;2}}{2} + Bt_{MN} + A = -\frac{t_{MN}^{\;2}}{2} + \left(D + t_f\right)t_{MN} + E + \frac{t_f^{\,2}}{2} - \left(D + t_f\right)t_f$$

$$x_2(t_{TM}) : t_{MN} + B = -t_{MN} + D + t_f$$

Obliczenie momentu przełączenia



$A := -2 \qquad B := -1 \qquad D := -1 \qquad E := 3$

$tmn := 2 \qquad tf := 4$

Given

$$\left[\frac{tmn^2}{2} + B \cdot tmn + A = \frac{-tmn^2}{2} + (D + tf) \cdot tmn + E + \frac{tf^2}{2} - (D + tf)tf\right]$$

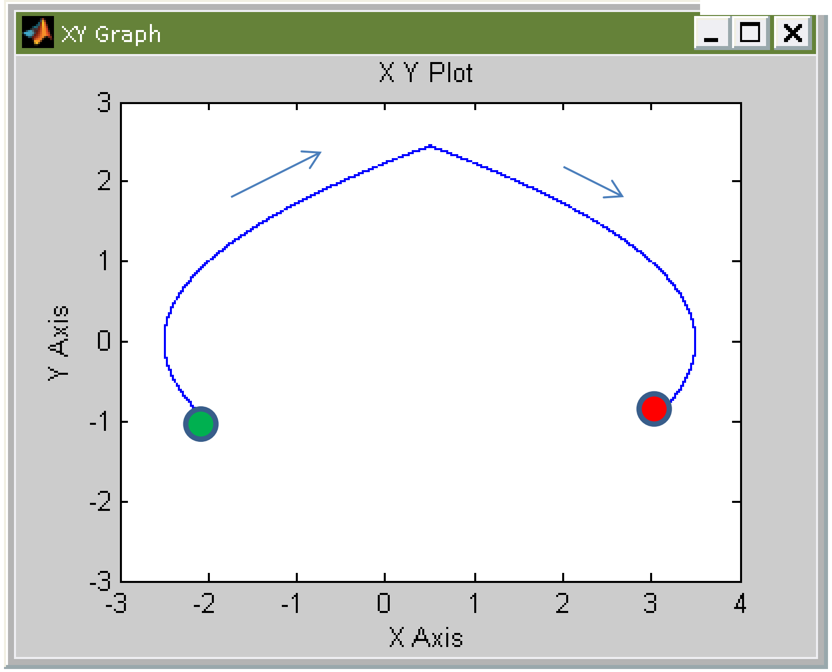
$$tmn + B = -tmn + D + tf$$

$$\mathrm{Find}(tmn, tf) = \begin{pmatrix} 3.449 \\ 6.899 \end{pmatrix}$$

Przykład

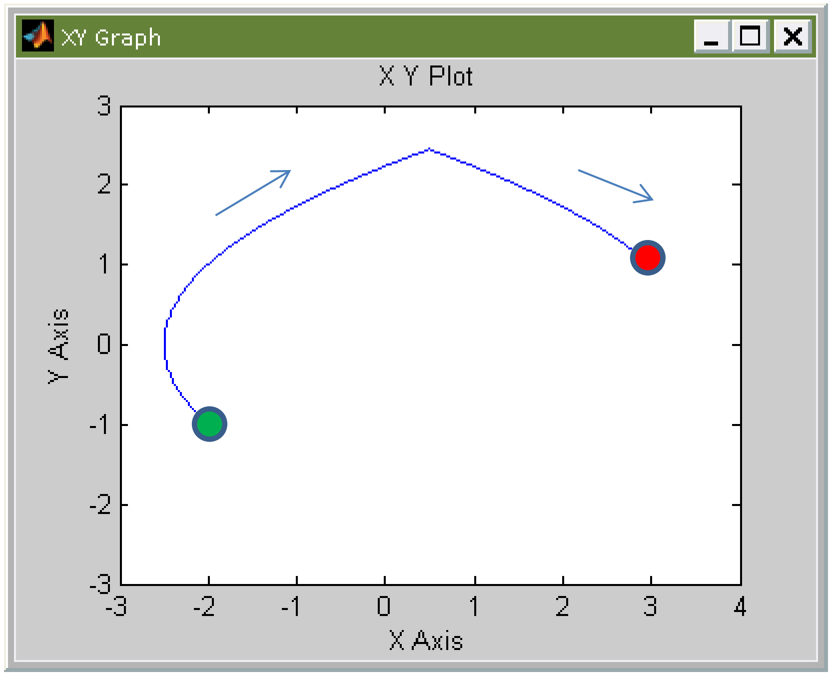
A := −2   B := −1   D := −1   E := 3

$$Find(tmn, tf) = \begin{pmatrix} 3.449 \\ 6.899 \end{pmatrix}$$

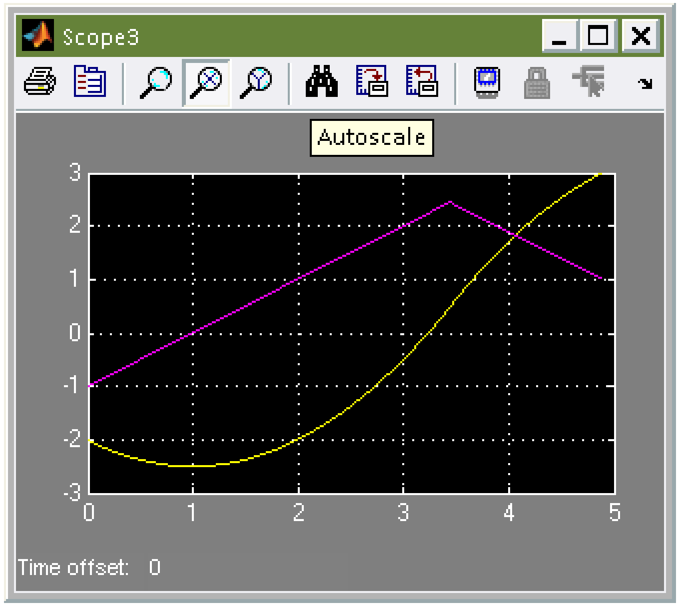


Przykład

$A := -2 \qquad B := -1 \qquad D := 1 \qquad E := 3$



$$\mathrm{Find}(tmn, tf) = \begin{pmatrix} 3.449 \\ 4.899 \end{pmatrix}$$



Badanie

$A := -2 \quad B := -1 \quad D := 1 \quad E := -2$

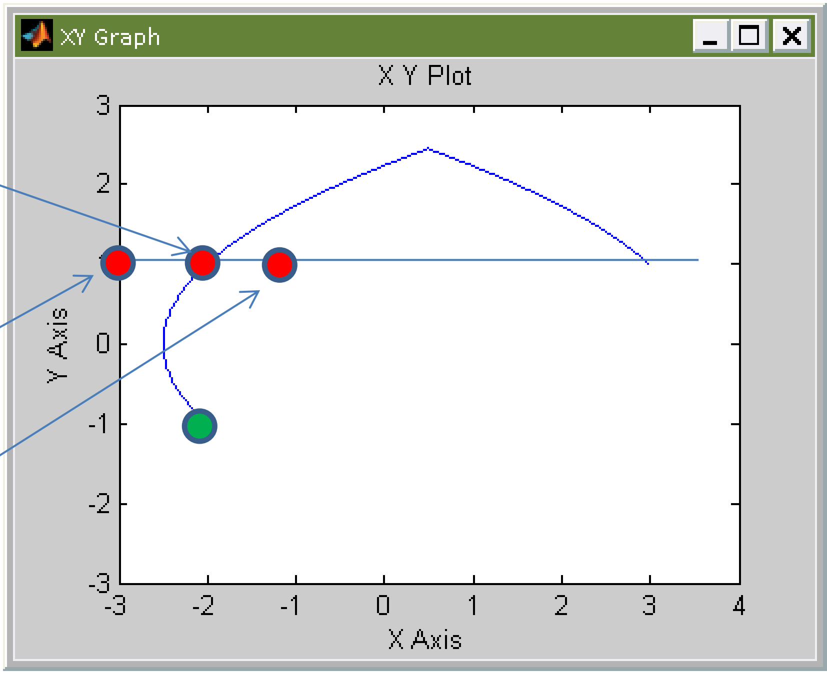
$$\mathrm{Find}(tmn, tf) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$A := -2 \quad B := -1 \quad D := 1 \quad E := -3$

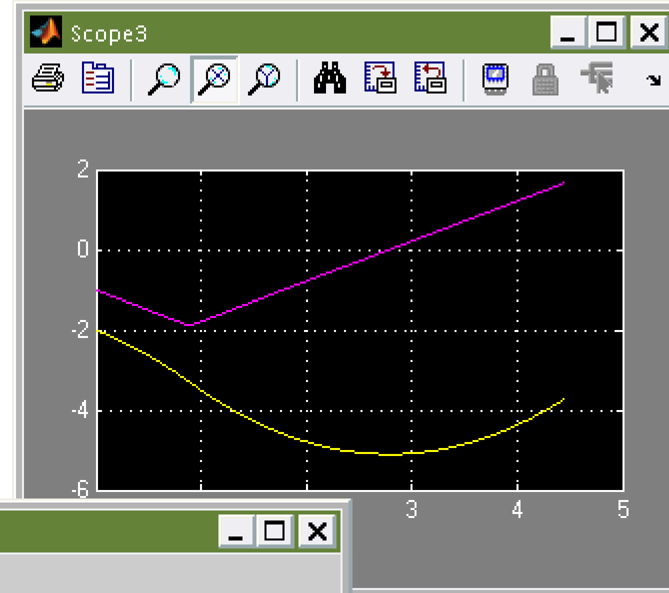
$$\mathrm{Find}(tmn, tf) = \begin{pmatrix} 1 \\ -1.288 \times 10^{-6} \end{pmatrix}$$

$A := -2 \quad B := -1 \quad D := 1 \quad E := -1$

$$\mathrm{Find}(tmn, tf) = \begin{pmatrix} 2.414 \\ 2.828 \end{pmatrix}$$

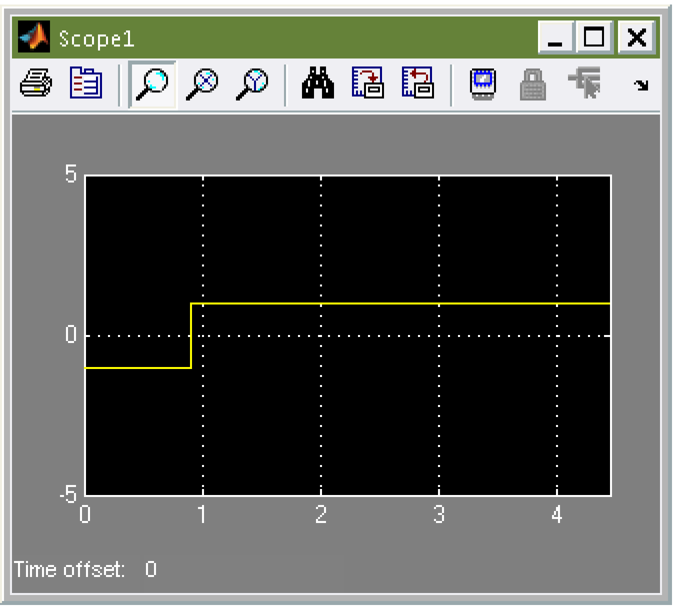


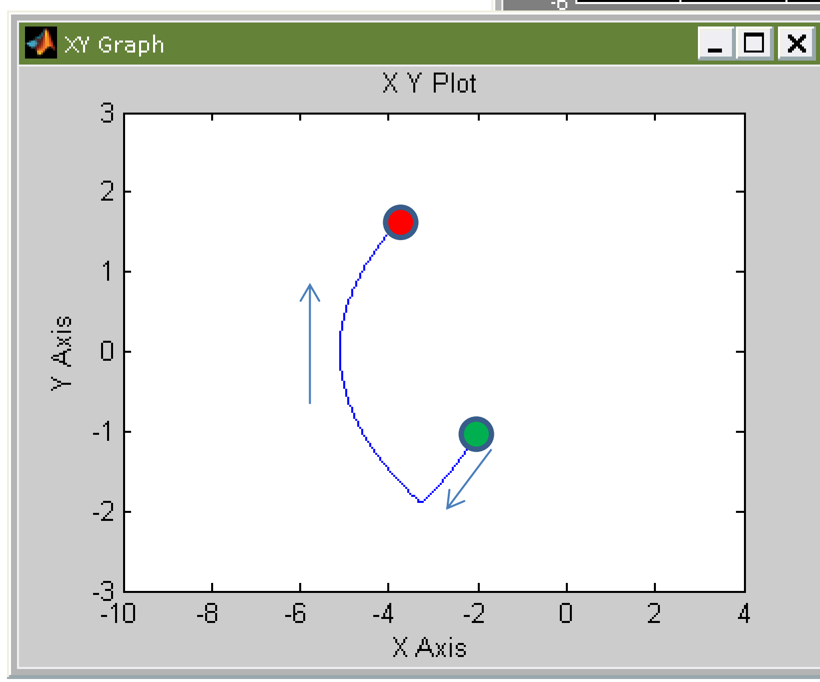
Zmiana kolejności sterowania



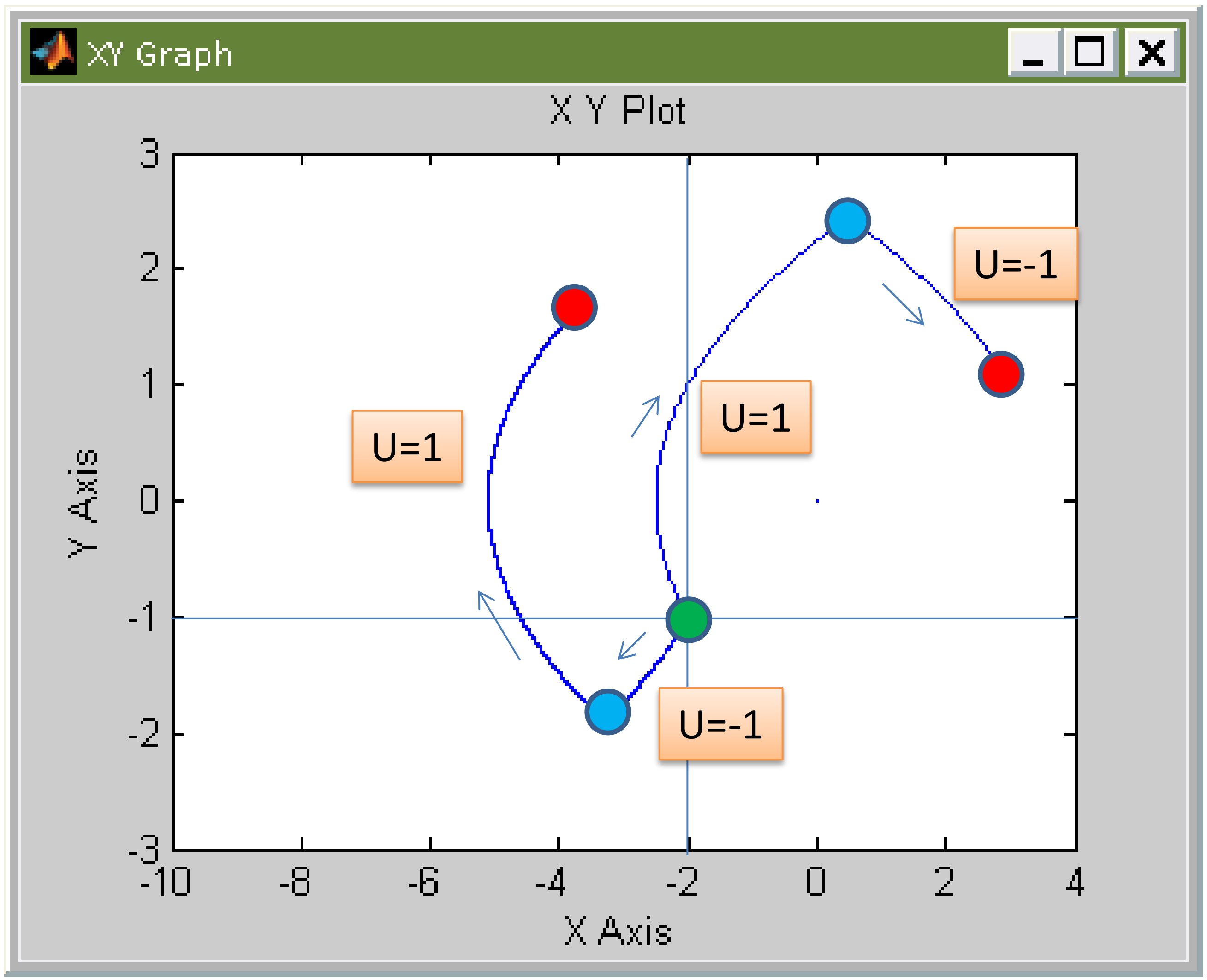
U0=-1 0.899

U1=1 4.449

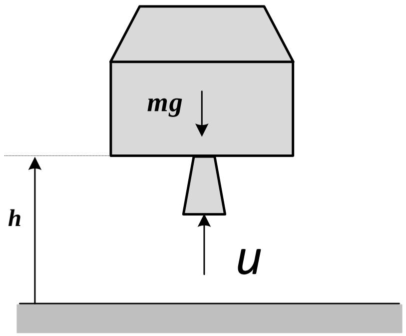




Minimalizacja ilości przełączeń



Lądowanie na Księżycu



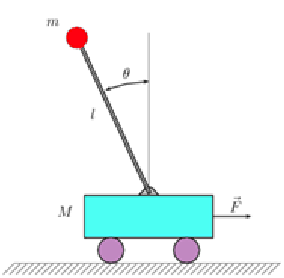
$$\min J(u(\cdot)) = -\int_0^{\tau} u(t)dt$$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \dfrac{u(t)}{m(t)} \\ \dot{m}(t) = -ku(t) \end{cases} \qquad \begin{cases} h(0) = h_0 \\ v(0) = v_0 \\ m(0) = m_0 \end{cases}$$

$$m\ddot{h} = -gm + \alpha$$

$h(t) \geq 0$ and $m(t) \geq 0$

$$u(t) \in A = [0,1]$$



SpaceX successfully launches and lands Starship, May 6, 2021

Lądowanie na Księżycu (cd.)

$$H(x,\lambda,u) = \mathbf{f}\cdot\lambda + L$$

$$= v(t)\lambda_h - g\lambda_v + \frac{u(t)}{m(t)}\lambda_v - ku(t)\lambda_m - u(t)$$

$$\begin{cases} \dot{\lambda}_h = -\dfrac{\partial H}{\partial h} = 0 \\ \dot{\lambda}_v = -\dfrac{\partial H}{\partial v} = \lambda_h \\ \dot{\lambda}_m = -\dfrac{\partial H}{\partial m} = \dfrac{u(t)}{m(t)^2}\lambda_v \end{cases}$$

Lądowanie na Księżycu (cd.)

$$u(t) \in A = [0,1]$$

$$H(\mathbf{x}(t), \boldsymbol{\lambda}(t), \mathbf{u}(t)) = \max_{0 \le a \le 1} H(\mathbf{x}(t), \boldsymbol{\lambda}(t), a)$$

$$= v(t)\lambda_h(t) - g\lambda_v(t) + \max_{0 \le a \le 1}\left[a\left(\frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 \right)\right]$$

$$= \begin{cases} v(t)\lambda_h(t) - g\lambda_v(\mathrm{t}) + \frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 & \text{if} \quad \frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 > 0 \\ v(t)\lambda_h(t) - g\lambda_v(\mathrm{t}) & \text{if} \quad \frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 < 0 \end{cases}$$

$$u(t) = \begin{cases} 1 & \text{if} \quad \frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 > 0 \\ 0 & \text{if} \quad \frac{1}{m(t)}\lambda_v(t) - k\lambda_m(t) - 1 < 0 \end{cases}$$

Lądowanie na Księżycu (cd.)

$$u(t) = \begin{cases} 0 & \text{if} \quad 0 \le \mathrm{t} \le \mathrm{t}_s \\ 1 & \text{if} \quad \mathrm{t}_s \le \mathrm{t} \le \tau \end{cases}$$

$$\begin{cases} h(\tau) = 0 \\ v(\tau) = 0 \\ m(t_s) = m_0 \end{cases}$$

$$u(t) = 0$$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g \\ \dot{m}(t) = 0 \end{cases}$$

$$\begin{cases} h_{free}(t) = -\frac{1}{2}gt^2 + tv_0 + h_0 \\ v_{free}(t) = -gt + v_0 \\ m_{free}(t) = m_0 \end{cases}$$

$$m(t_s) = m(0) = m_0$$

$$u(t) = 1$$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{1}{m(t)} \\ \dot{m}(t) = -k \end{cases}$$

Lądowanie na Księżycu (cd.)

$$\begin{cases} h_{powered}(t) = \tau g t - \frac{1}{2} g t^2 - \frac{\tau - t}{k} - \frac{1}{k^2} \log\left(\frac{m_0 - k(\tau - t_s)}{m_0 - k(t - t_s)} \right) (m_0 - k(t - t_s)) - \frac{1}{2} g \tau^2 \\ v_{powered}(t) = g(t - \tau) + \frac{1}{k} \log\left(\frac{m_0 + k(t_s - \tau)}{m_0 + k(t_s - t)} \right) \\ m_{powered}(t) = m_0 + k(t_s - t) \end{cases}$$

Lądowanie na Księżycu (cd.)

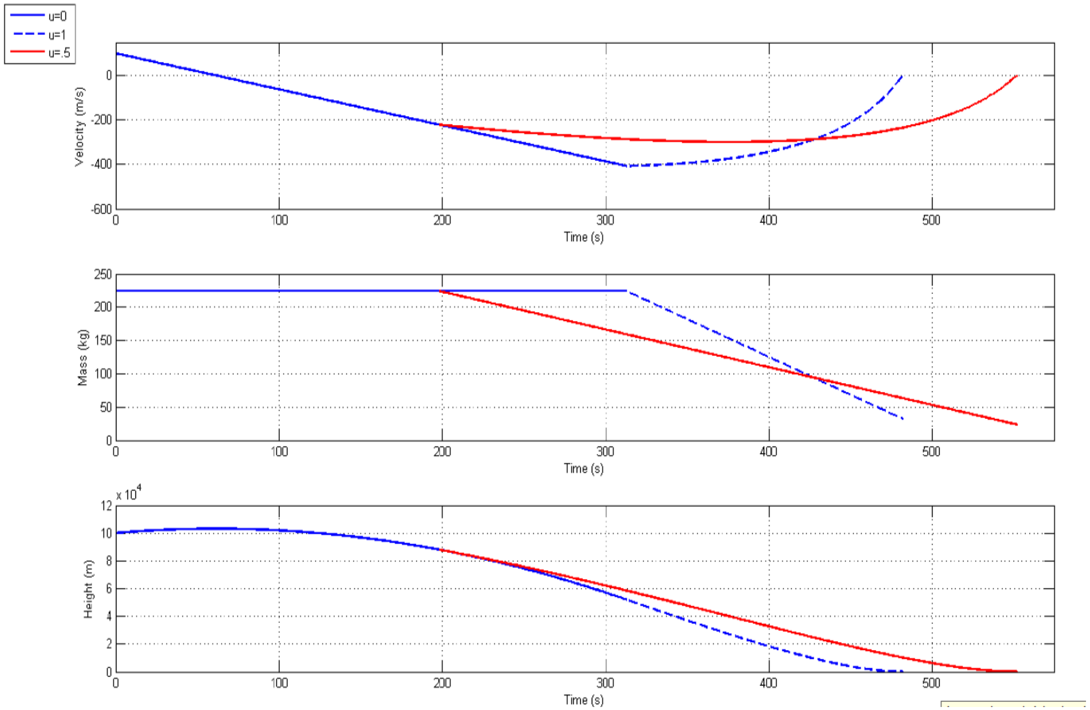
W punkcie startu silnika ts

$$\begin{cases} h(t_s) = -\frac{1}{2} g t_s^{\ 2} + t_s v_0 + h_0 \\ v(t_s) = -g t_s + v_0 \\ h(t_s) = -\frac{1}{2} g (t_s - \tau)^2 + \frac{t_s - \tau}{k} - \frac{m_0}{k^2} \log\left(\frac{m_0 - k(t_s - \tau)}{m_0}\right) \\ v(t_s) = g(t_s - \tau) + \frac{1}{k} \log\left(\frac{m_0 + k(t_s - \tau)}{m_0}\right) \end{cases}$$

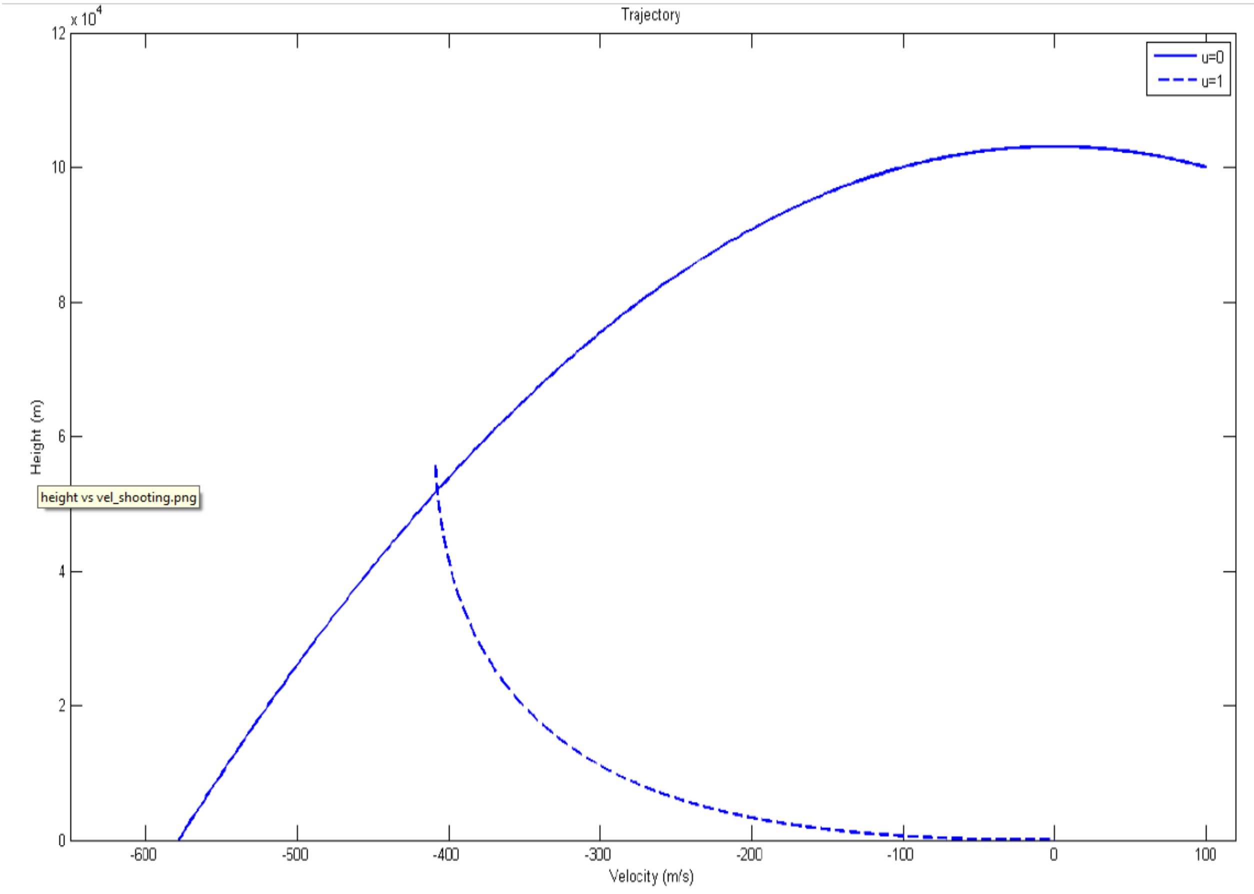
$$m(\tau) = m_0 + k(t_s - \tau)$$

Lądowanie na Księżycu (cd.)

$u_{\max}$	m_0	m_{fuel}	k	v_0	h_0	g_{moon}
400N	224kg	204kg	2.833×10^{-3}kg/s	100m/s	100000m	1.622m/s^2



Lądowanie na Księżyc (cd.)



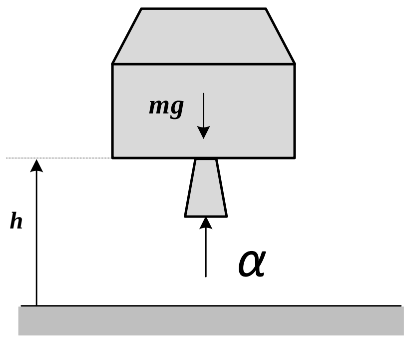
Lądowanie na Księżycu

$$m\ddot{h} = -gm + \alpha,$$

$$h(t) \geq 0, m(t) \geq 0$$

$$0 \leq \alpha(t) \leq 1$$

g=1,623 m/s²



$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t), \end{cases}$$

$$P[\alpha(\cdot)] = -\int_0^\tau \alpha(t)\,dt.$$

$$\begin{cases} h(0) = h_0 > 0 \\ v(0) = v_0 \\ m(0) = m_0 > 0 \end{cases}$$

Lądowanie na Księżycu (cd.)

$$\mathbf{x}(t) = \begin{pmatrix} h(t) \\ v(t) \\ m(t) \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} v \\ -g + a/m \\ -ka \end{pmatrix}$$

$$\begin{aligned} H(x, p, a) &= \mathbf{f} \cdot p + r \\ &= (v, -g + a/m, -ka) \cdot (p_1, p_2, p_3) - a \\ &= -a + p_1 v + p_2 \left(-g + \frac{a}{m}\right) + p_3(-ka) \end{aligned}$$

$$H_{x_1} = H_h = 0, \ H_{x_2} = H_v = p_1, \ H_{x_3} = H_m = -\frac{p_2 a}{m^2}$$

$$\begin{cases} \dot{p}^1(t) = 0 \\ \dot{p}^2(t) = -p^1(t) \\ \dot{p}^3(t) = \frac{p^2(t)\alpha(t)}{m(t)^2} \end{cases}$$

Lądowanie na Księżycu (cd.)

$$
\begin{aligned}
H(\mathbf{x}(t), \mathbf{p}(t), \alpha(t)) &= \max_{0 \le a \le 1} H(\mathbf{x}(t), \mathbf{p}(t), a) \\
&= \max_{0 \le a \le 1} \left\{ -a + p^1(t)v(t) + p^2(t)\left[-g + \frac{a}{m(t)}\right] + p^3(t)(-ka) \right\} \\
&= p^1(t)v(t) - p^2(t)g + \max_{0 \le a \le 1} \left\{ a\left(-1 + \frac{p^2(t)}{m(t)} - kp^3(t)\right) \right\}.
\end{aligned}
$$

$$
\alpha(t) = \begin{cases} 1 & \text{if} \quad 1 - \frac{p^2(t)}{m(t)} + kp^3(t) < 0 \\ 0 & \text{if} \quad 1 - \frac{p^2(t)}{m(t)} + kp^3(t) > 0 \end{cases}
$$

Lądowanie na Księżycu (cd.)

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le t^* \\ 1 & \text{for} \quad t^* \le t \le \tau \end{cases}$$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{\alpha(t)}{m(t)} \\ \dot{m}(t) = -k\alpha(t), \end{cases}$$

$$\alpha \equiv 0$$

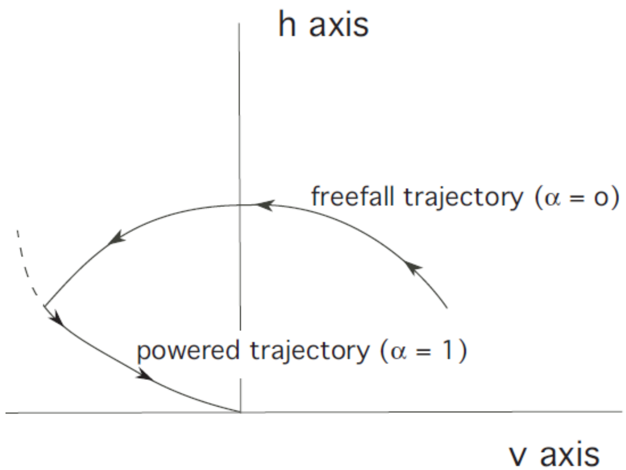
$$\begin{cases} \dot{h} = v \\ \dot{v} = -g \\ \dot{m} = 0; \end{cases}$$

$$\begin{cases} m(t) \equiv m_0 \\ v(t) = -gt + v_0 \\ h(t) = -\frac{1}{2}gt^2 + tv_0 + h_0 \end{cases}$$

Lądowanie na Księżycu (cd.)

$$h(t) = h_0 - \frac{1}{2g}(v^2(t) - v_0^2) \qquad (0 \leq t \leq t^*)$$

$$h = h_0 - \frac{1}{2g}(v^2 - v_0^2)$$



Lądowanie na Księżycu (cd.)

$$h(\tau) = v(\tau) = 0$$

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad 0 \le t \le t^* \\ 1 & \text{for} \quad t^* \le t \le \tau \end{cases}$$

$$t^* \le t \le \tau$$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{1}{m(t)} \\ \dot{m}(t) = -k \end{cases}$$

$$h(\tau) = 0, \; v(\tau) = 0, \; m(t^*) = m_0$$

$$\begin{cases} m(t) = m_0 + k(t^* - t) \\ v(t) = g(\tau - t) + \frac{1}{k} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)} \right] \\ h(t) = \text{ complicated formula.} \end{cases}$$

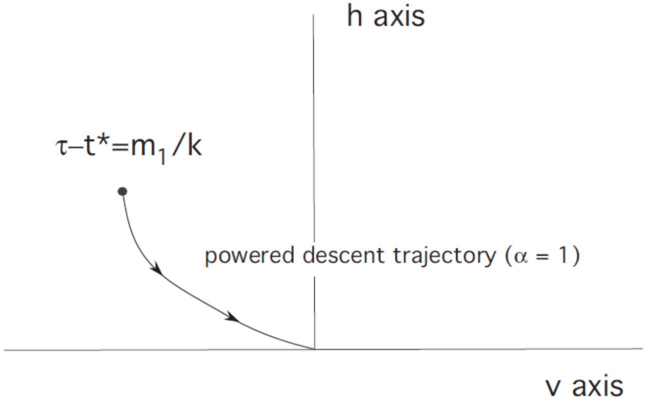
Lądowanie na Księżycu (cd.)

$$t = t^*$$

$$\begin{cases} m(t^*) = m_0 \\ v(t^*) = g(\tau - t^*) + \frac{1}{k}\log\left[\frac{m_0 + k(t^* - \tau)}{m_0}\right] \\ h(t^*) = -\frac{g(t^* - \tau)^2}{2} - \frac{m_0}{k^2}\log\left[\frac{m_0 + k(t^* - \tau)}{m_0}\right] + \frac{t^* - \tau}{k} \end{cases}$$

$k(\tau - t^*) \leq m_1$ Start z powrotem potrzebuje paliwa m1

$$0 \leq \tau - t^* \leq \frac{m_1}{k}$$



Lądowanie na Księżycu (cd.)

$$p^1(0) = \lambda_1,\ p^2(0) = \lambda_2,\ p^3(0) = \lambda_3.$$

$$\begin{cases} p^1(t) \equiv \lambda_1 & (0 \le t \le \tau) \\ p^2(t) = \lambda_2 - \lambda_1 t & (0 \le t \le \tau) \\ p^3(t) = \begin{cases} \lambda_3 & (0 \le t \le t^*) \\ \lambda_3 + \int_{t^*}^{t} \frac{\lambda_2 - \lambda_1 s}{(m_0 + k(t^* - s))^2}\, ds & (t^* \le t \le \tau). \end{cases} \end{cases}$$

$$r(t) := 1 - \frac{p^2(t)}{m(t)} + p^3(t)k$$

$$\dot{r} = -\frac{\dot{p}^2}{m} + \frac{p^2\dot{m}}{m^2} + \dot{p}^3 k = \frac{\lambda_1}{m} + \frac{p^2}{m^2}(-k\alpha) + \left(\frac{p^2\alpha}{m^2}\right)k = \frac{\lambda_1}{m(t)}$$

Lądowanie na Księżycu (cd.)

$\lambda_1 < 0$ $$r(t^*) = 1 - \frac{(\lambda_2 - \lambda_1 t^*)}{m_0} + \lambda_3 k$$

$$r > 0 \text{ on } [0, t^*).$$

$$r < 0 \text{ on } (t^*, \tau].$$

$$r(t^*) = 0.$$

$$\alpha(t) = \begin{cases} 1 & \text{if } r(t) < 0 \\ 0 & \text{if } r(t) > 0 \end{cases}$$