Metody numeryczne II. Rozwiązywanie równań różniczkowych.

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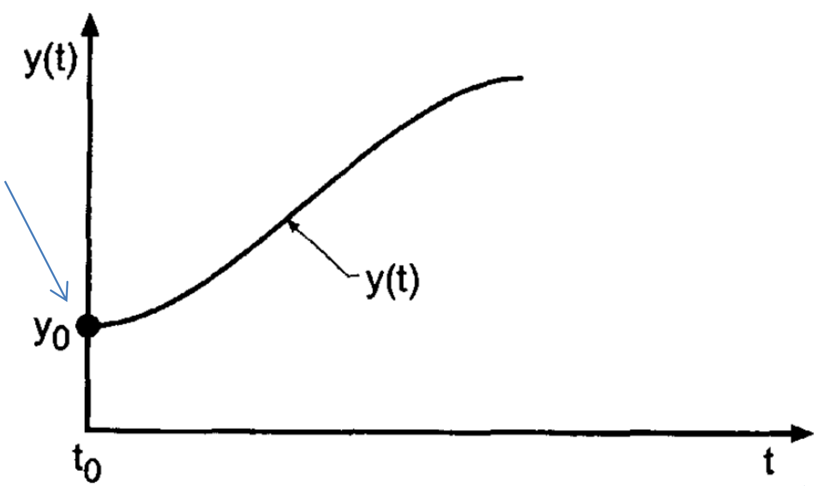
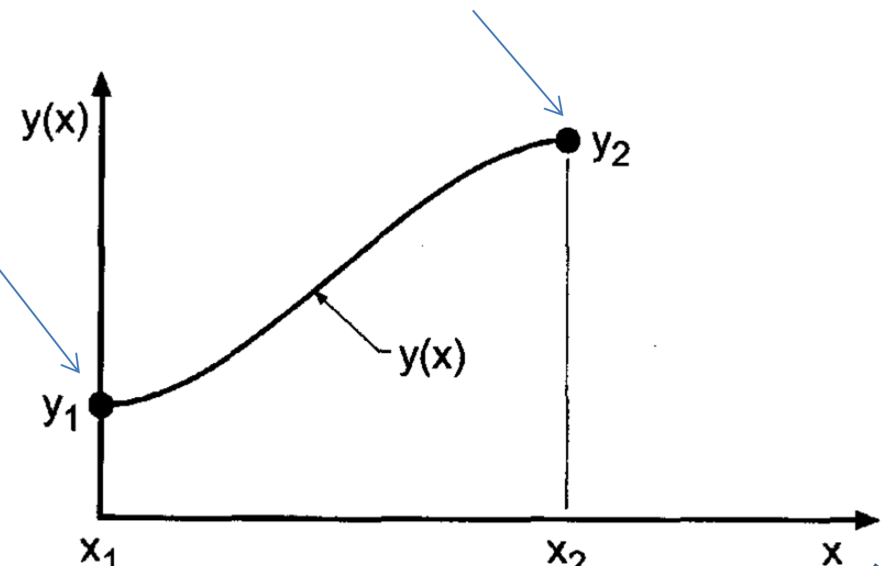
Zagadnienia

$$\frac{dy}{dt} = f(t, y)$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \mathrm{L} \; + a_2 y'' + a_1 y' + a_0 y = F(t)$$





Rodzaje problemów RR

Initial Value Problems (IVP): Zagadnienie Cauchy'ego (zagadnienie początkowe)

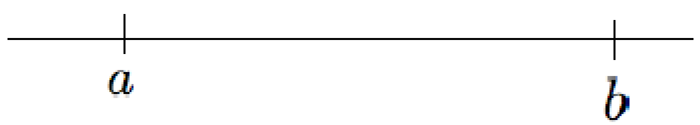
$$\frac{du}{dt} = f(t, u(t))\ ,\ \ t \geq t_0$$

$$u(t_0) = u_0$$

Boundary Value Problems (BVP): Zagadnienie brzegowe

$$y''(x) = f(x, y, y')\ ,\ \ a \leq x \leq b$$

$$y(a) = \alpha\ ,\ \ y(b) = \beta$$



Układ równań różniczkowych pierwszego rzędu

$$
\begin{aligned}
y_1' &= f_1(x, y_1, \ldots, y_n) & \qquad y_1(x_0) &= y_{10} \\
&\cdots & &\cdots \\
y_n' &= f_n(x, y_1, \ldots, y_n) & \qquad y_n(x_0) &= y_{n0}.
\end{aligned}
$$

Najstarsze równania różniczkowe

Newton (1671), Problema II, Solutio Casus II, Ex. I

$$y' = 1 - 3x + y + x^2 + xy$$

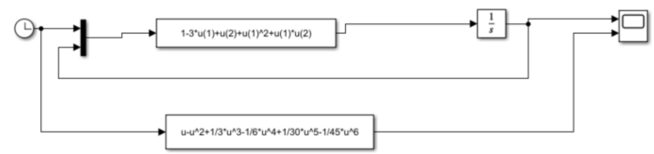
NewtonDifEq.slx

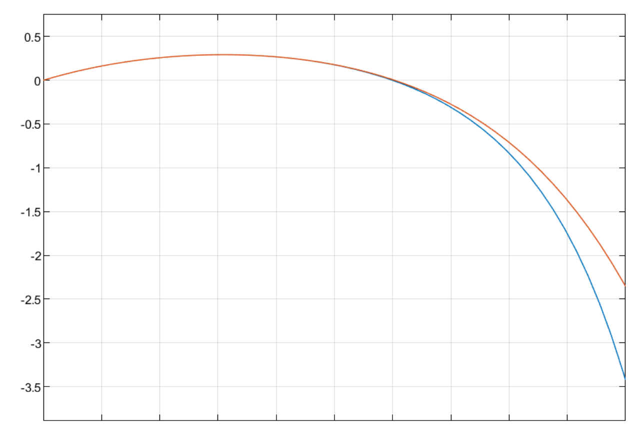
EXEMPL. I.

Sit Æquatio $\frac{\dot{y}}{\dot{x}} = 1 - 3x + y + xx + xy$, cujus Terminos $1 - 3x + xx$ non affectos *Relatâ* Quantitate diſpoſitos vides in lateralem Seriem primo loco, & reliquos y & xy in ſiniſtrâ Columnâ.

	$+1 - 3x + xx$
$+y$	$* + x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$; &c.
$+xy$	$* \quad x + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$; &c.
Aggreg.	$+1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$; &c.
$y =$	$+x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$; &c.

Nunc





$$y = x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6;\ \&c$$

Najstarsze równania różniczkowe (cd)

Odwrotne problemy styczne
Debeaune 1638 in a letter to Descartes
1674 by “Claudius Perraltus Medicus Parisinus” to Leibniz
Rozwiązanie:
Leibniz (1684) and the Jacob Bernoulli (1690)

ścieżka srebrnego zegarka kieszonkowego

Leibniz.slx

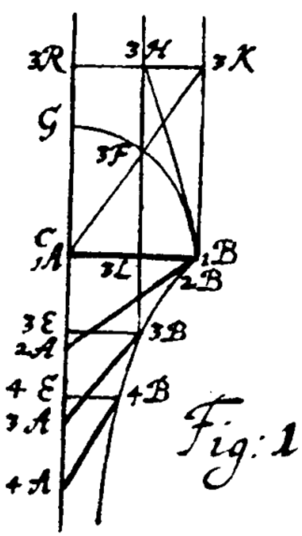
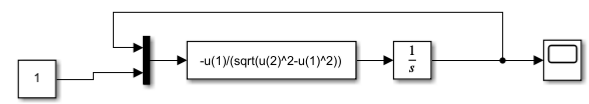
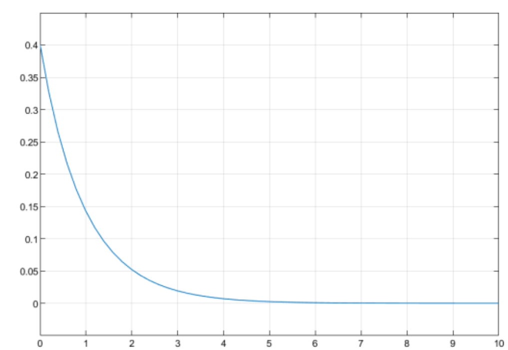


Fig. 2.2. Illustration from Leibniz (1693)

$$y' = -\frac{y}{\sqrt{a^2 - y^2}}$$





Najstarsze równania różniczkowe (cd)

Alexis Clairaut

Ruch prostokątnego klina

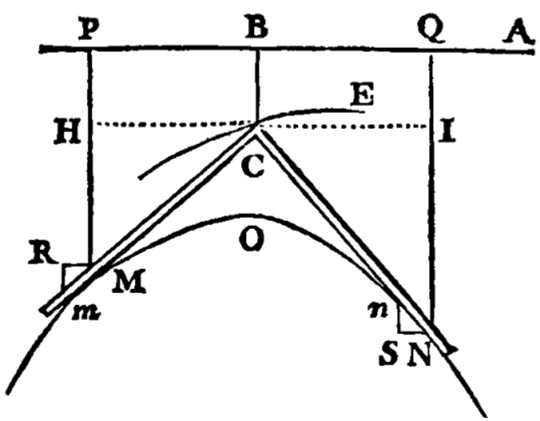


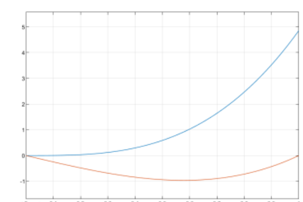
Fig. 2.5. Illustration from Clairaut (1734)

$$y - xy' + f(y') = 0$$

$$y = Cx - f(C)$$

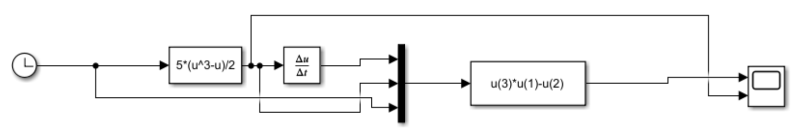
$$f(C) = 5(C^3 - C)/2$$

Alexis Claude de Clairault (1713 - 1765) – francuski matematyk.



$$x(p) = f'(p)$$

$$y(p) = pf'(p) - f(p)$$



Clairaut.slx

Rozwiązanie analityczne

$$y' = f(x)g(y) \Longrightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx + C$$

$$y' = f(x)y \Longrightarrow \begin{aligned} &y(x) = CR(x), \\ &R(x) = \exp\left(\int f(x)\, dx\right) \end{aligned}$$

$$\begin{aligned} &y' = f(x)y + g(x) \\ &y(x) = c(x)R(x) \end{aligned} \Longrightarrow y(x) = R(x)\left(\int_{x_0}^{x} \frac{g(s)}{R(s)}\, ds + C\right)$$

$$P(x,y) + Q(x,y)y' = 0 \Longrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Równania różniczkowe drugiego rzędu

$$y'' = a(x)y' + b(x)y \quad \Longrightarrow \quad \boxed{p^2 + p' = a(x)p + b(x)}$$

substitution (Riccati 1723, Euler 1728)

$$y = \exp\Big(\int p(x)\,dx\Big)$$

$$y' = p \cdot \exp\Big(\int p(x)\,dx\Big), \qquad y'' = (p^2 + p') \cdot \exp\Big(\int p(x)\,dx\Big)$$

Równania różniczkowe drugiego rzędu

$$y'' = f(x, y') \Longrightarrow v' = f(x, v) \qquad v' = f(x, v)$$

$$y'' = f(y, y') \Longrightarrow y'' = p'p = f(y, p)$$

$$y' = p(y)$$

Równania różniczkowe liniowe

$$\mathcal{L}(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

$$\Rightarrow \quad \boxed{y(x) = C_1 u_1(x) + \ldots + C_n u_n(x)}$$

funkcja jednorodna a_i

$$\boxed{y^{(n)}(x) = 0} \quad \Rightarrow \quad y^{(n-1)}(x) = C_1$$

$$y^{(n-2)}(x) = C_1 x + C_2$$

$$\boxed{y(x) = C_1 x^{n-1} + C_2 x^{n-2} + \ldots + C_n}$$

Równanie charakterystyczne

$$\boxed{y^{(n)} + A_{n-1}y^{(n-1)} + \ldots + A_0 y = 0}$$

$$y = \exp\left(\int p(x)\, dx\right) \Longrightarrow y = e^{px}$$ dla stałych A_i

$$\boxed{p^n + A_{n-1}p^{n-1} + \ldots + A_0 = 0}$$

Rozwiązania $p_1, \ldots, p_n$

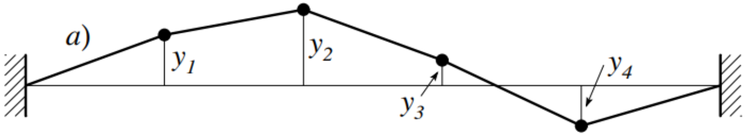
$$\boxed{y(x) = C_1 e^{p_1 x} + \ldots + C_n e^{p_n x}}$$

Układy równań

Ruch planet, Euler (1747)

$$m\frac{d^2x}{dt^2} = X, \qquad m\frac{d^2y}{dt^2} = Y, \qquad m\frac{d^2z}{dt^2} = Z,$$

Wibrująca struna i rozchodzenie się dźwięku, Brook Taylor ,1715, Johann Bernoulli, 1727, Daniel Bernoulli, 1732 and by Lagrange, 1762



$$\begin{aligned} y_1'' &= K^2(-2y_1 + y_2) \\ y_2'' &= K^2(y_1 - 2y_2 + y_3) \\ &\cdots \\ y_n'' &= K^2(y_{n-1} - 2y_n). \end{aligned}$$

d'Alembert 1747

$$y_{i-1} - 2y_i + y_{i+1} \approx c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Rozwiązanie

$$y_i'' = \sum_{j=1}^{n} a_{ij} y_j, \qquad i = 1, \ldots, n,$$

$$y_i = c_i e^{pt} \quad \Rightarrow \quad p^2 c_i = \sum_{j=1}^{n} a_{ij} c_j$$

$c = (c_1, \ldots, c_n)^T$ wektory własne

p^2 wartości własne

Pierwsze wspomnienie

Układy równań (cd)

przewodnictwo ciepła,
Biot 1804, Fourier 1807

Energia, którą cząstka przekazuje swoim sąsiadom, jest proporcjonalna do różnicy ich temperatur, tj.

$y_{i-1} - y_i$ w lewo i $y_{i+1} - y_i$ w prawo

$$y_i' = K^2(y_{i-1} - 2y_i + y_{i+1})$$

$$y_i' = \sum^{n} a_{ij} y_i, \qquad i = 1, \ldots, n, \qquad y_i = c_i e^{pt}$$

$$pc_i = \sum_{j=1}^{n} a_{ij} c_j, \qquad i = 1, \ldots, n.$$

$$y_j(t) = \sum_{k=1}^{n} a_k \sin \frac{jk\pi}{n+1} \exp(-w_k t), \qquad w_k = 4K^2 \left(\sin \frac{\pi k}{2n+2} \right)^2.$$

W przedziale

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Mechanika Lagrange'a

$$\mathcal{L} = T - U$$

$$T = m\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2}$$

$$\frac{\partial U}{\partial x} = -X, \qquad \frac{\partial U}{\partial y} = -Y, \qquad \frac{\partial U}{\partial z} = -Z$$

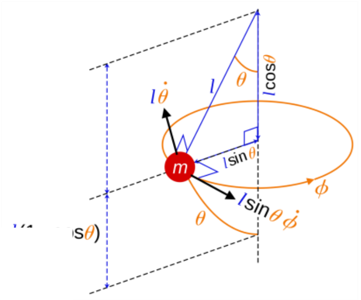
$$\int_{t_0}^{t_1} \mathcal{L}\, dt = \min$$

Wahadło sferyczne

$$\ell = 1 \qquad x = \sin\theta\cos\varphi$$

$$y = \sin\theta\sin\varphi$$

$$z = -\cos\theta.$$



$$m = g = 1$$

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\varphi}^2\right)$$

$$U = z = -\cos\theta$$

$$\mathcal{L}_\theta - \frac{d}{dt}\left(\mathcal{L}_{\dot{\theta}}\right) = -\sin\theta + \sin\theta\cos\theta \cdot \dot{\varphi}^2 - \ddot{\theta} = 0$$

$$\mathcal{L}_\varphi - \frac{d}{dt}\left(\mathcal{L}_{\dot{\varphi}}\right) = -\sin^2\theta \cdot \ddot{\varphi} - 2\sin\theta\cos\theta \cdot \dot{\varphi} \cdot \dot{\theta} = 0.$$

Metoda Eulera, 1768

$$\boxed{y' = f(x,y), \qquad y(x_0) = y_0, \qquad y(X) =?}$$

$$x_0, x_1, \ldots, x_{n-1}, x_n = X$$

$$y_1 - y_0 = (x_1 - x_0) f(x_0, y_0)$$

$$y_2 - y_1 = (x_2 - x_1) f(x_1, y_1)$$

$$\cdots$$

$$y_n - y_{n-1} = (x_n - x_{n-1}) f(x_{n-1}, y_{n-1}).$$

$$h = (h_0, h_1, \ldots, h_{n-1}) \qquad h_i = x_{i+1} - x_i$$

Wielokąt Eulera

$$y_h(x) = y_i + (x - x_i) f(x_i, y_i) \qquad \text{for} \quad x_i \le x \le x_{i+1}$$

$$\boxed{|y_h(x) - y_0| \le A \cdot |x - x_0|} \qquad |f| \le A$$

Metoda Eulera (cd)

$$\boxed{\begin{array}{lll} y_1' = f_1(x, y_1, \ldots, y_n), & y_1(x_0) = y_{10}, & y_1(X) = ? \\ \quad \cdots & \quad \cdots & \quad \cdots \\ y_n' = f_n(x, y_1, \ldots, y_n), & y_n(x_0) = y_{n0}, & y_n(X) = ? \end{array}}$$

$$y_{k,i+1} = y_{ki} + (x_{i+1} - x_i) \cdot f_k(x_i, y_{1i}, \ldots, y_{ni})$$

$$k = 1, \ldots, n \text{ and } i = 0, 1, 2, \ldots$$

$$|y_{ki} - y_{k0}| \leq A_k |x_i - x_0| \qquad \text{if} \qquad |f_k(x, y_1, \ldots, y_n)| \leq A_k$$

Układy liniowych równań różniczkowych

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + \ldots + b_0(x)y = g(x), \qquad b_i(x) = a_i(x)/a_n(x)$$

$$g(x) = f(x)/a_n(x) \qquad y = y_1\,,\; y' = y_2, \ldots, y^{(n-1)} = y_n$$

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & \ldots & 1 \\ -b_0(x) & -b_1(x) & \ldots & -b_{n-1}(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix}$$

$$\boxed{y' = A(x)y + f(x)}$$

$$A(x) = \big(a_{ij}(x)\big), \qquad f(x) = \big(f_i(x)\big), \qquad i, j = 1, \ldots, n$$

Układy ze stałymi współczynnikami

$$y_i' = f_i(y_1, \ldots, y_n) \qquad\qquad y_i'' = f_i(y_1, \ldots, y_n)$$

$$f_i(0, \ldots, 0) = 0$$

Szereg Taylora

$$y_i' = \sum_{k=1}^{n} \frac{\partial f_i}{\partial y_k}(0) y_k \qquad\qquad y_i'' = \sum_{k=1}^{n} \frac{\partial f_i}{\partial y_k}(0) y_k$$

$$y' = Ay \qquad\qquad y'' = Ay$$

$$y(x) = v \cdot e^{\lambda x}$$

$$Av = \lambda v \qquad\qquad Av = \lambda^2 v$$

Układy ze stałymi współczynnikami (cd)

$$\chi_A(\lambda) := \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) = 0.$$

$$A\Big(v_1, v_2, \dots, v_n\Big) = \Big(v_1, v_2, \dots, v_n\Big)\,\mathrm{diag}\Big(\lambda_1, \lambda_2, \dots, \lambda_n\Big)$$

$$T^{-1}AT = \mathrm{diag}\Big(\lambda_1, \lambda_2, \dots, \lambda_n\Big)$$

$$y(x) = Tz(x), \qquad y'(x) = Tz'(x)$$

$$z'(x) = \mathrm{diag}\Big(\lambda_1, \lambda_2, \dots, \lambda_n\Big) z(x)$$

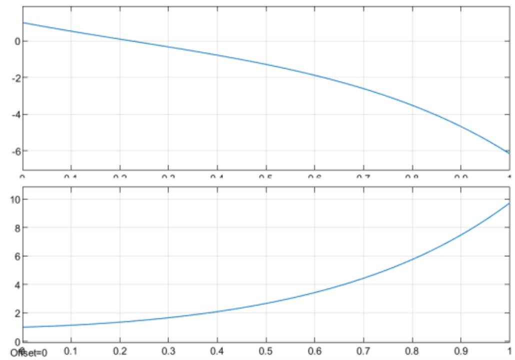
$$\boxed{y(x) = T\,\mathrm{diag}\Big(\exp(\lambda_1 x), \exp(\lambda_2 x), \dots, \exp(\lambda_n x)\Big)T^{-1}y_0.}$$

Przykład

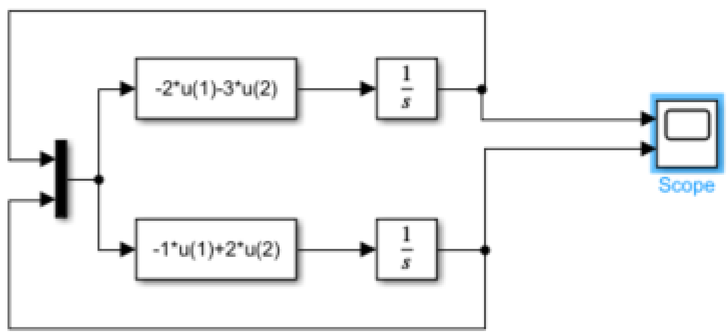
$$\begin{cases} \dot y_1 = -2y_1 - 3y_2 \\ \dot y_2 = -1y_1 - 2y_2 \end{cases}$$

$$\frac{\partial f_1}{\partial y_1} = -2 \qquad \frac{\partial f_1}{\partial y_2} = -3$$

$$\frac{\partial f_2}{\partial y_1} = -1 \qquad \frac{\partial f_2}{\partial y_2} = 2$$

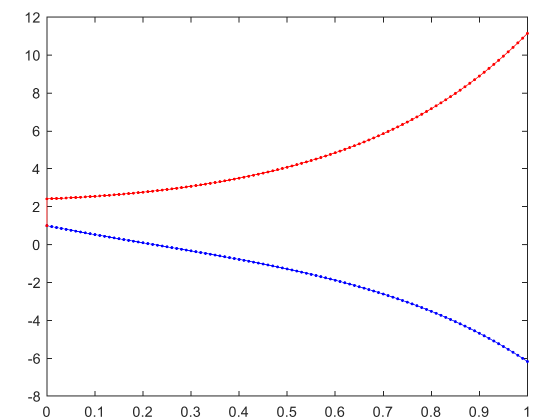



```
>> A

A =

    -2    -3
    -1     2

>> [v l]=eig(A)

v =

        -0.977608773427834          0.542476806985263
        -0.210430715716423         -0.840070779091306


l =

         -2.64575131106459                          0
                         0          2.64575131106459
```




```
A=[-2 -3;-1 2];
[T L]=eig(A);
y0=[1 1]';
T^-1*A*T-L
Out=[0 y0'];
for x=0:0.01:1
   y=T*exp(L*x)*T^-1*y0;
   Out=[Out;x y'];
end;
plot(Out(:,1),Out(:,2),'.-b');
hold on;
plot(Out(:,1),Out(:,3),'.-r');
```

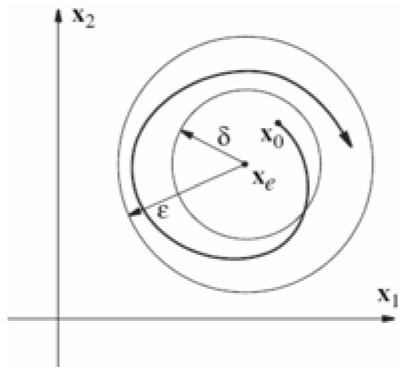
ex1W8skrypt.m
ex1W8.slx



Stabilność układów dynamicznych

Stabilność to jedna z najważniejszych właściwości systemów dynamicznych. Istnieje wiele interpretacji pojęcia stabilności, które w zasadzie są równoważne dobrze znanym pojęciom matematycznym, takim jak ograniczoność lub ciągłość. **Układ dynamiczny nazywamy stabilnym**, *gdy trajektorie stanów są ograniczone albo gdy zależą one w sposób ciągły od stanów początkowych lub sterowań*. Pojęcie stabilności układu można również definiować poprzez stawianie odpowiednich wymagań trajektoriom wyjścia układu.

Większość definicji stabilności odwołuje się do **pojęcia punktu/stanu równowagi**. Najczęściej spotykane definicje stabilności odnoszą się do układów opisywanych równaniem różniczkowym – mówi się wówczas o stabilności poszczególnych rozwiązań równania różniczkowego otrzymanych przy ustalonym sterowaniu **u**, przy czym przez stabilność rozwiązania rozumie się ciągłą zależność tego rozwiązania od warunku początkowego **x(0)**.



Stabilność w sensie Lapunowa

Rozważamy układ równań

$$\dot{x} = f(t, x) \tag{1}$$

z funkcją $f : \mathbb{R}^{m+1} \to \mathbb{R}^m$ klasy C^1. Niech $\bar{x}(t)$ będzie rozwiązaniem układu (1) na przedziale $[0, +\infty)$. Mówimy, że rozwiązanie $\bar{x}(t)$ jest *stabilne w sensie Lapunowa* dla $t \to \infty$ jeżeli dla każdego $\varepsilon > 0$ istnieją $t_0 \geq 0$ i $\eta > 0$ t. że każde rozwiązanie $x(t)$ układu 1 spełniające

$$|x(t_0) - \bar{x}(t_0)| < \eta$$

spełnia również

$$|x(t) - \bar{x}(t)| < \varepsilon \quad \text{dla } t > t_0.$$

Jeżeli dodatkowo

$$\lim_{t\to\infty} |x(t) - \bar{x}(t)| = 0,$$

to rozwiązanie $\bar{x}(t)$ nazywamy *asymptotycznie stabilnym*.

Dla układów liniowych

$$\det(\lambda I - A) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

$$\boxed{\mathrm{Re}\,(\lambda) \leq 0}$$

Przykład

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + x + b\left(\frac{dx}{dt}\right)^3 = 0\ , \quad a>0\ , \ x=0\ , \ \frac{dx}{dt}=0$$

$$x_1 = x\ , \quad x_2 = \dot{x} = \dot{x}_1$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - ax_2 - bx_2^3 \end{cases} \qquad \underline{F}(\underline{x}) = \begin{bmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \end{bmatrix}$$

$$\frac{\partial \underline{F}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -a-3bx_2^2 \end{bmatrix}$$

$$\left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{\underline{x}=\underline{0}} = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix} = \underline{A} \quad \Rightarrow \ddot{x} + a\dot{x} + x = 0$$

$$\det(s\underline{I} - \underline{A}) = \begin{vmatrix} s & -1 \\ 1 & s+a \end{vmatrix} = s(s+a) + 1 = s^2 + sa + 1$$

$$a>0 \Rightarrow Re(s_{1,2}) < 0$$

Twierdzenie Hurwitza

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n = 0, \qquad a_0 > 0$$

Każdy pierwiastek równania ma ujemną część rzeczywistą wtedy i tylko wtedy, gdy macierz

$$H = \begin{bmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-2} \\ 0 & 0 & 0 & 0 & \ldots & a_n \end{bmatrix}$$

Na głównej przekątnej wypisać współczynniki a1, . . . , an.
W każdym wierszu wypisać współczynniki w kolejności malejących numerów, przy czym współczynniki o numerach mniejszych od 0 lub większych od n zastąpić zerami

jest dodatnio określona.

Macierz H nazywamy **macierzą Hurwitza**

Przykład

$$x' = Ax$$

$$x^{(5)} + x^{(4)} + 7\,x^{(3)} + 4\,x'' + 10\,x' + 3\,x = 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -3 & -10 & -4 & -7 & -1 \end{bmatrix}$$

$$det\,(A - \lambda\,I) = 0.$$

$$\lambda^5 + \lambda^4 + 7\,\lambda^3 + 4\,\lambda^2 + 10\,\lambda + 3 = 0$$

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 7 & 1 & 1 & 0 \\ 3 & 10 & 4 & 7 & 1 \\ 0 & 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Pozostaje ustalić określoność macierzy H. Obliczamy minory główne macierzy H i otrzymujemy, że wszystkie te minory są dodatnie, skąd wynika, że macierz H jest dodatnio określona. Zatem na mocy twierdzenia Hutwitza wszystkie wartości własne macierzy A mają ujemne części rzeczywiste, co implikuje asymptotyczną stabilność punktu krytycznego x = (0, 0, 0, 0, 0) układu liniowego
z macierzą A, a stąd dalej asymptotyczną stabilność rozwiązania trywialnego x(t) ≡ 0 równania

$$x^{(5)} + x^{(4)} + 7\,x^{(3)} + 4\,x'' + 10\,x' + 3\,x = 0$$

Uwaga 1

Jeżeli funkcja $y(t)$ jest rozwiązaniem równania (1) to funkcja tożsamościowo równa zero jest rozwiązaniem równania:

$$x'(t) = f(t, x(t) + y(t)) - f(t, y(t)). \tag{2}$$

Rozwiązanie $y(t)$ jest stabilne (asymptotycznie stabilne) wtedy i tylko wtedy gdy rozwiązanie zerowe równania (2) jest stabilne (asymptotycznie stabilne).
Istotnie, rozwiązanie $y(t)$ jest stabilne w sensie Lapunowa to dla dowolnego $\varepsilon > 0$ istnieje $\delta > 0$ taka, że $\|x(t) - y(t)\| < \varepsilon$ gdy $x(t)$ jest rozwiązaniem równania (1) i $\|x(t_0) - y(t_0)\| < \delta$.
Niech $z(t) := x(t) - y(t)$. Ponieważ

$$z' = x' - y' = f(t, x) - f(t, y) = f(t, z + y) - f(t, y),$$

więc z jest rozwiązaniem równania (2) ponadto $\|z(t_0) - 0\| < \delta$ i $\|z(t) - 0\| < \varepsilon$. Zatem rozwiązanie zerowe jest stabilne w sensie Lapunowa. W drugą strone rozumowanie jest analogiczne.
Ponieważ

$$\lim_{t\to\infty} \|x(t) - y(t)\| = \lim_{t\to\infty} \|z(t) - 0\| = 0$$

więc jak rozwiązanie $y(t)$ równania (1) jest asymptotycznie stabilne to rozwiązanie zerowe równania (2) również jest asymptotycznie stabilne.

Przykład

$\boxed{x' + x = 0}$ $x(0) = x_0$ $x(t) = x_0 e^{-t}$

Bierzemy dowolne $\varepsilon > 0.$ Szukamy $\delta > 0,$ że prawdziwa będzie implikacja

$$|x_0 - 0| < \delta \Rightarrow |x_0 e^{-t} - 0| < \varepsilon$$

Ponieważ $|x_0 e^{-t}| = |x_0| e^{-t} \leq |x_0| \quad \text{dla} \;\; t \geq 0$

więc dla $\delta = \varepsilon$ implikacja jest prawdziwa, zatem rozwiązanie zerowe równania jest stabilne w sensie Lapunowa

Ponieważ $\lim\limits_{t \to \infty} |x_0 e^{-t} - 0| = |x_0| \lim\limits_{t \to \infty} e^{-t} = 0$

więc rozwiązanie zerowe równania jest asymptotycznie stabilne

Przykład 2

Rozważmy równanie

$$x' = t(x+1). \tag{6}$$

Rozwiązaniem tego równania z warunkiem początkowym $x(0) = y_0$ gdzie $y_0 \neq -1$ jest funkcja

$$y(t) = (y_0 + 1)e^{\frac{t^2}{2}} - 1.$$

Zbadać stabilność w sensie Lapunowa rozwiązania $y(t)$.
Funkcja po prawej stronie równania (6) jest postaci $f(t,x) = t(x+1)$, więc na podstawie uwagi 1 wystarczy zbadać stabilność rozwiązania zerowego następującego równania:

$$x' = f(t, x+y) - f(t,y) = tx. \tag{7}$$

Rozwiązanie równania (7) z warunkiem początkowym $x(0) = x_0$ ma postać $x(t) = x_0 e^{\frac{t^2}{2}}$.
Ponieważ $|x_0|e^{\frac{t^2}{2}}$ zmierza do nieskończoności jak t zmierza do nieskończoności, więc dla dowolnego $\varepsilon > 0$ nie można znaleźć $\delta > 0$ takiej, że prawdziwa byłaby implikacja:

$$|x_0| < \delta \Rightarrow |x_0 e^{\frac{t^2}{2}}| < \varepsilon \quad \text{dla} \quad t \geq 0.$$

Zatem rozwiązanie zerowe równania (7) nie jest stabilne w sensie Lapunowa.

Funkcja Lapunowa

$$x' = f(x) \qquad f : Q \to R^n$$

Zakładamy, że f(0) = 0, czyli x = 0 jest punktem osobliwym układu

Funkcję rzeczywistą $V : Q \to R$, gdzie $Q \subseteq R^n$ i $0 \in Q$, będziemy nazywać *dodatnio określoną* (*ujemnie określoną*) w zbiorze Q, jeśli $V(0) = 0$ oraz dla każdego $x \in Q \setminus \{0\}$ zachodzi warunek $V(x) > 0$ (odpowiednio $V(x) > 0$).

$$\dot{V}(x(t)) := \frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V(x(t))}{\partial x_i} x_i'(t) \qquad \text{Zatem } \dot{V} = \nabla V \cdot f$$

Niech $Q \subseteq R^n$ i $0 \in Q$. Funkcję $V : Q \to R$ spełniającą warunki:

(i) V jest klasy $C^(Q)$;

(ii) V jest dodatnio określona w Q;

(iii) $\dot{V} = \nabla V \cdot f$ jest ujemnie określona w Q,

nazywamy *silną funkcją Lapunowa* dla układu $x' = f(x)$

Przykład 2

$$\dot{x}_1 = x_2 - 3x_1$$
$$\dot{x}_2 = -x_2^3 - 2x_1$$

$$V(x) = 2x_1^2 + x_2^2 \Longrightarrow \begin{array}{ll} V(x) > 0 & x \neq 0 \\ V(x) = 0 & x = 0 \end{array}$$

$$\begin{aligned} \dot{V}(x) &= 4x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 4x_1(x_2 - 3x_1) + 2x_2(-x_2^3 - 2x_1) \\ &= 4x_1x_2 - 12x_1^2 - 2x_2^4 - 4x_1x_2 \\ &= -12x_1^2 - 2x_2^4 \end{aligned}$$

$$\Longrightarrow \begin{array}{ll} \dot{V}(x) < 0 & x \neq 0 \\ \dot{V}(x) = 0 & x = 0 \end{array}$$

jest słuszne dla dowolnego otoczenia x=0.
Stan jest globalnie asymptotycznie stabilny

Przykład 3

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2$$
$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1)$$

$$V(x) = x_1^2 + x_2^2 \Rightarrow \begin{array}{ll} V(x) > 0 & x \neq 0 \\ V(x) = 0 & x = 0 \end{array}$$

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1(x_1(x_1^2 + x_2^2 - 1) - x_2) + 2x_2(x_1 + x_2(x_1^2 + x_2^2 - 1)) \\ &= 2x_1^2(x_1^2 + x_2^2 - 1) - 2x_1x_2 + 2x_2x_1 + 2x_2^2(x_1^2 + x_2^2 - 1) \\ &= 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \end{aligned} \Rightarrow \quad \dot{V}(x) = 0 \qquad x = 0$$

$$N = \{x : \|x\| < 1\}, \Rightarrow \dot{V}(x) < 0 \qquad x \neq 0$$

Stan x=0 jest lokalnie asymptotycznie stabilny

Zadanie

$$x_1' = 1 - 3\,x_1 + 3\,x_1^2 + 2\,x_2^2 - x_1^3 - 2\,x_1\,x_2^2$$
$$x_2' = x_2 - 2\,x_1\,x_2 + x_1^2\,x_2 - x_2^3$$

$$V(y_1, y_2) = y_1^2 + y_1^2\,y_2^2 + y_2^4, \quad (y_1, y_2) \in R^2$$

i punktu krytycznego (1, 0)

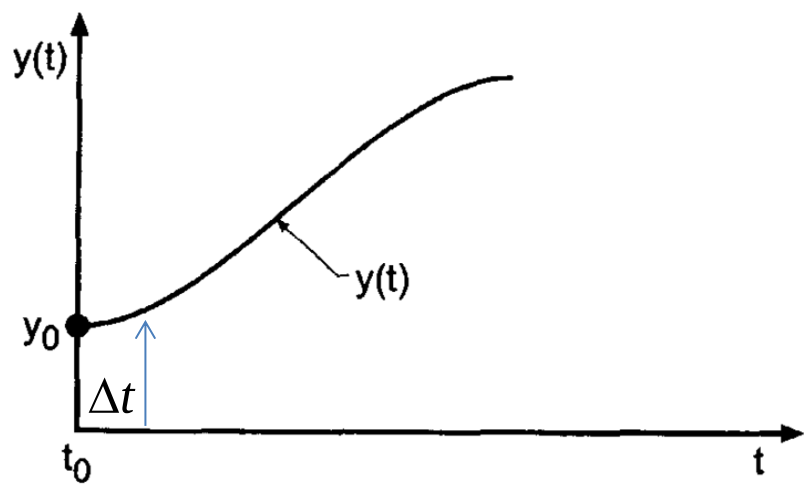
Metoda Taylora. Funkcja określona analitycznie

$$y' = \frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0$$

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{1}{2} y''(t_0)(t - t_0)^2 + \frac{1}{6} y'''(t_0)(t - t_0)^3$$

$$+ \cdots + \frac{1}{n!} y^{(n)}(t_0)(t - t_0)^n + \cdots$$

$$\Delta t = (t - t_0)$$



$$y'' = (y')' = \frac{d(y')}{dt}$$

$$d(y') = d(y'(t,y)) = \frac{\partial y'}{\partial t}\ dt + \frac{\partial y'}{\partial y}\ dy = dt\left(\frac{\partial y'}{\partial t} + \frac{\partial y'}{\partial y}\ \frac{dy}{dt}\right)$$

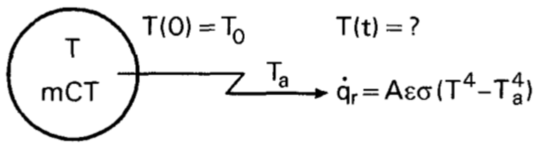
$$\boxed{\frac{d(y')}{dt} = \frac{\partial y'}{\partial t} + \frac{\partial y'}{\partial y}\ \frac{dy}{dt}} = y'_t + y'_y \frac{dy}{dt} \qquad \boxed{y' = \frac{dy}{dt} = f(t,y)}$$

$$\frac{d\left(y'\right)}{dt} = \frac{\partial y'}{dt} + \frac{\partial y'}{\partial y}\frac{dy}{dt} = y'_t + y'_y\frac{dy}{dt} = f'_t\left(t,y\right) + f'_y\left(t,y\right)\cdot f\left(t,y\right)$$

$$y''' = (y'')' = \frac{d(y'')}{dt} = \frac{\partial}{\partial t}(y'_t + y'_y y') + \frac{\partial}{\partial y}(y'_t + y'_y y')\frac{dy}{dt}$$

$$y''' = y''_{tt} + 2y''_{ty}y' + y'_t y'_y + \left(y'_y\right)^2 y' + y''_{yy}\left(y'\right)^2$$

Wymiana ciepła przez promieniowanie z bryły



Energia ***E*** zmagazynowana w bryle jest dana przez

$$\boxed{E = mCT}$$

m masa bryły(kg)

C ciepło właściwe materiału (J/kgK)

Przenoszenie ciepła z bryły **m** do otoczenia przez promieniowanie reguluje prawo promieniowania Stefana-Boltzmanna:

$$\boxed{\dot{q}_r = A\varepsilon\sigma(T^4 - T_a^4)}$$

$\dot{q}_r$ szybkość wymiany ciepła (J/s)

A powierzchnia zbitej masy (m2),

ε stała Stefana-Boltzmanna 5.67 x 10 -8 J/m^{2}K^{4}s

σ emisyjność ciała (bezwymiarowa) czyli stosunek rzeczywistego promieniowania do promieniowania ciała doskonale czarnego

T temperatura wewnętrzna bryły (K)

T_a temperatura otoczenia (K)

Bilans energetyczny stwierdza, że tempo, w jakim zmienia się energia zmagazynowana w bryle, jest równe tempu, z jakim ciepło jest przekazywane do otoczenia

$$\frac{d(mCT)}{dt} = -\dot{q}_r = -A\varepsilon\sigma(T^4 - T_a^4)$$

Znak minus w równaniu jest wymagany, aby szybkość zmian zmagazynowanej energii była ujemna, gdy T jest większe niż Ta. Dla stałych m i C mamy

$$\frac{dT}{dt} = T' = -\alpha(T^4 - T_a^4) \qquad \alpha = \frac{A\varepsilon\sigma}{mC}$$

$$T'(t) = \alpha\, C - \alpha\, T(t)^4$$

Differential equation solution:

$$c_1 + \alpha\, t = \frac{-\log\left(\sqrt[4]{C} - T(t)\right) + \log\left(\sqrt[4]{C} + T(t)\right) + 2\tan^{-1}\left(\frac{T(t)}{\sqrt[4]{C}}\right)}{4\, C^{3/4}}$$

Przykład

$$\boxed{T' = f(t, T)} = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$\boxed{T(t) = T_0 + T'|_0 t + \tfrac{1}{2} T''|_0 t^2 + \tfrac{1}{6} T'''|_0 t^3 + \tfrac{1}{24} T^{(4)}|_0 t^4 + \cdots}$$

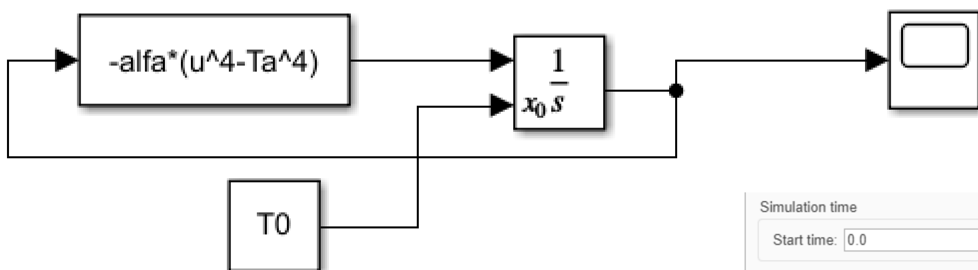
$$T'|_0 = -\alpha(T^4 - T_a^4)|_0 = -(4.0 \times 10^{-12})(2500.0^4 - 250.0^4) = -156.234375$$

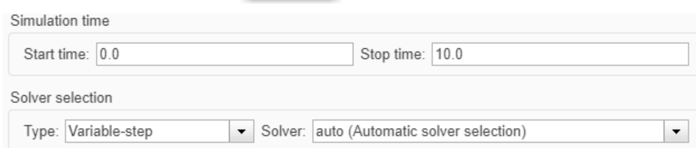
$$T'' = (T')' = \boxed{\frac{\partial T'}{\partial t} + \frac{\partial T'}{\partial T} T'} = 0.0 - 4\alpha T^3 T'$$

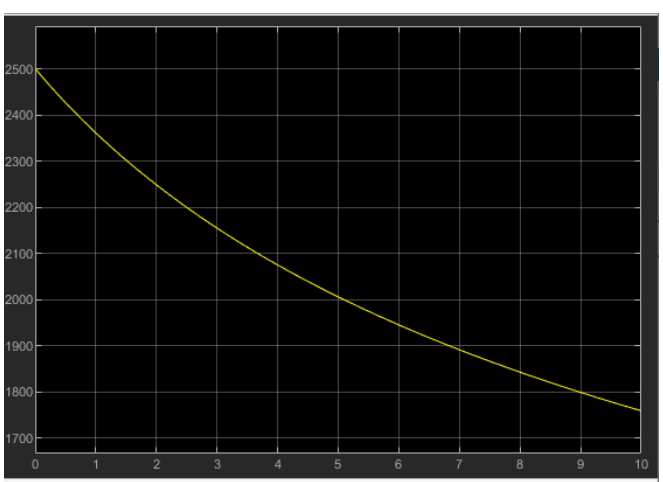
$$T_0'' = -4(4.0 \times 10^{-12})2500.0^3(-156.234375) = 39.058594$$

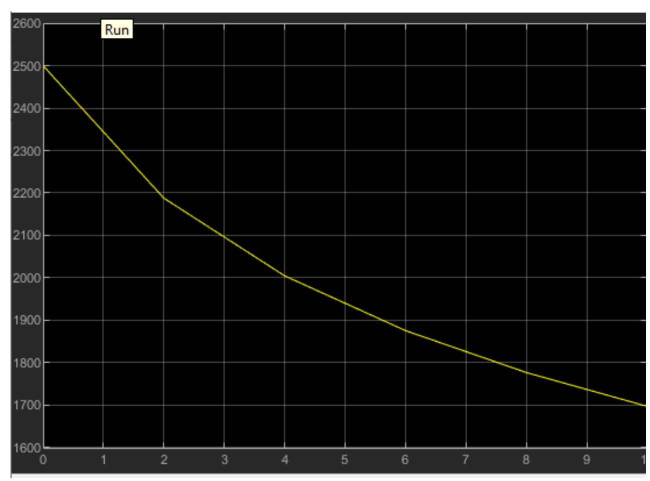
$$T'' = 4\alpha^2(T^7 - T^3 Ta^4)$$

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$









$$T''' = (T'')' = \boxed{\frac{\partial T''}{\partial t} + \frac{\partial T''}{\partial T} T'} = 0.0 + 4\alpha^2(7T^6 - 3T^2 T_a^4)T'$$

$$T_0''' = 4(4.0 \times 10^{-12})^2$$
$$\times (7 \times 2500.0^6 - 3 \times 2500.0^2 \times 250.0^4)(-156.234375)$$
$$= -17.087402$$

$$T''' = -4\alpha^3(7T^{10} - 10T^6 T_a^4 + 3T^2 T_a^8)$$

$$T^{(4)} = (T''')' = \boxed{\frac{\partial T'''}{\partial t} + \frac{\partial T'''}{\partial T} T'} = 0.0 - 4\alpha^3(70T^9 - 60T^5 T_a^4 + 6TT_a^8)T'$$

$$T^{(4)} = -4(4.0 \times 10^{-12})^3(70 \times 2500.0^9 - 60 \times 2500.0^5 \times 250.0^4 + 6$$
$$\times 2500.0 \times 250.0^8)(-156.234375) = 10.679169$$

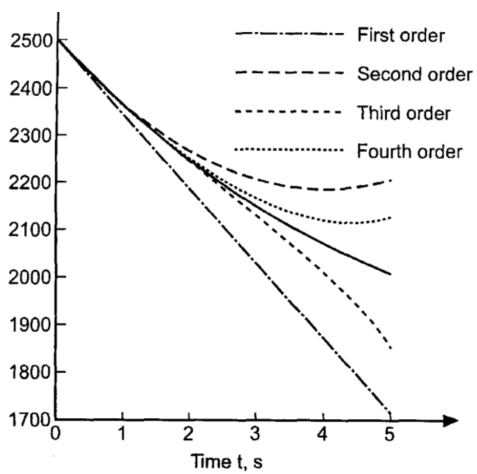
$$\boxed{T(t) = 2500.0 - 156.284375t + 19.529297t^2 - 2.847900t^3 + 0.444965t^4}$$

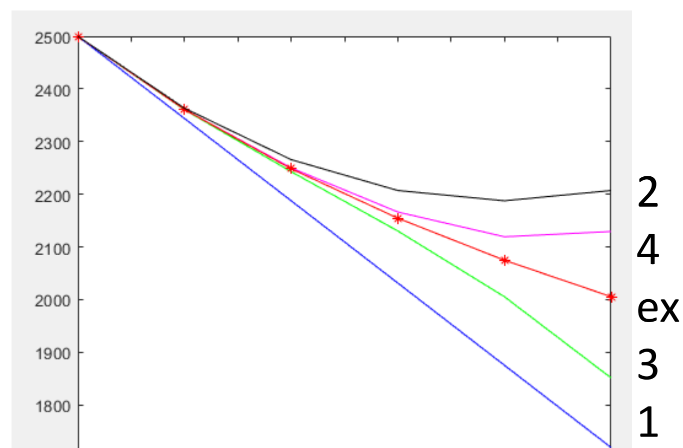
$$\boxed{T(t) = T_0 + T'|_0 t + \tfrac{1}{2} T''|_0 t^2 + \tfrac{1}{6} T'''|_0 t^3 + \tfrac{1}{24} T^{(4)}|_0 t^4 + \cdots}$$

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$\alpha = 4.0 \times 10^{-12}$$

t, s	$T_{exact}(t)$, K	$T(t)^{(1)}$, K	$T(t)^{(2)}$, K	$T(t)^{(3)}$, K	$T(t)^{(4)}$, K
0.0	2500.00	2500.00	2500.00	2500.00	2500.00
1.0	2360.83	2343.77	2363.29	2360.45	2360.89
2.0	2248.25	2187.53	2265.65	2242.87	2249.98
3.0	2154.47	2031.30	2207.06	2130.17	2166.21
4.0	2074.61	1875.06	2187.53	2005.27	2119.18
5.0	2005.42	1718.83	2207.06	1851.07	2129.18





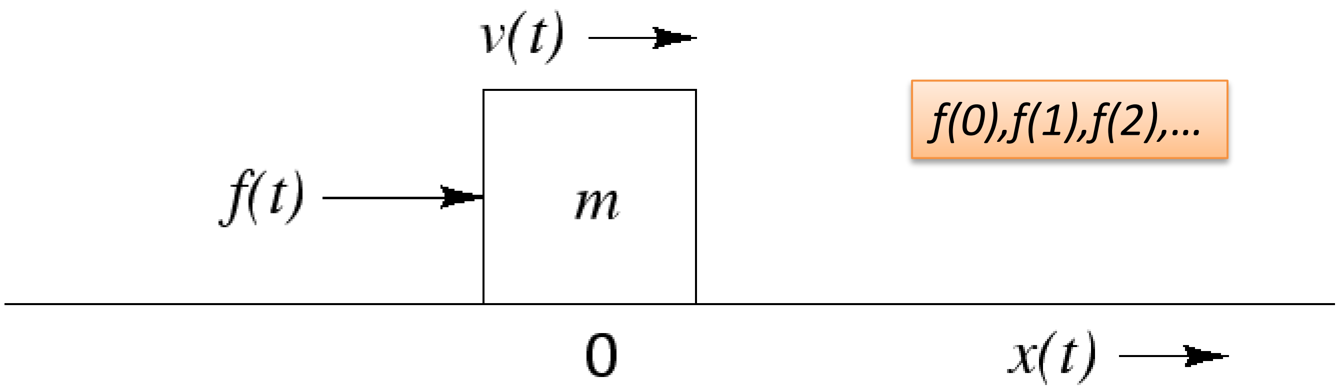
$$\boxed{T(t) = 2500.0 - 156.284375t + 19.529297t^2 - 2.847900t^3 + 0.444965t^4}$$

Rownrozn1.m

Metoda różnicowa
(Funkcja określona w punktach)

- Dyskretyzacja argumentu funkcji
- Aproksymacja pochodnej w punktach (**FDA** – **finite difference approximation**)
- Zastępowanie przez równanie różnicowe
- Rozwiązywanie równań algebraicznych

Finite Difference Approximation (FDA)



f(0),f(1),f(2),...

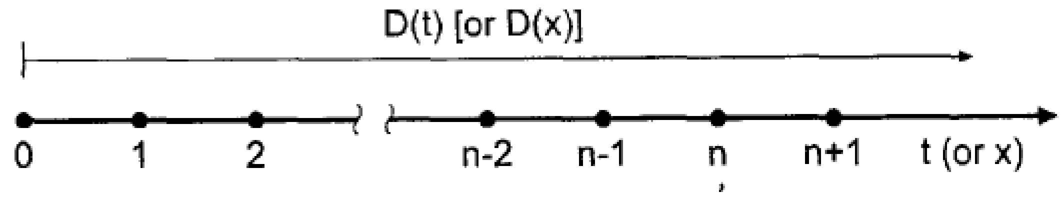
$$f(t) = m\frac{dv}{dt}$$

$$\begin{aligned} \frac{dv}{dt} &\approx \frac{v_n - v_{n-1}}{T} && (\text{“backward difference”}) \\ &\approx \frac{v_{n+1} - v_{n-1}}{2T} && (\text{“centered difference”}) \end{aligned}$$

$$v_n = v_{n-1} + \frac{T}{m} f_n, \quad n = 0, 1, 2, \ldots .$$

Metoda siatek

$$y' = \frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0$$



$$y(t_n) = y_n$$

$$\frac{dy}{dt}(t_n) = y'(t_n) = f(t_n, y_n) = y'\big|_n$$

Aproksymacja pochodnej

Szereg Taylora

$$\bar{y}_{n+1} = \bar{y}_n + \bar{y}'|_n \,\Delta t + \tfrac{1}{2}\bar{y}''|_n \,\Delta t^2 + \tfrac{1}{6}\bar{y}'''|_n \,\Delta t^3 + \cdots$$

z ograniczoną dokładnością

$$\bar{y}_{n+1} = \bar{y}_n + \bar{y}'|_n \,\Delta t + \tfrac{1}{2}\bar{y}''|_n \,\Delta t^2 + \cdots + \frac{1}{m!}\bar{y}^{(m)}|_n \,\Delta t^m + R^{m+1}$$

$$R^{m+1} = \frac{1}{(m+1)!}\bar{y}^{(m+1)}(\tau)\,\Delta t^{m+1} \qquad t \le \tau \le t + \Delta t$$

$$\bar{y}'|_n = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \tfrac{1}{2}\bar{y}''|_n \,\Delta t - \tfrac{1}{6}\bar{y}'''|_n \,\Delta t^2 - \cdots$$

W kierunku t+

$$\boxed{y'|_n = \frac{y_{n+1} - y_n}{\Delta t}} \qquad 0(\Delta t)$$

Aproksymacja pochodnej (cd.)

W kierunku t-

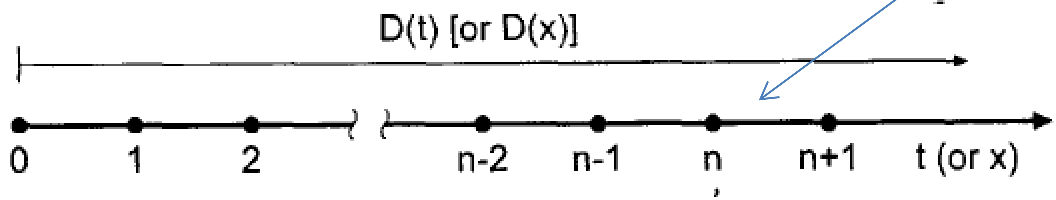
$$\bar{y}_n = \bar{y}_{n+1} + \bar{y}'_{n+1}(-\Delta t) + \frac{1}{2}\bar{y}''_{n+1}(-\Delta t)^2 + \mathrm{L}$$

$$\bar{y}'|_{n+1} = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} + \tfrac{1}{2}\bar{y}''(\tau)\,\Delta t$$

$$\boxed{y'|_{n+1} = \frac{y_{n+1} - y_n}{\Delta t}} \qquad 0(\Delta t)$$

Aproksymacja pochodnej (cd.)

$$n + \frac{1}{2}$$



$$\bar{y}_{n+1} = \bar{y}_{n+1/2} + \bar{y}'_{n+1/2}(\Delta t/2) + \tfrac{1}{2}\bar{y}''_{n+1/2}(\Delta t/2)^2 + \tfrac{1}{6}\bar{y}'''_{n+1/2}(\Delta t/2)^3 + \cdots$$

$$\bar{y}_{n} = \bar{y}_{n+1/2} + \bar{y}'_{n+1/2}(-\Delta t/2) + \tfrac{1}{2}\bar{y}''_{n+1/2}(-\Delta t/2)^2 + \tfrac{1}{6}\bar{y}'''_{n+1/2}(-\Delta t/2)^3 + \cdots$$

$$\bar{y}'|_{n+1/2} = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \frac{1}{24}\bar{y}'''(\tau)\,\Delta t^2$$

$$\boxed{y'|_{n+1/2} = \frac{y_{n+1} - y_n}{\Delta t}} \qquad 0(\Delta t^2)$$

Aproksymacja pochodnej (cd.)

$$\mathrm{FDA} = \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + y'|_n \, \Delta t + \tfrac{1}{2} y''|_n \, \Delta t^2 + \cdots$$

$$\mathrm{FDA} = \frac{y_n + y'|_n \, \Delta t + \tfrac{1}{2} y'|_n \, \Delta t^2 + \cdots - y_n}{\Delta t} = y'|_n + \tfrac{1}{2} y''|_n \, \Delta t + \cdots$$

$$\boxed{\Delta t \to 0, \ \mathrm{FDA} \to y'|_n}$$

Równania różnicowe

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(0) = \bar{y}_0$$

$$y'_n = \frac{y_{n+1} - y_n}{\Delta t}$$

lub

$$y'_{n+1} = \frac{y_{n+1} - y_n}{\Delta t}$$

$$y'_n = \frac{y_{n+1} - y_n}{\Delta t} = f(t_n, y_n) = f_n$$

$$y'_{n+1} = \frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+1}, y_{n+1}) = f_{n+1}$$

Równania różnicowe (cd.)

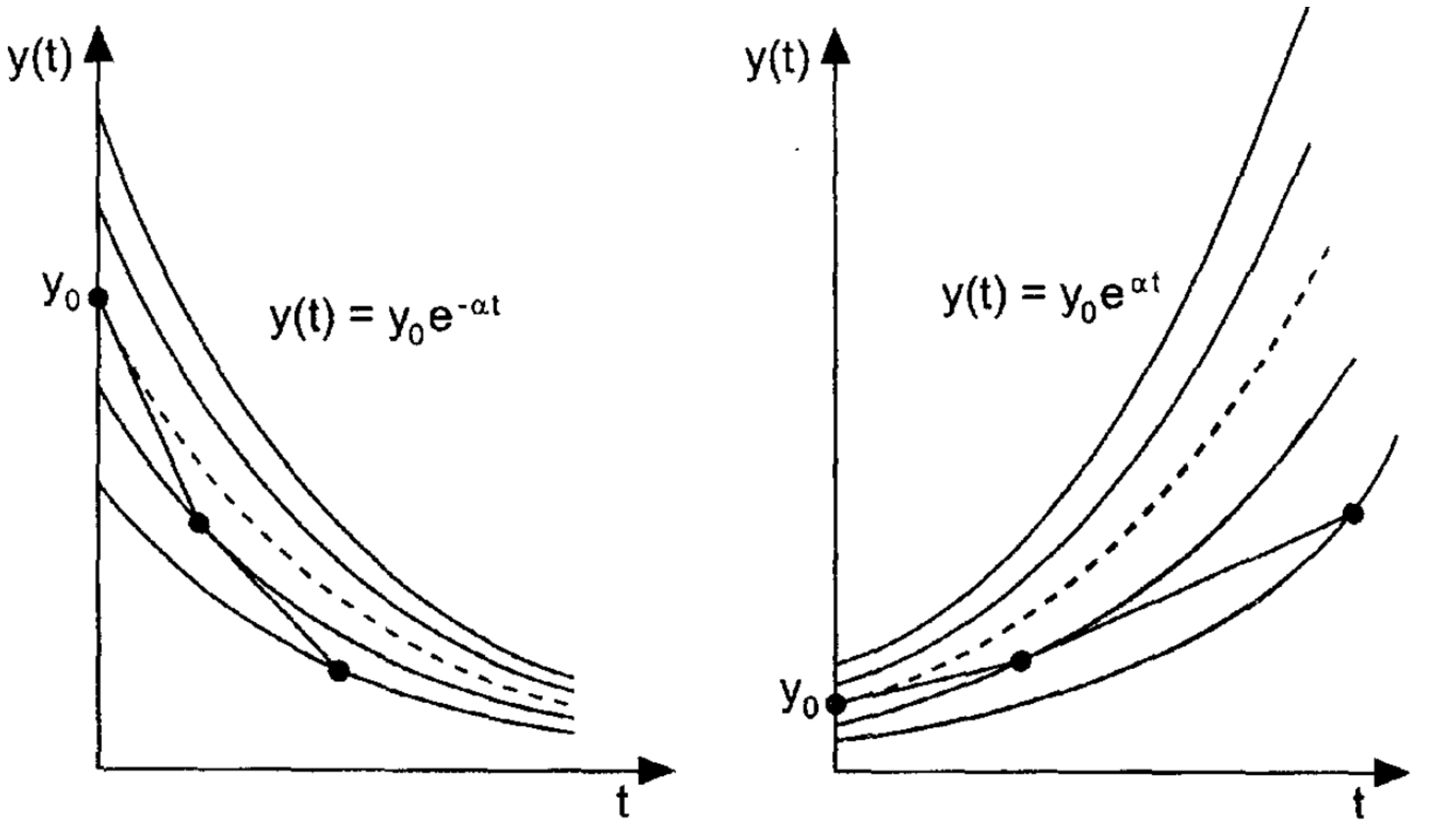
W kierunku t+

$$y_{n+1} = y_n + \Delta t f(t_n, y_n) = y_n + \Delta t f_n$$

W odwrotnym kierunku t-

$$y_{n+1} = y_n + \Delta t f(t_{n+1}, y_{n+1}) = y_n + \Delta t f_{n+1}$$

Stabilność

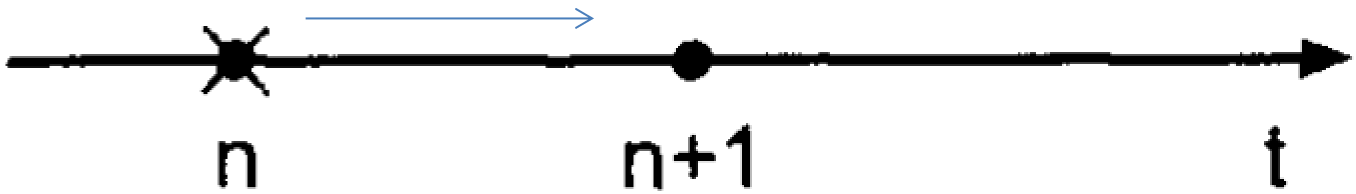


(a) $y' + \alpha y = 0.$ (b) $y' - \alpha y = 0.$

Jawna metoda Eulera

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$

$$\bar{y}'|_n = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \tfrac{1}{2}\bar{y}''(\tau_n)\,\Delta t$$



$$\frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \tfrac{1}{2}\bar{y}''(\tau_n)\,\Delta t = f(t_n, \bar{y}_n) = \bar{f}_n$$

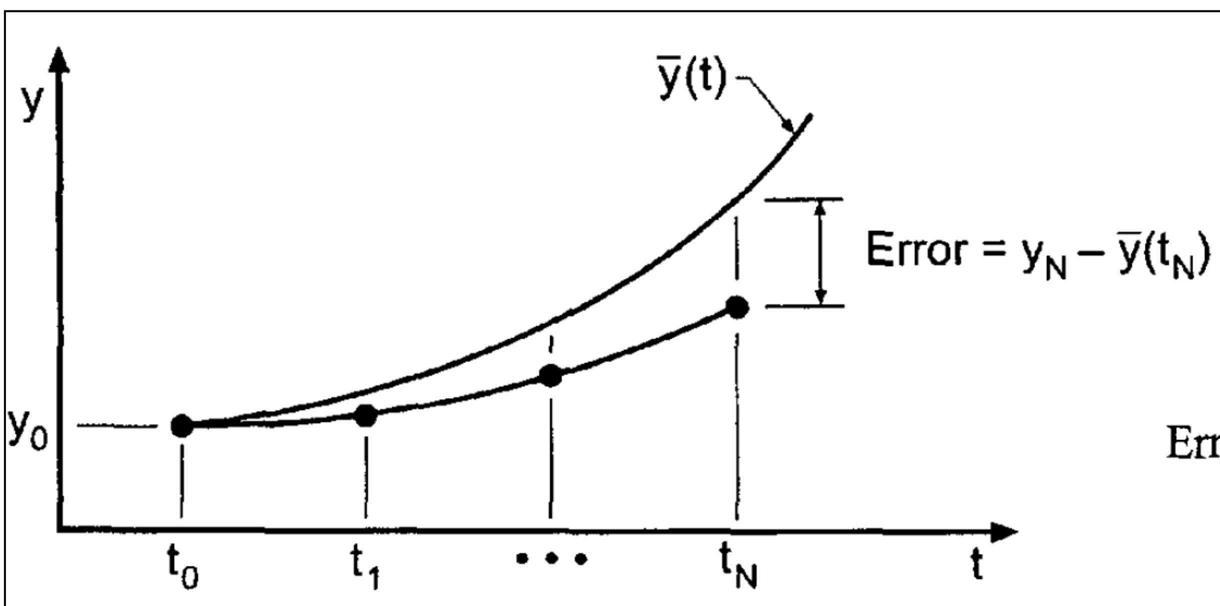
$$\bar{y}_{n+1} = \bar{y}_n + \Delta t \bar{f}_n + \tfrac{1}{2}\bar{y}''(\tau_n)\,\Delta t^2 = \bar{y}_n + \Delta t \bar{f}_n + 0(\Delta t^2)$$

$$\boxed{y_{n+1} = y_n + \Delta t f_n \qquad 0(\Delta t^2)}$$

Metoda Eulera(cd.)

$$y_N = y_0 + \sum_{n=0}^{N-1}(y_{n+1} - y_n) = y_0 + \sum_{n=0}^{N-1} \Delta y_{n+1}$$

$$\text{Error} = \sum_{n=0}^{N-1} \tfrac{1}{2} y''(\tau_n)\, \Delta t^2 = N \tfrac{1}{2} y''(\tau)\, \Delta t^2$$



$$t_0 \le \tau \le t_N.$$

$$N = \frac{t_N - t_0}{\Delta t}$$

$$\text{Error} = \tfrac{1}{2}(t_N - t_0) y''(\tau)\, \Delta t = 0(\Delta t)$$

Przykład

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$f(t, T) = -\alpha(T^4 - T_a^4)$$

$$T_{n+1} = T_n - \Delta t\, \alpha(T_n^4 - T_a^4)$$

$$\Delta t = 2.0 \text{ s}$$

$$f_0 = -(4.0 \times 10^{-12})(2500.0^4 - 250.0^4) = -156.234375$$

$$T_1 = 2500.0 + 2.0(-156.234375) = 2187.531250$$

t, s	$T_{\text{exact}}(t)$, K	$T(t)^{(1)}$, K
0.0	2500.00	2500.00
1.0	2360.83	2343.77
2.0	2248.25	2187.53

Przykład (cd.)

t, s	$T_{exact}(t)$, K
0.0	2500.00
1.0	2360.83
2.0	2248.25
3.0	2154.47
4.0	2074.61
5.0	2005.42

KROK 2

t_n / t_{n+1}	T_n / T_{n+1}	f_n	$\tilde{T}_{n+1}$	Error
0.0	2500.000000	−156.234375		
2.0	2187.531250	−91.580490	2248.247314	−60.716064
4.0	2004.370270	−64.545606	2074.611898	−70.241628
6.0	1875.279058	−49.452290	1944.618413	−69.339355
8.0	1776.374478	−39.813255	1842.094508	−65.720030
10.0	1696.747960		1758.263375	−61.515406

KROK 1

t_n / t_{n+1}	T_n / T_{n+1}	f_n	$\tilde{T}_{n+1}$	Error
0.0	2500.000000	−156.234375		
1.0	2343.765625	−120.686999	2360.829988	−17.064363
2.0	2223.078626	−97.680938	2248.247314	−25.168688
3.0	2125.397688	−81.608926	2154.470796	−29.073108
......	....	....	....	....
9.0	1768.780668	−39.136553	1798.227867	−29.447199
10.0	1729.644115		1758.263375	−28.619260

Przykład (cd.)

$$E(\Delta t = 2.0) = \tfrac{1}{2}(t_N - t_0)T''(\tau)(2.0)$$
$$E(\Delta t = 1.0) = \tfrac{1}{2}(t_N - t_0)T''(\tau)(1.0)$$

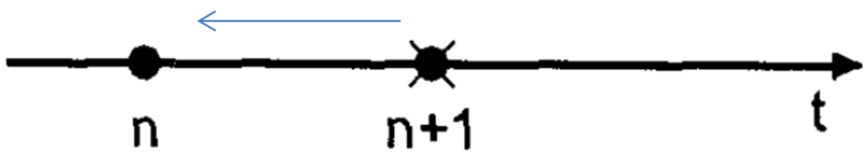
Teoretycznie $$\text{Ratio} = \frac{E(\Delta t = 2.0)}{E(\Delta t = 1.0)} = \frac{2.0}{1.0} = 2.0$$

$$t = 10.0\,\text{s}$$

$$\text{Ratio} = \frac{E(\Delta t = 2.0)}{E(\Delta t = 1.0)} = \frac{-61.515406}{-28.619260} = 2.15$$

Niejawna metoda Eulera

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$



$$\bar{y}'|_{n+1} = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} + \tfrac{1}{2}\bar{y}''(\tau_{n+1})\,\Delta t$$

$$\frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} + \tfrac{1}{2}\bar{y}''(\tau_{n+1})\,\Delta t = f(t_{n+1}, \bar{y}_{n+1}) = \bar{f}_{n+1}$$

$$\bar{y}_{n+1} = \bar{y}_n + \Delta t\bar{f}_{n+1} - \tfrac{1}{2}\bar{y}''(\tau_{n+1})\,\Delta t^2 = \bar{y}_n + \Delta t\bar{f}_{n+1} + 0(\Delta t^2)$$

$$\boxed{y_{n+1} = y_n + \Delta t f_{n+1} \qquad 0(\Delta t^2)}$$

Przykład

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$f(t, T) = -\alpha(T^4 - T_a^4)$$

Równanie algebraiczne

$$\boxed{T_{n+1} = T_n - \Delta t\alpha(T_{n+1}^4 - T_a^4)}$$

$$\Delta t = 2.0\ \mathrm{s}$$

$$T_1 = 2500.0 - 2.0(4.0 \times 10^{-12})(T_1^4 - 250.0^4)$$

$$T_1 = 2373.145960$$

Przykład (cd.)

t_n t_{n+1}	T_n T_{n+1}	$\bar{T}_{n+1}$	Error
0.0	2500.000000		
2.0	2282.785819	2248.247314	34.538505
4.0	2120.934807	2074.611898	46.322909
6.0	1994.394933	1944.618413	49.776520
8.0	1891.929506	1842.094508	49.834998
10.0	1806.718992	1758.263375	48.455617
0.0	2500.000000		
1.0	2373.145960	2360.829988	12.315972
2.0	2267.431887	2248.247314	19.184573
......	....	....	....
9.0	1824.209295	1798.227867	25.981428
10.0	1783.732059	1758.263375	25.468684

Przykład (cd.)

$$t = 10.0\,\mathrm{s}$$

$$\text{Ratio} = \frac{E(\Delta t = 2.0)}{E(\Delta t = 1.0)} = \frac{48.455617}{25.468684} = 1.90$$

$$0(\Delta t)$$

Porównywanie metod

Stabilność algorytmu

$$\boxed{y_{n+1} = G y_n}$$

$$y_N = G^N y_0 \qquad |G| \leq 1$$

Przykład

$$\bar{y}' + \bar{y} = 0 \qquad \bar{y}(0) = 1$$

$$\bar{f}(t, \bar{y}) = -\bar{y}$$

$$\bar{y}(t) = e^{-t}$$

Porównywanie metod (cd.)

Jawna metoda

$$y_{n+1} = y_n + \Delta t f_n = y_n + \Delta t(-y_n)$$

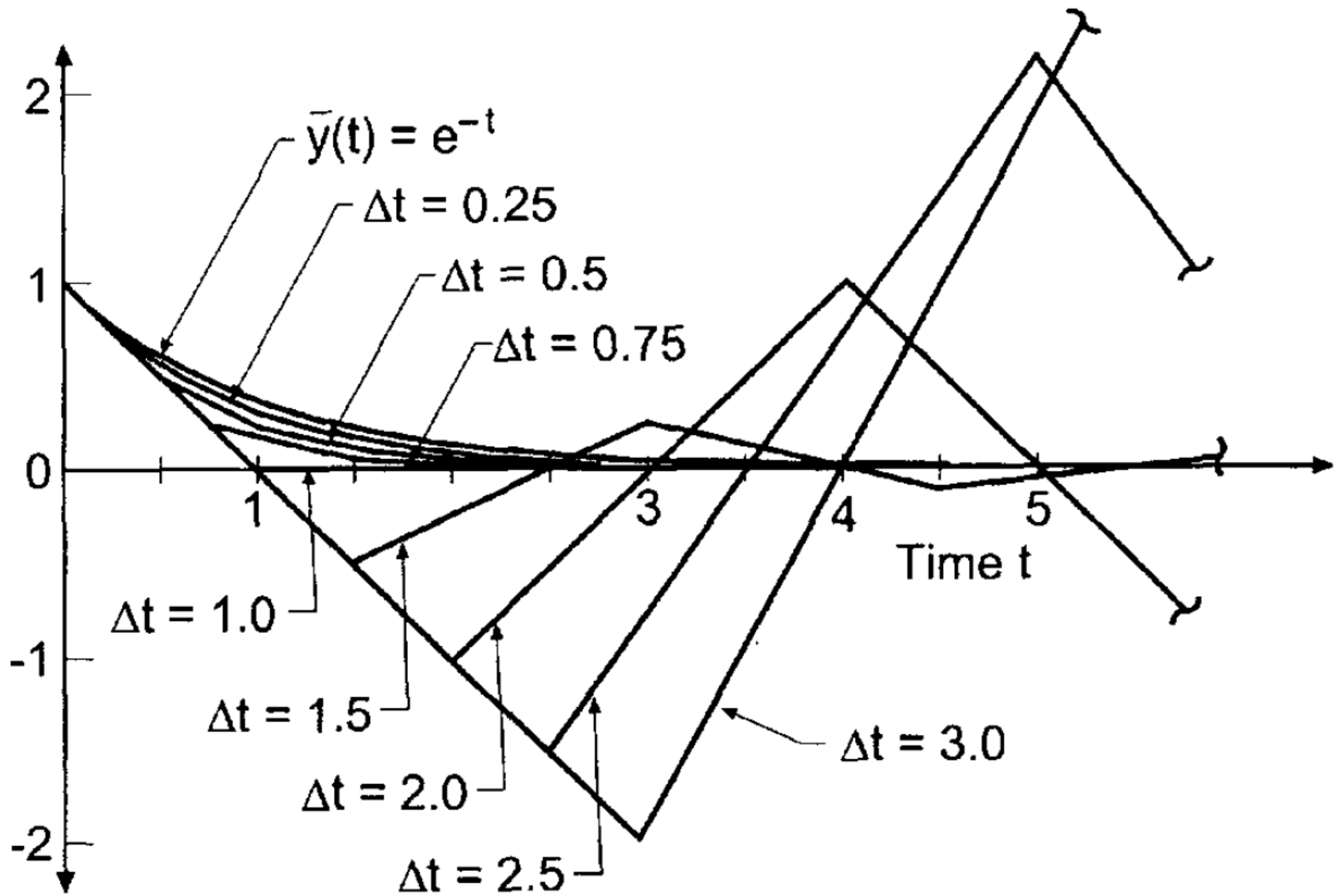
$$y_{n+1} = (1 - \Delta t) y_n$$

$$\Delta t \leq 1.0 \quad \bar{y}(\infty) = 0$$

$$1.0 < \Delta t < 2.0 \quad \bar{y}(\infty) = 0$$ oscylacja

$$\Delta t > 2.0 \quad \bar{y}(\infty) = \infty$$ oscylacja

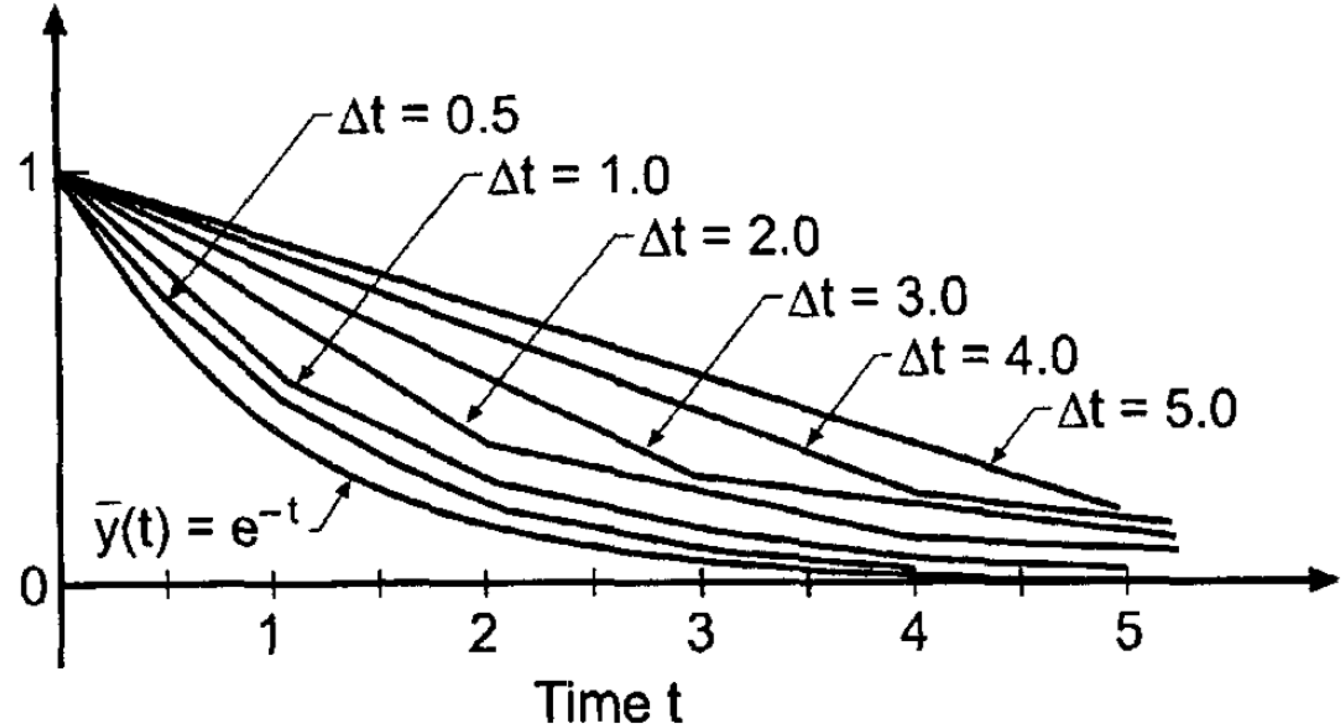
Porównywanie metod (cd.)



Porównywanie metod (cd.)

$$y_{n+1} = y_n + \Delta t f_{n+1} = y_n + \Delta t(-y_{n+1})$$

$$\bar{y}(t) = e^{-t} \quad y_{n+1} = \frac{y_n}{1 + \Delta t}$$



Stabilność jawnej metody Eulera

$$y_{n+1} = y_n + \Delta t\, f_n$$

$$f(t, \bar{y}) = -\alpha \bar{y}, \quad \alpha > 0$$

$$y_{n+1} = y_n + \Delta t(-\alpha y_n) = (1 - \alpha\, \Delta t) y_n = G y_n$$

$$G = (1 - \alpha\, \Delta t) \qquad |G| \leq 1$$

$$-1 \leq (1 - \alpha\, \Delta t) \leq 1 \qquad \alpha\, \Delta t \leq 2$$

$$\boxed{\Delta t \leq 2/\alpha}$$

Stabilność niejawnej metody Eulera

$$y_{n+1} = y_n + \Delta t\, f_{n+1}$$

$$y_{n+1} = y_n + \Delta t(-\alpha y_{n+1})$$

$$y_{n+1} = \frac{1}{1 + \alpha\,\Delta t} y_n = G y_n$$

$$G = \frac{1}{1 + \alpha\,\Delta t}$$

$|G| \leq 1$ dla dowolnego $\alpha\,\Delta t$

$\alpha > 0$

Inne metody jednokrokowe

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$

$$n + 1/2$$

$$\bar{y}_{n+1} = \bar{y}_{n+1/2} + \bar{y}'|_{n+1/2}\left(\frac{\Delta t}{2}\right) + \tfrac{1}{2}\bar{y}''|_{n+1/2}\left(\frac{\Delta t}{2}\right)^2 + \tfrac{1}{6}\bar{y}'''|_{n+1/2}\left(\frac{\Delta t}{2}\right)^3 + \cdots$$

$$\bar{y}_{n} = \bar{y}_{n+1/2} + \bar{y}'|_{n+1/2}\left(-\frac{\Delta t}{2}\right) + \tfrac{1}{2}\bar{y}''|_{n+1/2}\left(-\frac{\Delta t}{2}\right)^2 + \tfrac{1}{6}\bar{y}'''|_{n+1/2}\left(-\frac{\Delta t}{2}\right)^3 + \cdots$$

$$\bar{y}'|_{n+1/2} = \frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} - \tfrac{1}{24} y'''(\tau)\, \Delta t^2$$

$$t_n \leq \tau \leq t_{n+1}$$

Inne metody jednokrokowe (cd.)

$$\frac{\bar{y}_{n+1} - \bar{y}_n}{\Delta t} + 0(\Delta t^2) = f(t_{n+1/2}, \bar{y}_{n+1/2}) = \bar{f}_{n+1/2}$$

$$\bar{y}_{n+1} = \bar{y}_n + \Delta t\, \bar{f}_{n+1/2} + 0(\Delta t^3)$$

$$\boxed{y_{n+1} = y_n + \Delta t\, f_{n+1/2} \qquad 0(\Delta t^3)}$$

Predykcja
$$y^P_{n+1/2} = y_n + \frac{\Delta t}{2} f_n$$

Korekcja
$$y^C_{n+1} = y_n + \Delta t\, f^P_{n+1/2}$$

Przykład

$$\Delta t/2 \qquad \bar{y}' + \alpha\bar{y} = 0 \qquad\qquad f(t,\bar{y}) = -\alpha\bar{y}$$

$$\bar{y}_{n+1/2} = \bar{y}_n + \frac{\Delta t}{2}(-\alpha\bar{y}_n) + 0(\Delta t^2) = \left(1 - \frac{\alpha\,\Delta t}{2}\right)\bar{y}_n + 0(\Delta t^2)$$

$$\boxed{y_{n+1} = y_n + \Delta t\, f_{n+1/2} \qquad 0(\Delta t^3)}$$

$$\bar{y}_{n+1} = \bar{y}_n - \alpha\,\Delta t\left[\left(1 - \frac{\alpha\,\Delta t}{2}\right)\bar{y}_n + 0(\Delta t^2)\right] + 0(\Delta t^3)$$

$$\bar{y}_{n+1} = \left[1 - \alpha\,\Delta t + \frac{(\alpha\,\Delta t)^2}{2}\right]\bar{y}_n + 0(\Delta t^3)$$

$$\boxed{y_{n+1} = \left[1 - \alpha\,\Delta t + \frac{(\alpha\,\Delta t)^2}{2}\right]y_n}$$

Przykład (cd.)

$$G = y_{n+1}/y_n \qquad G = 1 - \alpha\,\Delta t + \frac{(\alpha\,\Delta t)^2}{2}$$

$\alpha\,\Delta t$	G	$\alpha\,\Delta t$	G
0.0	1.000	1.5	0.625
0.5	0.625	2.0	1.000
1.0	0.500	2.1	1.105

$|G| \le 1$ if $\alpha\,\Delta t \le 2$.

Przykład

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$f_n = f(t_n, T_n) = -\alpha(T_n^4 - 250.0^4)$$

$$T_{n+1/2}^P = T_n + \frac{\Delta t}{2} f_n$$

$$f_{n+1/2}^P = f(t_{n+1/2}, T_{n+1/2}^P) = -\alpha[(T_{n+1/2}^P)^4 - 250.0^4]$$

$$T_{n+1}^C = T_n + \Delta t\, f_{n+1/2}^P$$

Przykład (cd.)

$$\Delta t = 2.0\,\mathrm{s}$$

$$f_0 = -(4.0 \times 10^{-12})(2500.0^4 - 250.0^4) = -156.234375$$
$$T_{1/2}^P = 2500.0 + (2.0/2)(-156.234375) = 2343.765625$$

$$f_{1/2}^P = -(4.0 \times 10^{-12})(2343.765625^4 - 250.0^4) = -120.686999$$
$$T_1^C = 2500.0 + 2.0(-120.686999) = 2258.626001$$

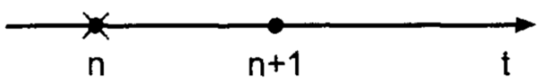
$$\text{Ratio} = \frac{E(\Delta t = 2.0)}{E(\Delta t = 1.0)} = \frac{8.855320}{1.908094} = 4.64$$

Przykład (cd.)

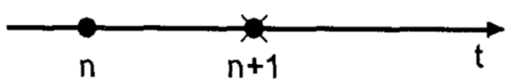
t_n t_{n+1}	T_n $T_{n+1/2}$ T_{n+1}	f_n $f_{n+1/2}$	$\bar{T}_{n+1}$	Error
0.0	2500.000000	−156.234375		
	2343.765625	−120.686999		
2.0	2258.626001	−104.081152	2248.247314	10.378687
	2154.544849	−86.179389		
4.0	2086.267223	−75.761780	2074.611898	11.655325
	2010.505442	−65.339704		
6.0	1955.587815	−58.486182	1944.618413	10.969402
	1897.101633	−51.795424		
8.0	1851.996968	−47.041033	1842.094508	9.902460
	1804.955935	−42.439137		
10.0	1767.118695		1758.263375	8.855320
0.0	2500.000000	−156.234375		
	2421.882812	−137.601504		
1.0	2362.398496	−124.571344	2360.829988	1.568508
	2300.112824	−111.942740		
2.0	2250.455756	−102.583087	2248.247314	2.208442
...	....	....	....	....
9.0	1800.242702	−41.997427	1798.227867	2.014835
	1779.243988	−40.071233		
10.0	1760.171468		1758.263375	1.908094

Metody Eulera jednokrokowe

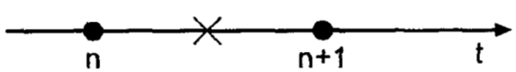
$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$



$$\boxed{y_{n+1} = y_n + \Delta t f_n \qquad 0(\Delta t^2)}$$



$$\boxed{y_{n+1} = y_n + \Delta t f_{n+1} \qquad 0(\Delta t^2)}$$



$$\boxed{y_{n+1} = y_n + \Delta t\, f_{n+1/2} \qquad 0(\Delta t^3)}$$

$$\boxed{\begin{aligned} y^P_{n+1/2} &= y_n + \frac{\Delta t}{2} f_n \\ y^C_{n+1} &= y_n + \Delta t\, f^P_{n+1/2} \end{aligned}}$$

Metody Rungego-Kutty

$$\boxed{\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(0) = \bar{y}_0}$$

$$y_{n+1} = y_n + \Delta y_n = y_n + (y_{n+1} - y_n)$$

$$\boxed{t_n \leq t \leq t_{n+1}}$$

$$\Delta y = C_1\, \Delta y_1 + C_2\, \Delta y_2 + C_3\, \Delta y_3 + \cdots$$

Metoda Rungego-Kutty 2 rzędu

$$y_{n+1} = y_n + C_1\,\Delta y_1 + C_2\,\Delta y_2$$

$$\Delta y_1 = \Delta t\, f(t_n, y_n) = \Delta t\, f_n$$

$$t_n \leq t \leq t_{n+1}$$

$$\Delta y_2 = \Delta t\, f[t_n + (\alpha\,\Delta t), y_n + (\beta\,\Delta y_1)]$$

$$\Delta t = h$$

$$y_{n+1} = y_n + C_1(hf_n) + C_2 hf[t_n + (\alpha h), y_n + (\beta\,\Delta y_1)]$$

Metoda Rungego-Kutty 2 rzędu (cd.)

Szereg Taylora

$$\boxed{f(t,\bar{y}) = \bar{f}_n + \bar{f}_t|_n h + \bar{f}_y|_n \,\Delta y + \cdots}$$

$$t = t_n + (\alpha h) \qquad \Delta t = \alpha h$$

$$y = y_n + (\beta\,\Delta y_n) \qquad \Delta y = \beta h f_n$$

$$f(t_n + (\alpha h), y_n + (\beta\,\Delta y_n)) =$$

$$f_n + (\alpha h)\,f_t|_n + (\beta h f_n)\,f_y|_n + 0(h^2)$$

Metoda Rungego-Kutty 2 rzędu (cd.)

$$y_{n+1} = y_n + C_1\,\Delta y_1 + C_2\,\Delta y_2$$

$$\Delta y_1 = \Delta t\, f(t_n, y_n) = \Delta t\, f_n$$

$$\Delta y_2 = \Delta t\, f[t_n + (\alpha\,\Delta t), y_n + (\beta\,\Delta y_1)]$$

$$\Delta y = \beta h f_n$$

$$f(t_n + (\alpha h), y_n + (\beta\,\Delta y_n)) =$$

$$f_n + (\alpha h)\, f_t|_n + (\beta h f_n)\, f_y|_n + 0(h^2)$$

Metoda Rungego-Kutty 2 rzędu (cd.)

$$y_{n+1} = y_n + (C_1 + C_2)hf_n + h^2(\alpha C_2 f_t|_n + \beta C_2 f_n f_y|_n) + 0(h^3)$$

$$\bar{y}_{n+1} = \bar{y}_n + \bar{y}'|_n h + \tfrac{1}{2}\bar{y}''|_n h^2 + \cdots$$

$$\bar{y}'|_n = \bar{f}(t_n, \bar{y}_n) = \bar{f}_n$$

$$\bar{y}''|_n = (\bar{y}')'|_n = \bar{f}'|_n = \left.\frac{d\bar{f}}{dt}\right|_n = \bar{f}_t|_n + \bar{f}_{\bar{y}}|_n \bar{y}'|_n + \cdots$$

$$\bar{y}_{n+1} = \bar{y}_n + h\bar{f}_n + \tfrac{1}{2}h^2(f_t|_n + \bar{f}_n \bar{f}_y|_n) + 0(h^3)$$

$$C_1 + C_2 = 1 \qquad \alpha C_1 = \tfrac{1}{2} \qquad \beta C_2 = \tfrac{1}{2}$$

Metoda Rungego-Kutty 2 rzędu (cd.)

Na przykład $C_1 = \frac{1}{2}\ \ C_2 = \frac{1}{2}, \alpha = 1\ \ \beta = 1$

$$\Delta y_1 = hf(t_n, y_n) = hf_n$$

$$\Delta y_2 = hf(t_{n+1}, y_{n+1}) = hf_{n+1}$$

$$y_{n+1} = y_n + \tfrac{1}{2}\Delta y_1 + \tfrac{1}{2}\Delta y_2 = y_n + \frac{h}{2}(f_n + f_{n+1})$$

Zmodyfikowana metoda Eulera

Metoda Rungego-Kutty 2 rzędu (cd.)

$$C_1 = 0 \quad C_2 = 1, \alpha = \tfrac{1}{2} \quad \beta = \tfrac{1}{2}$$

$$\Delta y_1 = hf(t_n, y_n) = hf_n$$

$$\Delta y_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_1}{2}\right) = hf_{n+1/2}$$

$$y_{n+1} = y_n + (0)\,\Delta y_1 + (1)\,\Delta y_2 = y_n + hf_{n+1/2}$$

Metoda Rungego-Kutty 2 rzędu (cd.)

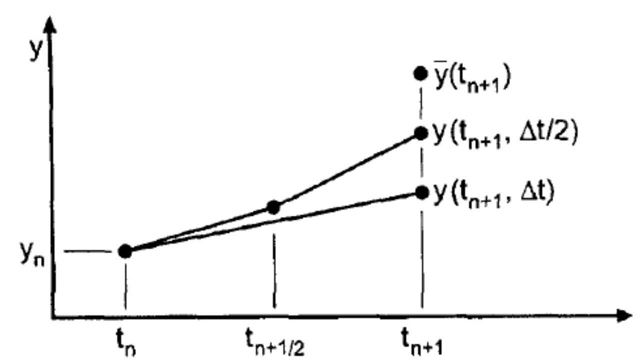
$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2)$$
$$k_1 = hf(t_n, y_n) = hf_n$$
$$k_2 = hf(t_n + \Delta t, y_n + k_1) = hf_{n+1}$$

Metoda Rungego-Kutty 4 rzędu

$$\boxed{y_{n+1} = y_n + \tfrac{1}{6}(\Delta y_1 + 2\,\Delta y_2 + 2\,\Delta y_3 + \Delta y_4)}$$

$$\Delta y_1 = hf(t_n, y_n) \qquad \Delta y_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_1}{2}\right)$$

$$\Delta y_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{\Delta y_2}{2}\right) \qquad \Delta y_4 = hf(t_n + h, y_n + \Delta y_3)$$



Przykład

$$\bar{y}' + \alpha\bar{y} = 0 \qquad f(t,\bar{y}) = -\alpha\bar{y}$$

$$\Delta y_1 = hf(t_n, y_n) = h(-\alpha y_n) = -(\alpha h)y_n$$

$$\Delta y_2 = hf\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_1}{2}\right) = h(-\alpha\{y_n + \tfrac{1}{2}[-(\alpha h)y_n]\})$$

$$\Delta y_2 = -(\alpha h)y_n\left(1 - \frac{(\alpha h)}{2}\right)$$

$$\Delta y_3 = hf\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_2}{2}\right) = h\left(-\alpha\left\{y_n + \frac{1}{2}\left[-(\alpha h)y_n\left(1 - \frac{(\alpha h)}{2}\right)\right]\right\}\right)$$

$$\Delta y_3 = -(\alpha h)y_n\left[1 - \frac{(\alpha h)}{2} + \frac{(\alpha h)^2}{4}\right]$$

$$\Delta y_4 = hf(t_n + \Delta t, y_n + \Delta y_3)$$

$$= h\left(-\alpha\left\{y_n + \left[-(\alpha h)y_n\left(1 - \frac{(\alpha h)}{2} + \frac{(\alpha h)^2}{4}\right]\right)\right\}\right)$$

$$\Delta y_4 = -(\alpha h)y_n\left[1 - (\alpha h) + \frac{(\alpha h)^2}{2} - \frac{(\alpha h)^3}{4}\right]$$

$$\boxed{y_{n+1} = y_n - (\alpha h)y_n + \tfrac{1}{2}(\alpha h)^2 y_n - \tfrac{1}{6}(\alpha h)^3 y_n + \tfrac{1}{24}(\alpha h)^4 y_n}$$

Przykład (cd.)

$$G = y_{n+1} | y_n$$

$$G = 1 - (\alpha h) + \frac{1}{2}(\alpha h)^2 - \frac{1}{6}(\alpha h)^3 + \frac{1}{24}(\alpha h)^4$$

(αh)	G	(αh)	G
0.0	1.000000	2.5	0.648438
0.5	0.606771	2.6	0.754733
1.0	0.375000	2.7	0.878838
1.596 ...	0.270395	2.785 ...	1.000000
2.0	0.333333	2.8	1.022400

$$|G| \leq 1 \text{ if } (\alpha h) \leq 2.785$$

Przykład

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$T_{n+1} = T_n + \tfrac{1}{6}(\Delta T_1 + 2\,\Delta T_2 + 2\,\Delta T_3 + \Delta T_4)$$

$$\Delta T_1 = \Delta t\, f(t_n, T_n) = \Delta t\, f_n \qquad \Delta T_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, T_n + \frac{\Delta T_1}{2}\right)$$

$$\Delta T_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, T_n + \frac{\Delta T_2}{2}\right) \qquad \Delta T_4 = \Delta t\, f(t_n + \Delta t, T_n + \Delta T_3)$$

Przykład (cd.)

$\Delta t = 2.0\,\mathrm{s}$

$$\Delta T_1 = (2.0)(-4.0 \times 10^{-12})(2500.0^4 - 250.0^4) = -312.46875000$$

$$\Delta T_2 = (2.0)(-4.0 \times 10^{-12})\left[\left(2500.0 - \frac{312.46875000}{2}\right)^4 - 250.0^4\right]$$

$$= -241.37399871$$

$$\Delta T_3 = (2.0)(-4.0 \times 10^{-12})\left[\left(2500.0 - \frac{241.37399871}{2}\right)^4 - 250.0^4\right]$$

$$= -256.35592518$$

$$\Delta T_4 = (2.0)(-4.0 \times 10^{-12})[(2500.0 - 256.35592518)^4 - 250.0^4]$$

$$= -202.69306346$$

$$T_1 = 2500.0 + \tfrac{1}{6}[-312.46875000 + 2(-241.37399871)$$

$$+ 2(-256.35592518) - 202.69306346]$$

$$= 2248.22972313$$

Przykład (cd.)

t_n / t_{n+1}	T_n / T_{n+1}	ΔT_1 / ΔT_3	ΔT_2 / ΔT_4	Error
0.0	2500.000000000	−312.468750000	−241.373998706	
		−256.355925175	−202.693063461	
2.0	2248.229723129	−204.335495214	−169.656671860	−0.017590925
		−175.210831240	−147.710427814	
4.0	2074.596234925	−148.160603003	−128.100880073	−0.015662958
		−130.689901700	−114.201682488	
6.0	1944.605593419	−114.366138259	−101.492235905	−0.012819719
		−102.884298188	−92.010660768	
8.0	1842.083948884	−92.083178136	−83.213391686	−0.010558967
		−84.038652301	−76.389312398	
10.0	1758.254519132			−0.008855569
0.0	2500.000000000	−156.234375000	−137.601503560	
		−139.731281344	−124.122675002	
1.0	2360.829563365	−124.240707542	−111.669705704	−0.000425090
		−112.896277161	−102.123860057	
2.0	2248.246807810			−0.000506244
...	....	....	....	....
9.0	1798.227583359	−41.809631378	−39.898370905	−0.000283433
		−39.984283934	−38.211873099	
10.0	1758.263114333			−0.000260369

$$\text{Ratio} = \frac{E(\Delta t = 2.0)}{E(\Delta t = 1.0)} = \frac{-0.008855569}{-0.000260369} = 34.01$$

Tabela Batchera

ogólna

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

dla metod jawnych

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array}$$

w wersji ogólnej
(niejawnej = sumowanie do s)

$$k_i = f(t_{n-1} + c_i \Delta t, u_{n-1} + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j)$$

$$u_n = u_{n-1} + \Delta t \sum_{i=1}^{s} b_i k_i$$

$$u_n = u_{n-1} + \Delta t \sum_{i=1}^{s} b_i f(t_{n-1} + c_i \Delta t, U_i)$$

$$U_i = u_{n-1} + \Delta t \sum_{j=1}^{i-1} a_{ij} f(t_{n-1} + c_j \Delta t, U_j)$$

$$c_i = \sum_{j=1}^{i-1} a_{ij}$$

Przykłady

$$
\begin{aligned}
y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i \\
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + c_2 h, y_n + h(a_{21}k_1)), \\
k_3 &= f(t_n + c_3 h, y_n + h(a_{31}k_1 + a_{32}k_2)), \\
&\ \vdots \\
k_i &= f\left(t_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j\right).
\end{aligned}
$$

Forward Euler

$$
\begin{array}{c|c}
0 & 0 \\
\hline
 & 1
\end{array}
$$

Explicit midpoint method

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1/2 & 1/2 & 0 \\
\hline
 & 0 & 1
\end{array}
$$

Heun's method

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & 1/2 & 1/2
\end{array}
$$

Classic fourth-order method

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
$$

Ralston's method

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
2/3 & 2/3 & 0 \\
\hline
 & 1/4 & 3/4
\end{array}
$$

Kutta's third-order method

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 \\
1 & -1 & 2 & 0 \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

Metoda kolokacji

As the first we will present derivation of collocation methods. First methods as Implicit Euler, implicit midpoint rule or trapezoidal rule are collocation methods. To derive the collocation method we start from differential equation

$$y'(t) = f(t, y(t))$$

and we convert it to integral equation by integrating over interval $[t_n, t]$

$$\int_{t_n}^{t} y'(r)\mathrm{d}r = \int_{t_n}^{t} f(r, y(r))\mathrm{d}r$$

and we compute the integral on the left hand side we obtain

$$y(t) = y(t_n) + \int_{t_n}^{t} f(r, y(r))\mathrm{d}r.$$

Now we replace the exact value $y(t_n)$ by approximated one y_n and the integrand by a polynomial interpolation

$$y(t) \approx y_n + \int_{t_n}^{t} p(r)\mathrm{d}r, \tag{3.1}$$

where $p(r)$ is a unique interpolation polynomial of degree $< s$ which interpolates points $[t_{n,i}, f(t_{n,i}, y(t_{n,i})]$, $i = 1, 2, ..., s$, where $t_{n,i} \equiv t_n + \tau_i h$, $i = 1, ..., s$. and $0 \leq \tau_1 < ... < \tau_s \leq 1$.

Using Lagrange interpolating polynomial we have

$$p(r) = \sum_{j=1}^{s} f(t_{n,j}, y(t_{n,j}))l_j(r), \tag{3.2}$$

where $l_j(r)$ are fundamental Lagrange polynomials defined as

$$l_j(r) = \prod_{\substack{i=1 \\ i \neq j}}^{s} \left(\frac{r - r_i}{r_j - r_i} \right), \qquad j = 1, ..., s.$$

Plug (3.2) into (3.1) we obtain

$$y(t) \approx y_n + \sum_{j=1}^{s} f(t_{n,j}, y(t_{n,j})) \int_{t_n}^{t} l_j(r) \mathrm{d}r.$$

Now the key of collocation method will come. We force equality of above expression at all the points $t_{n,j}$, $i = 1, 2, ..., s$. Thus approximations $y_{n,j}$ for $j = 1, ..., s$ of exact values $y(t_{n,j})$ at collocation node points are determined by following system of non-linear equations

$$y_{n,i} = y_n + \sum_{j=1}^{s} f(t_{n,i}, y_{n,j}) \int_{t_n}^{t_{n,i}} l_j(r) \mathrm{d}r, \qquad i = 1, ..., s. \tag{3.3}$$

In case $\tau_s = 1$ we put $y_{n+1} = y_{n,s}$. Otherwise we set

$$y_{n+1} = y_n + \sum_{j=1}^{s} f(t_{n,j}, y_{n,j}) \int_{t_n}^{t_{n+1}} l_j(r) dr. \tag{3.4}$$

For example, case $s = 2$ we will have Lagrange interpolation polynomial

$$p(r) = f(t_{n,1}, y(t_{n,1})) \frac{(t_{n,2} - r)}{h(\tau_2 - \tau_1)} + f(t_{n,2}, y(t_{n,2})) \frac{(r - t_{n,1})}{h(\tau_2 - \tau_1)}$$

and thus the resulting method will have Butcher tableau in the following form

$$\begin{array}{c|cc} \tau_1 & \frac{2\tau_2 - \tau_1^2}{2(\tau_2 - \tau_1)} & -\frac{\tau_1^2}{2(\tau_2 - \tau_1)} \\ \tau_2 & \frac{\tau_2^2}{2(\tau_2 - \tau_1)} & \frac{\tau_2^2 - 2\tau_1}{2(\tau_2 - \tau_1)} \\ \hline & \frac{2\tau_2 - 1}{2(\tau_2 - \tau_1)} & \frac{1 - 2\tau_1}{2(\tau_2 - \tau_1)} \end{array}$$

where coefficients a_{ij} and b_i are obtained from integrals in (3.3) and (3.4) respectively. The particular examples of collocation methods will be presented in the following section.

Niejawna metoda Rungego-Kutty

Niejawna metoda Eulera

$$\mathbf{y}_1 = \mathbf{y}_0 + hf(t_1, \mathbf{y}_1).$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h\sum_{i=1}^{s} b_i \mathbf{k}_i$$

$$\mathbf{k}_1 = f(t_0 + c_1 h, \mathbf{y}_0 + h\sum_{\ell=1}^{s} a_{1\ell}\mathbf{k}_\ell)$$

$$\vdots$$

$$\mathbf{k}_s = f\Big(t_0 + c_s h, \mathbf{y}_0 + h\sum_{\ell=1}^{s} a_{s\ell}\mathbf{k}_\ell\Big),$$

Tabela Butchera

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array} := \begin{array}{c|cccc} c_1 & a_{11} & \cdots & \cdots & a_{1,s} \\ c_2 & a_{21} & \cdots & \cdots & a_{2,s} \\ \vdots & \vdots & & & \vdots \\ c_s & a_{s1} & \cdots & \cdots & a_{ss} \\ \hline & b_1 & \cdots & \cdots & b_s \end{array}$$

Metody jawne vs niejawne/półjawne

$$\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1\nu} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2\nu} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{\nu 1} & a_{\nu 2} & \cdots & a_{\nu\nu} & c_\nu \\ \hline b_1 & b_2 & \cdots & b_\nu & \end{array}$$

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -1 & 2 & 0 & 1 \\ \hline \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{array}$$

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} & \frac{1}{2} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 1 \\ \hline \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{array}$$

$$\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \hline \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \end{array}$$

Przykłady

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

$$\begin{aligned} k_1 &= hf(t_n + h, y + hk_1) \\ y_{n+1} &= y_n + k_1 \end{aligned}$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Wielopunktowe metody

$$\bar{y}' = \frac{d\bar{y}}{dt} = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$

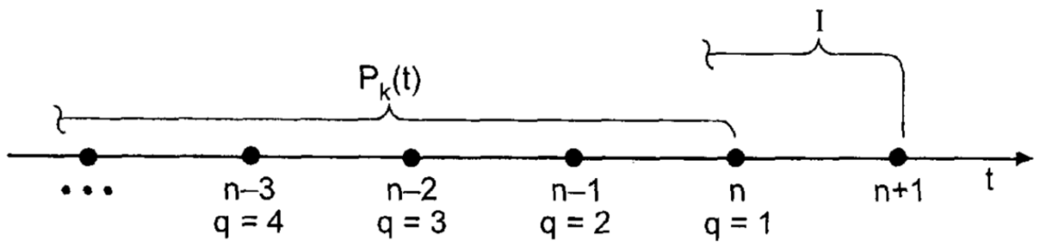
$$d\bar{y} = f(t, \bar{y})\, dt = f[t, \bar{y}(t)]\, dt = F(t)\, dt$$

Obliczenie przez całkowanie

Jawna metoda

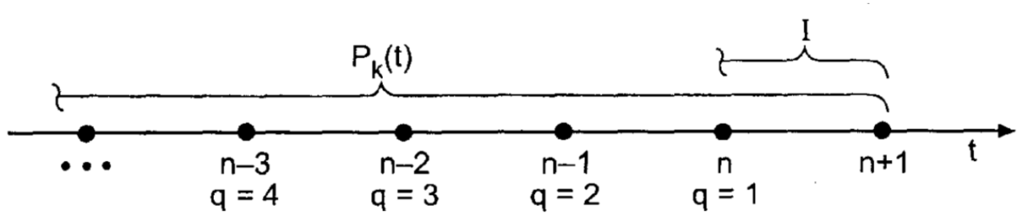
$$I = \int_{\bar{y}_{n+1-q}}^{\bar{y}_{n+1}} d\bar{y} = \int_{t_{n+1-q}}^{t_{n+1}} [P_k(t)]_n \, dt$$

P - wielomian



Niejawna metoda

$$I = \int_{\bar{y}_{n+1-q}}^{\bar{y}_{n+1}} d\bar{y} = \int_{t_{n+1-q}}^{t_{n+1}} [P_k(t)]_{n+1} \, dt$$



Jawna metoda Newtona interpolacji wielomianowej

Interpolacja – metoda numeryczna polegająca na wyznaczaniu w danym przedziale tzw. funkcji interpolacyjnej, która przyjmuje w nim z góry zadane wartości, w ustalonych punktach nazywanych węzłami (**mamy n+1 węzłów**)

$$P_n(x) = f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots + \frac{s(s-1)(s-2)\cdots[s-(n-1)]}{n!}\Delta^n f_0$$

$$s = \frac{x - x_0}{\Delta x} = \frac{x - x_0}{h} \qquad x = x_0 + sh$$

Symbol Newtona

$$\binom{s}{i} = \frac{s(s-1)(s-2)\cdots(s-[i-1])}{i!}$$

$$\boxed{P_n(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \binom{s}{3}\Delta^3 f_0 + \cdots}$$

Tablica różnic

$$\Delta f(x_i) = \Delta f_i = (f_{i+1} - f_i)$$

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

Przykład

$$f(x) = 1/x$$

$$f(3.44) = 1/3.44 = 0.290698\ldots.$$

$$h = 0.1$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
3.1	0.322581					
		-0.010081				
3.2	0.312500		0.000611			
		-0.009470		-0.000053		
3.3	0.303030		0.000558		0.000003	
		-0.008912		-0.000050		0.000007
3.4	0.294118		0.000508		0.000010	
		-0.008404		-0.000040		-0.000010
3.5	0.285714		0.000468		0.000000	
		-0.007936		-0.000040		0.000008
3.6	0.277778		0.000428		0.000008	
		-0.007508		-0.000032		-0.000008
3.7	0.270270		0.000396		0.000000	
		-0.007112		-0.000032		
3.8	0.263158		0.000364			
		-0.006748				
3.9	0.256410					

Przykład (cd.)

$$f(x) = 1/x \qquad\qquad x_0 = 3.40$$

$$P(3.44) = f(3.4) + s\,\Delta f(3.4) + \frac{s(s-1)}{2!}\Delta^2 f(3.4) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(3.4) + \cdots$$

$$s = \frac{x - x_0}{h} = \frac{3.44 - 3.40}{0.1} = 0.4$$

$$P(3.44) = 0.294118 + (0.4)(-0.008404) + (0.4)\frac{(0.4-1)}{2}(0.000468) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.000040) + \cdots$$

$P(3.44) = 0.290756$	linear interpolation	$\text{Error}(3.44) = 0.000058$
$= 0.290700$	quadratic interpolation	$= 0.000002$
$= 0.290698$	cubic interpolation	$= 0.000000$

Niejawna metoda Newtona interpolacji wielomianowej

$$P_n(x) = f_0 + s\,\nabla f_0 + \frac{s(s+1)}{2!}\nabla^2 f_0 + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_0 + \cdots + \frac{s(s+1)\cdots[s+(n-1)]}{n!}\nabla^n f_0$$

$$s = \frac{x - x_0}{\Delta x} = \frac{x - x_0}{h}$$

$$x = x_0 + sh$$

$$\binom{s^+}{i} = \frac{s(s+1)(s+2)\cdots(s+[i-1])}{i!}$$

$$P_n(x) = f_0 + \binom{s^+}{1}\nabla f_0 + \binom{s^+}{2}\nabla^2 f_0 + \binom{s^+}{3}\nabla^3 f_0 + \cdots$$

Tablica różnic

$$\nabla f(x_{i+1}) = \nabla f_{i+1} = (f_{i+1} - f_i)$$

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$
x_{-3}	f_{-3}			
		∇f_{-2}		
x_{-2}	f_{-2}		$\nabla^2 f_{-1}$	
		∇f_{-1}		$\nabla^3 f_0$
x_{-1}	f_{-1}		$\nabla^2 f_0$	
		∇f_0		
x_0	f_0			

Jawny i niejawny operator różnicowania

$$\Delta f(x_i) = \Delta f_i = (f_{i+1} - f_i)$$

$$\nabla f(x_{i+1}) = \nabla f_{i+1} = (f_{i+1} - f_i)$$

x	$f(x)$			
x_0	f_0			
		$(f_1 - f_0)$		
x_1	f_1		$(f_2 - 2f_1 + f_0)$	
		$(f_2 - f_1)$		$(f_3 - 3f_2 + 3f_1 - f_0)$
x_2	f_2		$(f_3 - 2f_2 + f_1)$	
		$(f_3 - f_2)$		
x_3	f_3			

Przykład

$$f(x) = 1/x$$

$$f(3.44) = 1/3.44 = 0.290698\ldots.$$

$$h = 0.1$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
3.1	0.322581					
		−0.010081				
3.2	0.312500		0.000611			
		−0.009470		−0.000053		
3.3	0.303030		0.000558		0.000003	
		−0.008912		−0.000050		0.000007
3.4	0.294118		0.000508		0.000010	
		−0.008404		−0.000040		−0.000010
3.5	0.285714		0.000468		0.000000	
		−0.007936		−0.000040		0.000008
3.6	0.277778		0.000428		0.000008	
		−0.007508		−0.000032		−0.000008
3.7	0.270270		0.000396		0.000000	
		−0.007112		−0.000032		
3.8	0.263158		0.000364			
		−0.006748				
3.9	0.256410					

Przykład

$$x_0 = 3.50 \qquad P(3.44)$$

$$s = \frac{x - x_0}{h} = \frac{3.44 - 3.50}{0.1} = -0.6$$

$$P(3.44) = f(3.5) + s\,\nabla f(3.5) + \frac{s(s+1)}{2!}\nabla^2 f(3.5) + \frac{s(s+1)(s+2)}{3!}\nabla^3 f(3.5) + \cdots$$

$$P(3.44) = 0.285714 + (-0.6)(-0.008404) + \frac{(-0.6)(-0.6+1)}{2}(0.000508) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-0.000050) + \cdots$$

$P(3.44) = 0.290756$	linear interpolation	Error =	0.000058
$= 0.290695$	quadratic interpolation	=	−0.000003
$= 0.290698$	cubic interpolation	=	0.000000

Metoda Adamsa-Bashfortha 4 rzędu

$$q = 1$$

$$I = \int_{\tilde{y}_n}^{\tilde{y}_{n+1}} d\bar{y} = \int_{t_n}^{t_{n+1}} [P_3(t)]_n \, dt$$

$$k = 3$$

$$P_3(s) = \bar{f}_n + s\,\nabla\bar{f}_n + \frac{(s+1)s}{2}\nabla^2\bar{f}_n + \frac{(s+2)(s+1)s}{6}\nabla^3\bar{f}_n + \frac{(s+3)(s+2)(s+1)s}{24}h^4\bar{f}^{(4)}(\tau)$$

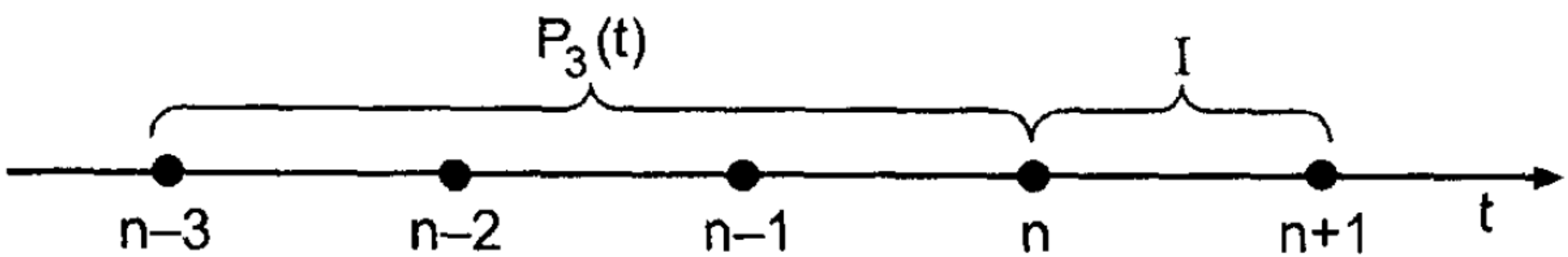
Metoda Adamsa-Bashfortha (cd.)

$$s = \frac{t - t_n}{h} \rightarrow t = t_n + sh \rightarrow dt = h\,ds$$

$$t_{n+1} \rightarrow s = 1$$

$$I = \int_{t_n}^{t_{n+1}} [P_3(t)]_n \, dt$$

$$t_n \rightarrow s = 0$$



Metoda Adamsa-Bashfortha (cd.)

$$I = \int_{\bar{y}_n}^{\bar{y}_{n+1}} d\bar{y} = \int_{t_n}^{t_{n+1}} [P_3(t)]_n \, dt$$

$$\bar{y}_{n+1} - \bar{y}_n = h \int_0^1 P_3(s)\, ds + h \int_0^1 \mathrm{Error}(s)\, ds$$

$$\bar{y}_{n+1} - \bar{y}_n = h \int_0^1 \left(\bar{f}_n + s\, \nabla \bar{f}_n + \frac{s^2 + s}{2} \nabla^2 \bar{f}_n + \frac{s^3 + 3s^2 + 2s}{6} \nabla^3 \bar{f}_n \right) ds$$

$$+ h \int_0^1 \frac{s^4 + 6s^3 + 11s^2 + 6s}{24} h^4 \bar{f}^{(4)}(\tau)\, ds$$

$$\boxed{\bar{y}_{n+1} = \bar{y}_n + h\left(\bar{f}_n + \tfrac{1}{2} \nabla \bar{f}_n + \tfrac{5}{12} \nabla^2 \bar{f}_n + \tfrac{3}{8} \nabla^3 \bar{f}_n\right) + \tfrac{251}{720} h^5 \bar{f}^{(4)}(\tau)}$$

Metoda Adamsa-Bashfortha (cd.)

t	$\bar{f}$	$\nabla\bar{f}$	$\nabla^2\bar{f}$	$\nabla^3\bar{f}$
t_{n-3}	$\bar{f}_{n-3}$			
		$(\bar{f}_{n-2} - \bar{f}_{n-3})$		
t_{n-2}	$\bar{f}_{n-2}$		$(\bar{f}_{n-1} - 2\bar{f}_{n-2} + \bar{f}_{n-3})$	
		$(\bar{f}_{n-1} - \bar{f}_{n-2})$		$(\bar{f}_n - 3\bar{f}_{n-1} + 3\bar{f}_{n-2} - \bar{f}_{n-3})$
t_{n-1}	$\bar{f}_{n-1}$		$(\bar{f}_n - 2\bar{f}_{n-1} + \bar{f}_{n-2})$	
		$(\bar{f}_n - \bar{f}_{n-1})$		$(\bar{f}_{n+1} - 3\bar{f}_n + 3\bar{f}_{n-1} - \bar{f}_{n-2})$
t_n	$\bar{f}_n$		$(\bar{f}_{n+1} - 2\bar{f}_n + \bar{f}_{n-1})$	
		$(\bar{f}_{n+1} - \bar{f}_n)$		
t_{n+1}	$\bar{f}_{n+1}$			

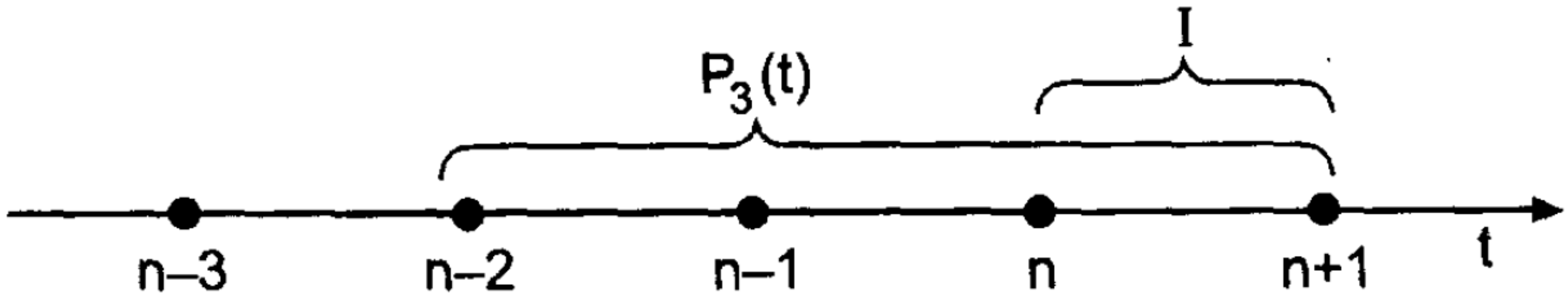
$$\bar{y}_{n+1} = \bar{y}_n + h[\bar{f}_n + \tfrac{1}{2}(\bar{f}_n - \bar{f}_{n-1}) + \tfrac{5}{12}(\bar{f}_n - \bar{2}f_{n-1} + \bar{f}_{n-2}) + \tfrac{3}{8}(\bar{f}_n - 3\bar{f}_{n-1} + 3\bar{f}_{n-2} - \bar{f}_{n-3})] + \tfrac{251}{720}h^5\bar{y}^{(5)}(\tau)$$

$$\boxed{y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})} \quad 0(\Delta t^5)$$

$0(\Delta t^4)$ globally

$\alpha\,\Delta t \lesssim 0.3$

Metoda Adamsa-Moultona



$$I = \int_{\bar{y}_n}^{\bar{y}_{n+1}} d\bar{y} = \int_{t_n}^{t_{n+1}} [P_3(t)]_{n+1}\, dt$$

$$P_3(s) = \bar{f}_{n+1} + s\,\nabla\bar{f}_{n+1} + \frac{(s+1)s}{2}\nabla^2\bar{f}_{n+1} + \frac{(s+2)(s+1)s}{6}\nabla^3\bar{f}_{n+1} + \frac{(s+3)(s+2)(s+1)s}{24}h^4\bar{f}^{(4)}(\tau)$$

Metoda Adamsa-Moultona (cd.)

$$I = \int_{\bar{y}_n}^{\bar{y}_{n+1}} d\bar{y} = \int_{t_n}^{t_{n+1}} [P_3(t)]_{n+1}\, dt$$

$$t_n \to s = -1 \qquad t_{n+1} \to s = 0$$

$$\bar{y}_{n+1} - \bar{y}_n = h \int_{-1}^{0} P_3(s)\, ds + h \int_{-1}^{0} \mathrm{Error}(s)\, ds$$

$$\bar{y}_{n+1} = \bar{y}_n + \frac{h}{24}(9\bar{f}_{n+1} + 19\bar{f}_n - 5\bar{f}_{n-1} + \bar{f}_{n-2}) - \tfrac{19}{720} h^5 \bar{y}^{(5)}(\tau)$$

$$\boxed{y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})} \quad 0(\Delta t^5) \qquad 0(\Delta t^4) \text{ globally}$$

$$\alpha\, \Delta t \lesssim 0.3$$

Metoda Adamsa-Moultona (cd.)

$$y_{n+1}^{P} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

$$y_{n+1}^{C} = y_n + \frac{h}{24}(9f_{n+1}^{P} + 19f_n - 5f_{n-1} + f_{n-2})$$

Przykład

$$f_n = f(t_n, T_n) = -\alpha(T_n^4 - 250.0^4)$$

$$T_{n+1}^P = T_n + \frac{\Delta t}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

$$f_{n+1}^P = f(t_{n+1}, T_{n+1}^P) = -\alpha[(T_{n+1}^P)^4 - 250.0^4]$$

$$T_{n+1}^C = T_n + \frac{\Delta t}{24}(9f_{n+1}^P + 19f_n - 5f_{n-1} + f_{n-2})$$

Przykład (cd.)

$$t_4 = 4.0\,\mathrm{s}$$

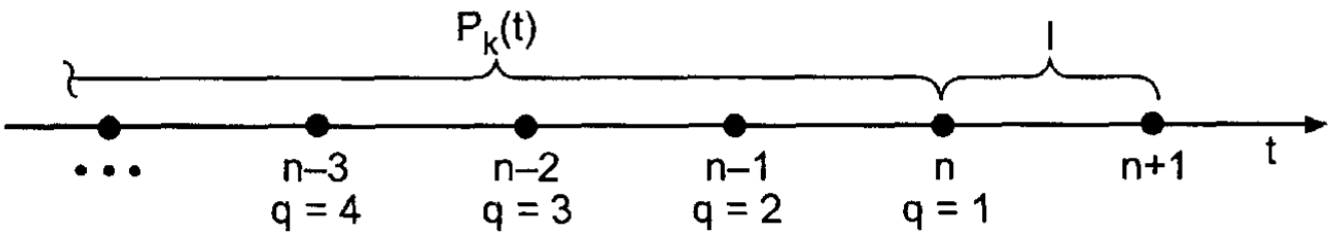
$$\begin{aligned}
T_4^P &= 2154.47079576 + \frac{1.0}{24}[55(-86.16753966) - 59(-102.18094603) \\
&\quad + 37(-124.24079704) - 9(-158.23437500)] \\
&= 2075.24833822 \\
f_4^P &= -(4.0 \times 10^{-12})(2075.24833822^4 - 250^4) \\
&= -74.17350708 \\
T_4^C &= 2154.47079576 + \frac{1.0}{24}[9(-74.17350708) + 19(-86.16753966) \\
&\quad - 5(-102.18094603) + (-124.24079704)] \\
&= 2074.55075892
\end{aligned}$$

Przykład (cd.)

t_n t_{n+1}	T_n T^P_{n+1} T^C_{n+1}	f_n f^P_{n+1}	$\bar{T}_{n+1}$	Error
0.0	2500.00000000	−156.23437500		
1.0	2360.82998845	−124.24079704	exact	
2.0	2248.24731405	−102.18094603	solution	
3.0	2154.47079576	−86.16753966		
	2075.24833822	−74.17350708		
4.0	2074.55075892	−74.07380486	2074.61189788	−0.06113896
	2005.68816488	−64.71557210		
5.0	2005.33468855	−64.66995205	2005.41636581	−0.08167726
	1944.70704983	−57.19500703		
6.0	1944.53124406	−57.17432197	1944.61841314	−0.08716907
...	....	....	....	....
9.0	1798.14913834	−41.80233360	1798.22786679	−0.07872845
	1758.21558333	−38.20946276		
10.0	1758.18932752		1758.26337470	−0.07404718

Ogólna jawna metoda Adamsa

$$\int_{\bar{y}_n}^{\bar{y}_{n+1}} d\bar{y} = h \int_0^1 [P_k(s)]_n \, ds$$

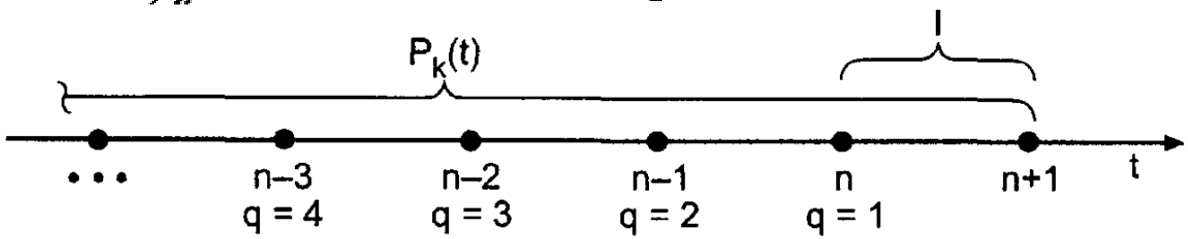


$$y_{n+1} = y_n + \beta h(\alpha_0 f_n + \alpha_{-1} f_{n-1} + \alpha_{-2} f_{n-2} + \cdots) \quad 0(h^n), \quad \alpha \, \Delta t \leq C$$

k	β	α_0	α_{-1}	α_{-2}	α_{-3}	α_{-4}	α_{-5}	n	C
0	1	1						1	2.0
1	1/2	3	−1					2	1.0
2	1/12	23	−16	5				3	0.5
3	1/24	55	−59	37	−9			4	0.3
4	1/720	1901	−2774	2616	−1274	251		5	0.2
5	1/1440	4277	−7923	9982	−7298	2877	−475	6	

Ogólna niejawna metoda Adamsa

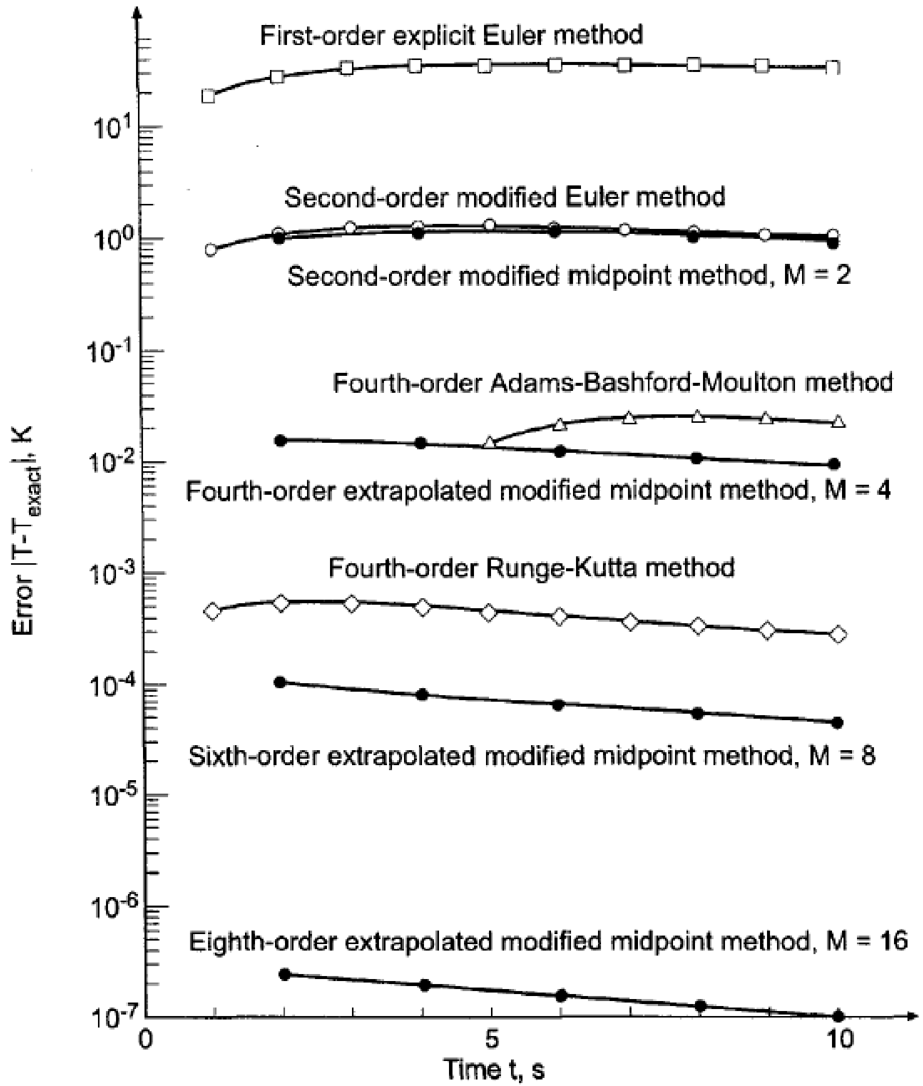
$$\int_{\bar{y}_n}^{\bar{y}_{n+1}} d\bar{y} = h \int_{-1}^{0} [P_k(s)]_{n+1} \, ds$$



$$y_{n+1} = y_n + \beta h(\alpha_1 f_{n+1} + \alpha_0 f_n + \alpha_{-1} f_{n-1} + \cdots) \qquad 0(h^n), \quad \alpha \, \Delta t \leq C$$

k	β	α_1	α_0	α_{-1}	α_{-2}	α_{-3}	α_{-4}	n	C
0	1	1						1	∞
1	1/2	1	1					2	∞
2	1/12	5	8	−1				3	6.0
3	1/24	9	19	−5	1			4	3.0
4	1/720	251	646	−264	106	−19		5	1.9
5	1/1440	475	1427	−798	482	−173	27	6	

Porównywanie metod



Nieliniowe równania

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y}(t_0) = \bar{y}_0$$

Linearyzacja w czasie

$$y_{n+1} = y_n + \Delta t\, f_{n+1}$$

Szereg Taylora

$$\bar{f}_{n+1} = \bar{f}_n + \bar{f}_t|_n\, \Delta t + \bar{f}_y|_n (y_{n+1} - y_n) + \cdots$$

$$y_{n+1} = y_n + \Delta t(f_n + f_t|_n\, \Delta t + f_y|_n (y_{n+1} - y_n)]$$

$$\boxed{y_{n+1} = \frac{y_n + \Delta t\, f_n + \Delta t^2\, f_t|_n - \Delta t\, y_n\, f_y|_n}{1 - \Delta t\, f_y|_n}}$$

Przykład

$$T' = f(t, T) = -\alpha(T^4 - T_a^4) \qquad T(0.0) = 2500.0 \qquad T_a = 250.0$$

$$T_{n+1} = T_n + \Delta t\, f_{n+1}$$

t_n t_{n+1}	T_n T_{n+1}	f_n	$f_T\vert_n$	$\bar{T}_{n+1}$	Error
0.0	2500.000000	−156.234375	−0.250000		
2.0	2291.687500	−110.311316	−0.192569	2248.247314	43.440186
4.0	2132.409031	−82.691332	−0.155143	2074.611898	57.797133
6.0	2006.190233	−64.780411	−0.129192	1944.618413	61.571820
8.0	1903.232170	−52.468392	−0.110305	1842.094508	61.137662
10.0	1817.261400			1758.263375	58.998026
0.0	2500.000000	−156.234375	−0.250000		
1.0	2375.012500	−127.253656	−0.214347	2360.829988	14.182512
2.0	2270.220672	−106.235194	−0.187208	2248.247314	21.973358
...	....	....	....	....	....
9.0	1827.215365	−44.572473	−0.097609	1798.227867	28.987498
10.0	1786.606661			1758.263375	28.343286

$$f(t,T) = -\alpha(T^4 - T_a^4) \qquad y_{n+1} = \frac{y_n + \Delta t\, f_n + \Delta t^2\, f_t|_n - \Delta t\, y_n\, f_y|_n}{1 - \Delta t\, f_y|_n}$$

$$f_t = 0 \quad f_T = -4\alpha T^3 \quad T_{n+1} = \frac{T_n + \Delta t\, f_n - \Delta t\; T_n\, f_T|_n}{1 - \Delta t\, f_T|_n}$$

$$\Delta t = 2.0\ \text{s.}$$

$$f_0 = -(4.0\times 10^{-12})(2500.0^4 - 250.0^4) = -156.234375$$

$$f_T|_0 = -4(4.0\times 10^{-12})2500.0^3 = -0.250000$$

$$T_1 = \frac{2500.0 + 2.0(-156.234375) - 2.0(2500.0)(-0.250000)}{1 - 2.0(-0.250000)}$$

$$= 2291.687500$$

Niejawna metoda Newtona

$$y_{n+1} = G(y_{n+1})$$

$$F(y_{n+1}) = y_{n+1} - G(y_{n+1}) = 0$$

Szereg Taylora

$$F(y^*_{n+1}) = F(y_{n+1}) + F'(y_{n+1})(y^*_{n+1} - y_{n+1}) + \cdots = 0$$

$$y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \frac{F(y_{n+1}^{(k)})}{F'(y_{n+1}^{(k)})}$$

Przykład

$$T_{n+1} = T_n + \Delta t\, f_{n+1} = T_n - \alpha\, \Delta t(T_{n+1}^4 - T_a^4)$$

$$F(T_{n+1}) = T_{n+1} - T_n + \Delta t\, \alpha(T_{n+1}^4 - T_a^4) = 0$$

$$F'(T_{n+1}) = 1 + 4\, \Delta t\, \alpha T_{n+1}^3$$

$$T_{n+1}^{(k+1)} = T_{n+1}^{(k)} - \frac{F(T_{n+1}^{(k)})}{F'(T_{n+1}^{(k)})}$$

Przykład (cd.)

$$\Delta t = 2.0\,\mathrm{s}$$

$$F(T_1) = T_1 - 2500.0 + (2.0)(4.0 \times 10^{-12})(T_1^4 - 250.0^4)$$
$$F'(T_1) = 1 + 4(2.0)(4.0 \times 10^{-12})T_1^3$$

$$T_1^{(0)} = 2500.0\,\mathrm{K}$$

$$\begin{aligned} F(T_1^{(0)}) &= 2500.0 - 2500.0 + (2.0)(4.0 \times 10^{-12})(2500.0^4 - 250.0^4) \\ &= 312.468250 \end{aligned}$$
$$F'(T_1^{(0)}) = 1 + 4(2.0)(4.0 \times 10^{-12})2500.0^3 = 1.500000$$

Przykład (cd.)

$$T_1^{(1)} = 2500.0 - \frac{312.468750}{1.500000} = 2291.687500$$

$$T_{n+1}^{(k+1)} = T_{n+1}^{(k)} - \frac{F(T_{n+1}^{(k)})}{F'(T_{n+1}^{(k)})}$$

Po trzeciej iteracji

$$T_1^{(4)} = 2282.785819$$

Przykład (cd.)

t_n t_{n+1} t_{n+1}	k	T_n T_{n+1} T_{n+1}	F_n	F'_n	$\overline{T}_{n+1}$	Error
0.0	0	2500.000000				
	1	2500.000000	312.468750	1.500000		
	2	2291.687500	12.310131	1.385138		
	3	2282.800203	0.019859	1.380674		
	4	2282.785819	0.000000	1.380667		
2.0		2282.785819			2248.247314	34.538505
4.0		2120.934807			2074.611898	46.322909
6.0		1994.394933			1944.618413	49.776520
8.0		1891.929506			1842.094508	49.834998
10.0		1806.718992			1758.263375	48.455617

Układy równań różniczkowych

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$

$$\bar{y}(t_0) = \bar{y}_0 \qquad \text{and} \qquad \bar{y}^{(i)}(t_0) = \bar{y}_0^{(i)}$$

$$(i = 1, 2, \dots, n-1)$$

$$\boxed{\begin{aligned} y_1 &= y \\ y_2 &= y' = y_1' \\ y_3 &= y'' = y_2' \\ &\cdots\cdots\cdots\cdots \\ y_n &= y^{(n-1)} = y_{n-1}' \end{aligned}}$$

$$y_n' = y^{(n)}$$

Układy równań różniczkowych (cd.)

$$y_1' = y_2 \qquad y_1(0) = y_0$$

$$y_2' = y_3 \qquad y_2(0) = y_0'$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

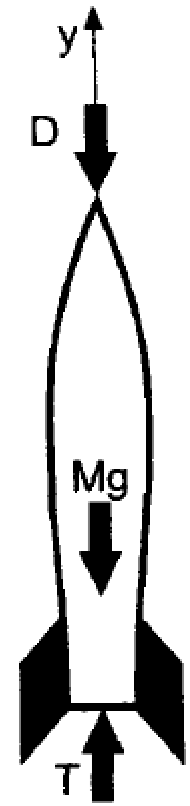
$$y_{n-1}' = y_n \qquad y_{n-1}(0) = y_0^{(n-2)}$$

$$y_n' = F(t, y_1, y_2, \ldots, y_n) \qquad y_n(0) = y_0^{(n-1)}$$

Ogólnie

$$\boxed{\begin{aligned} &\bar{y}_i' = \bar{f}_i(t, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \qquad (i = 1, 2, \ldots, n) \\ &\bar{y}_i(0) = Y_i \qquad (i = 1, 2, \ldots, n) \end{aligned}}$$

Przykład



$$y' = V \qquad y(0.0) = 0.0$$

$$V' = \frac{T}{M_0 - \dot{m}t} - g \qquad V(0.0) = 0.0$$

Rozwiązanie analityczne

$$V(t) = -\frac{T}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - gt$$

$$y(t) = \frac{M_0}{\dot{m}}\left(\frac{T}{\dot{m}}\right)\left(1 - \frac{\dot{m}t}{M_0}\right)\ln\left(1 - \frac{\dot{m}t}{M_0}\right) + \frac{Tt}{\dot{m}} - \frac{1}{2}gt^2$$

Przykład (cd.)

$$T = 10{,}000\,\mathrm{N},\ M_0 = 100.0\,\mathrm{kg},\ \boxed{\dot{m} = 5.0\,\mathrm{kg/s},}$$

$$g = 9.8\,\mathrm{m/s^2}$$

$$y' = f(t, y, V) = V \qquad y(0.0) = 0.0$$

$$V' = g(t, y, V) = \frac{10{,}000.0}{100.0 - 5.0t} - 9.8 \qquad V(0.0) = 0.0$$

$$V(t) = -1{,}000 \ln(1 - 0.05t) - 9.8t$$

$$y(t) = 10{,}000(1 - 0.05t)\ln(1 - 0.05t) + 2000t - 4.9t^2$$

Przykład (cd.)

Metoda Rungego-Kutty $\Delta t = 1.0\,\mathrm{s}$

$$y_{n+1} = y_n + \tfrac{1}{6}(\Delta y_1 + 2\,\Delta y_2 + 2\,\Delta y_3 + \Delta y_4)$$

$$V_{n+1} = V_n + \tfrac{1}{6}(\Delta V_1 + 2\,\Delta V_2 + 2\,\Delta V_3 + \Delta V_4)$$

$$\Delta y_1 = \Delta t\, f(t_n, y_n, V_n) \qquad \Delta V_1 = \Delta t\, g(t_n, y_n, V_n)$$

$$\Delta y_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_1}{2}, V_n + \frac{\Delta V_1}{2}\right)$$

$$\Delta V_2 = \Delta t\, g\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_1}{2}, V_n + \frac{\Delta V_1}{2}\right)$$

$$\Delta y_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_2}{2}, V_n + \frac{\Delta V_2}{2}\right)$$

$$\Delta V_3 = \Delta t\, g\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta y_2}{2}, V_n + \frac{\Delta V_2}{2}\right)$$

$$\Delta y_4 = \Delta t\, f(t_n + \Delta t, y_n + \Delta y_3, V_n + \Delta V_3)$$

$$\Delta V_4 = \Delta t\, g(t_n + \Delta t, y_n + \Delta y_3, V_n + \Delta V_3)$$

$$f(t, y, V) = V$$

$$g(t, y, V) = \frac{10{,}000.0}{100.0 - 5.0t} - 9.8$$

Przykład (cd.)

$$\Delta y_1 = \Delta t\ (V_n) \qquad \Delta V_1 = \Delta t\left(\frac{10{,}000.0}{100.0 - 5.0t_n} - 9.8\right)$$

$$\Delta y_2 = \Delta t\left(V_n + \frac{\Delta V_1}{2}\right) \qquad \Delta V_2 = \Delta t\left[\frac{10{,}000.0}{100.0 - 5.0(t_n + \Delta t/2)} - 9.8\right]$$

$$\Delta y_3 = \Delta t\left(V_n + \frac{\Delta V_2}{2}\right) \qquad \Delta V_3 = \Delta t\left[\frac{10{,}000.0}{100.0 - 5.0(t_n + \Delta t/2)} - 9.8\right]$$

$$\Delta y_4 = \Delta t\left(V_n + \Delta V_3\right) \qquad \Delta V_3 = \Delta t\left[\frac{10{,}000.0}{100.0 - 5.0(t_n + \Delta t)} - 9.8\right]$$

Przykład (cd.)

$$\Delta y_1 = 1.0(0.0) = 0.000000$$

$$\Delta V_1 = 1.0\left[\frac{10{,}000.0}{100.0 - 5.0(0.0)} - 9.8\right] = 90.200000$$

$$\Delta y_2 = 1.0\left(0.0 + \frac{90.200000}{2}\right) = 45.100000$$

$$\Delta V_2 = 1.0\left[\frac{10{,}000.0}{100.0 - 5.0(0.0 + 1.0/2)} - 9.8\right] = 95.463158$$

$$\Delta y_3 = 1.0\left(0.0 + \frac{95.463158}{2}\right) = 47.731579$$

$$\Delta V_3 = 1.0\left[\frac{10{,}000.0}{100.0 - 5.0(0.0 + 1.0/2)} - 9.8\right] = 95.463158$$

$$\Delta y_4 = 1.0(0.0 + 95.463158) = 95.463158$$

$$\Delta V_4 = 1.0\left[\frac{10{,}000.0}{100.0 - 5.0(0.0 + 1.0)} - 9.8\right] = 101.311111$$

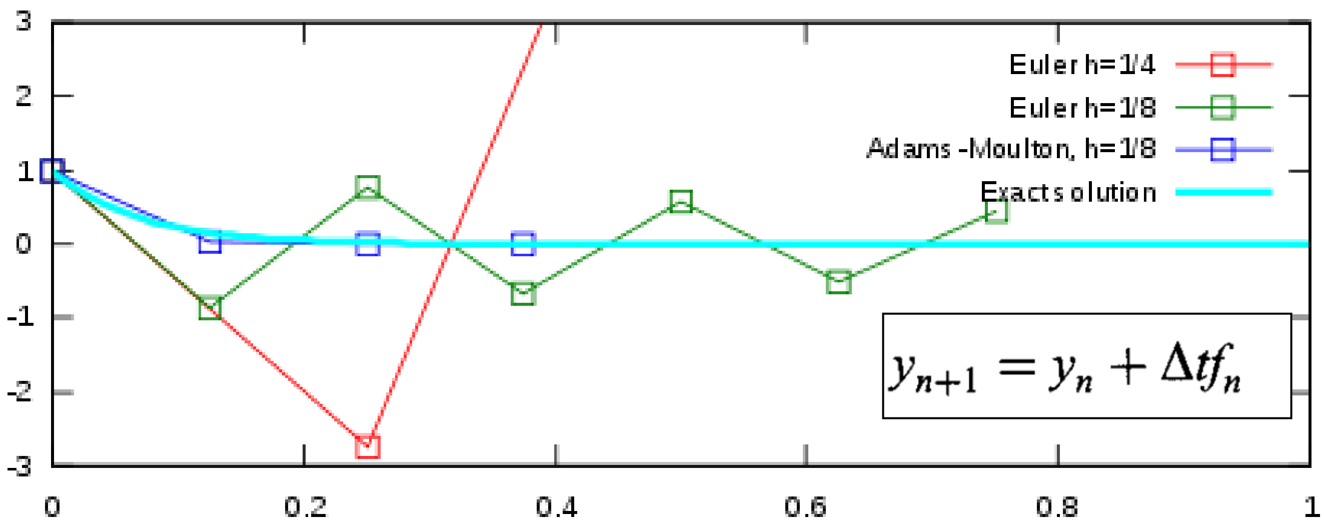
$$\boxed{\begin{aligned} y_1 &= \tfrac{1}{6}[0.0 + 2(45.100000 + 47.731579) + 95.463158] = 46.854386 \text{ m} \\ V_1 &= \tfrac{1}{6}[90.200000 + 2(95.463158 + 95.463158) + 101.311111] \\ &= 95.560624 \text{ m/s} \end{aligned}}$$

Równania sztywne

Równania sztywne – równania różniczkowe, których rozwiązania niektórymi metodami numerycznymi mimo małego kroku całkowania są niestabilne.

$$y'(t) = -15y(t), \quad t \geq 0, y(0) = 1.$$

Rozwiązanie analityczne $y(t) = e^{-15t}$ $\quad y(t) \to 0 \quad t \to \infty$



Metody niejawne są potrzebne

stiffEx1OS.m

An Improved Stiffly Stable Method for Direct Integration of Nonlinear Structural Dynamic Equations K. C. Park, J. Appl. Mech. 42, 464 (1975), DOI:10.1115/1.3423600

Przykład

$$\bar{y}' = f(t, \bar{y}) = -\alpha(\bar{y} - F(t)) + F'(t), \qquad \bar{y}(t_0) = \bar{y}_0$$

Rozwiązanie analityczne

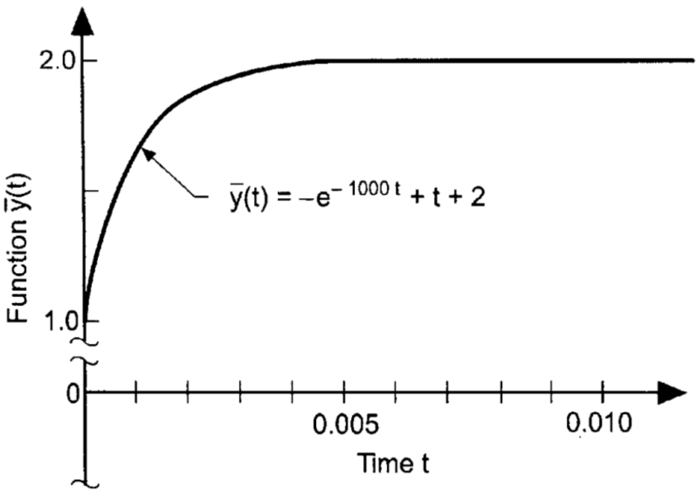
$$\bar{y}(t) = (\bar{y}_0 - F(0))e^{-\alpha t} + F(t)$$

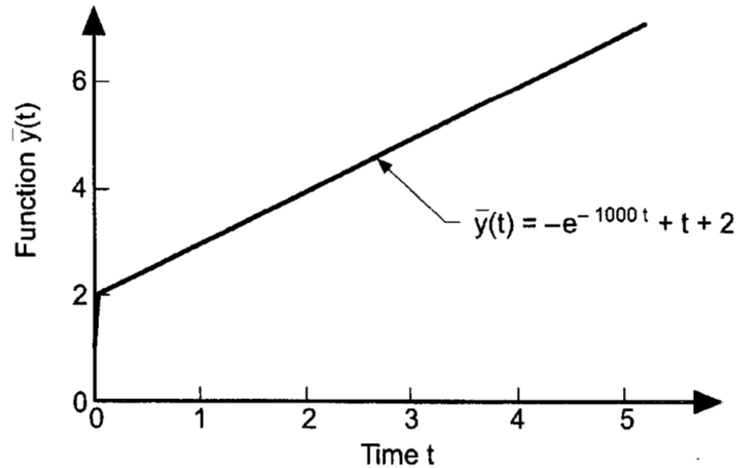
$$\alpha = 1000, F(t) = t + 2 \qquad \bar{y}(0) = 1$$

$$\bar{y}' = \bar{f}(t, \bar{y}) = -1000(\bar{y} - (t + 2)) + 1, \qquad y(0) = 1$$

$$\bar{y}(t) = -e^{-1000t} + t + 2$$

Przykład (cd.)





Rozwiązanie analityczne

Przykład (cd.)

Jawna metoda Eulera

$$y_{n+1} = y_n + \Delta t\, f_n$$

Warunek stabilności dla jawnej metody $\alpha\, \Delta t \leq 2$

$$\Delta t < 2/\alpha = 2/1000 = 0.002$$

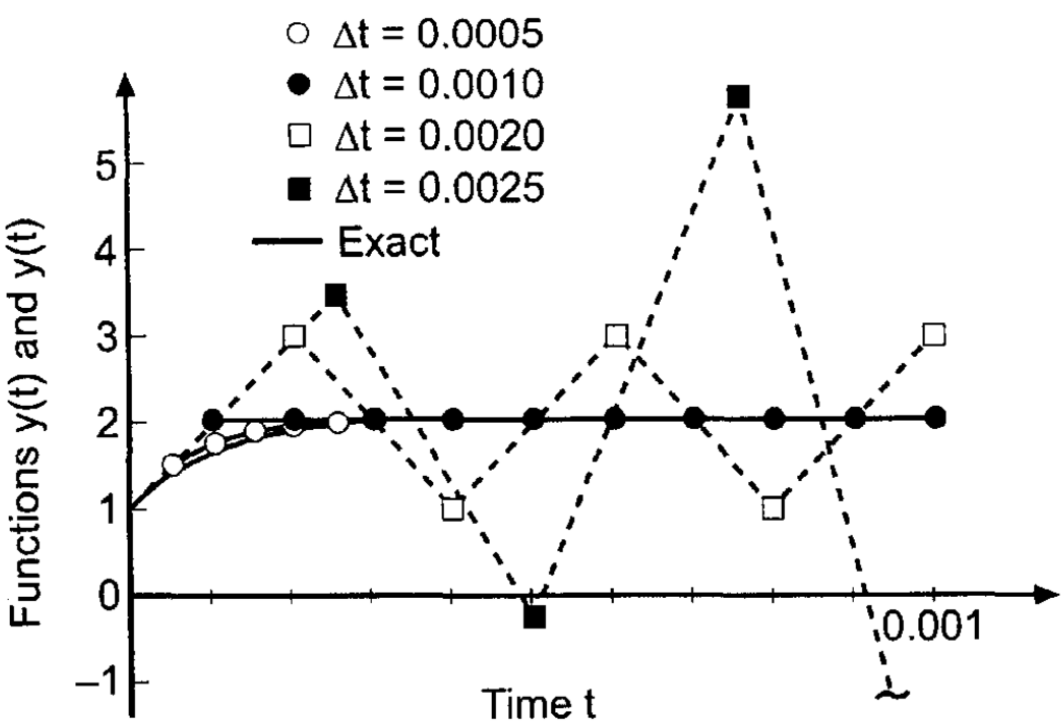
$$\Delta t < 0.002/2 = 0.001$$

$$\Delta t \leq 0.002/10 = 0.0002$$

$$t = 5,\ N = 5/0.0002 = 25{,}000$$

Przykład (cd.)

$$y_{n+1} = y_n + \Delta t(-1000(y_n - (t_n + 2)) + 1)$$



Przykład (cd.)

$\Delta t = 0.0005$			$\Delta t = 0.001$		
t_n	y_n	$\bar{y}_n$	t_n	y_n	$\bar{y}_n$
0.0000	1.000000		0.000	1.000000	
0.0005	1.500500	1.393969	0.001	2.001000	1.633121
0.0010	1.751000	1.633121	0.002	2.002000	1.866665
0.0015	1.876500	1.778370	0.003	2.003000	1.953213
0.0020	1.939500	1.866665	0.004	2.004000	1.985684
...	....	....	...	....	
0.0100	2.009999	2.009955	0.010	2.010000	2.009955

$\Delta t = 0.002$			$\Delta t = 0.0025$		
t_n	y_n	$\bar{y}_n$	t_n	y_n	$\bar{y}_n$
0.000	1.000000		0.0000	1.000000	
0.002	3.002000	1.866665	0.0025	3.502500	1.920415
0.004	1.004000	1.985684	0.0050	−0.245000	1.998262
0.006	3.006000	2.003521	0.0075	5.382500	2.006947
0.008	1.008000	2.007665	0.0100	−3.052500	2.009955
0.010	3.010000	2.009955			

Przykład (cd.)

Niejawna metoda Eulera

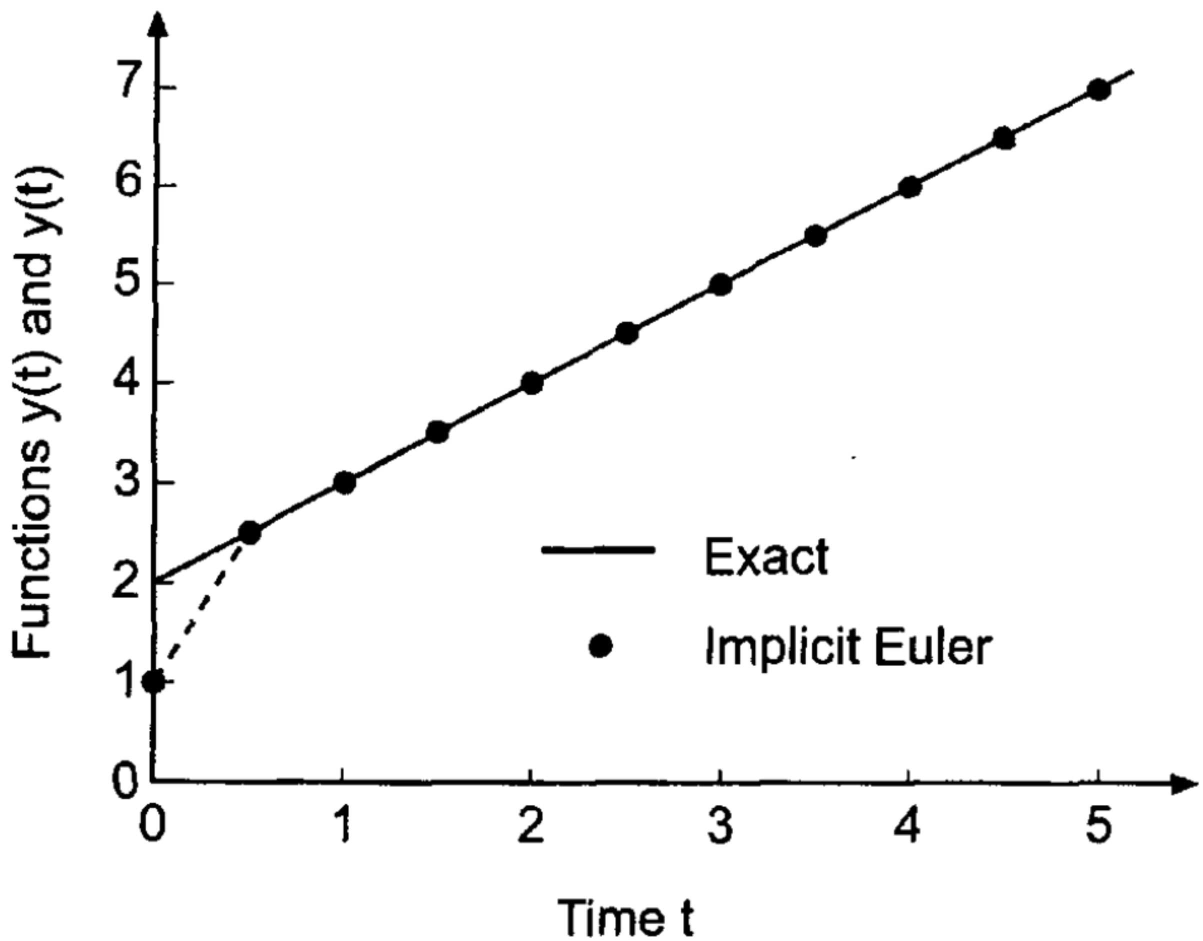
$$y_{n+1} = y_n + \Delta t\, f_{n+1}$$

$$y_{n+1} = y_n + \Delta t(-1000(y_{n+1} - (t_{n+1} + 2)) + 1)$$

$$y_{n+1} = \left(\frac{1}{1 + 1000\,\Delta t}\right)(y_n + 1000(t_{n+1} + 2)\,\Delta t + \Delta t)$$

Przykład (cd.)

$$\Delta t = 0.01,\ 0.05,\ 0.10, \text{ and } 0.5$$



Przykład (cd.)

$\Delta t = 0.01$		
t_n	y_n	$\bar{y}_n$
0.00	1.000000	
0.01	1.919091	2.009955
0.02	2.011736	2.020000
0.03	2.029249	2.030000
0.04	2.039932	2.040000
0.05	2.049994	2.050000
0.06	2.059999	2.060000
0.07	2.070000	2.070000
0.08	2.080000	2.080000
0.09	2.090000	2.090000
0.10	2.100000	2.100000

$\Delta t = 0.05$		
t_n	y_n	$\bar{y}_n$
0.00	1.000000	
0.05	2.030392	2.050000
0.10	2.099616	2.100000

$\Delta t = 0.1$		
t_n	y_n	$\bar{y}_n$
0.0	1.000000	
0.1	2.090099	2.100000

Układy równań liniowych

$$\bar{y}'_i = \bar{f}_i(t, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \qquad (i = 1, 2, \ldots, n)$$

$$\bar{\mathbf{y}}' = \mathbf{A}\bar{\mathbf{y}} + \mathbf{F} \qquad \bar{\mathbf{y}}^{\mathrm{T}} = [\bar{y}_1 \bar{y}_2 \ldots \bar{y}_n]$$

$\mathbf{A}$ macierz rzędu $n \times n$ $\qquad \mathbf{F}^T = [F_1 F_2 \ldots F_n]$

$\alpha_i \ (i = 1, \ldots, n)$ Wartości własne

$$\text{Stiffness ratio} = \frac{\text{Max}|\text{Re}(\alpha_i)|}{\text{Min}|\text{Re}(\alpha_i)|}$$

Przykład

$$u' = F(t, u, v)$$

$$v' = G(t, u, v)$$

$$y' = -y, \qquad y(0) = 1$$

$$z' = -1000z, \qquad z(0) = 1$$

Metoda jawna $\alpha \, \Delta t \leq C$

$$\alpha_1 = 1 \qquad \alpha_2 = 1000$$

Dla $C = 1$ $\Delta t_1 \leq \frac{C}{1} = C \quad \Delta t_2 \leq \frac{C}{1000} = 0.001C$

Przykład (cd.)

$$u' = F(t, u, v)$$
$$v' = G(t, u, v)$$

$$\Delta t = \Delta t_2 \leq 0.001C$$

$$y(t) = e^{-t}$$
$$z(t) = e^{-1000t}$$

Inaczej trzeba używać metod niejawnych