Coding and error detection / correction : a linear-algebraic view.

$$\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \xrightarrow{C} \begin{bmatrix} y_1 \\ \vdots \\ y_7 \end{bmatrix}$$

$\bar{x}$ $\qquad$ $\bar{y}$

$$C = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & & 1 \\ \hdashline 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \\ y_4 &= x_4 \end{aligned}\right\} \text{code}$$

$$\left.\begin{aligned} y_5 &= x_1 + x_2 + x_4 \\ y_6 &= x_1 + x_3 + x_4 \\ y_7 &= x_2 + x_3 + x_4 \end{aligned}\right\} \text{parity bits}$$

Transmission $\qquad \bar{y} \xrightarrow{\Phi} \bar{y}' \qquad \bar{y}' = \begin{cases} \bar{y} & \text{no error} \\ \bar{y} + \bar{e}_i & \text{error(s)} \end{cases}$

Error detection $\qquad S\bar{y}' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ iff $\bar{y}' = \bar{y}$

$$S\bar{y}' = \bar{u} = S(\bar{y} + \bar{e}) = S\bar{y} + S\bar{e} = \bar{0} + \underbrace{S\bar{e}}_{\bar{u}}$$

$$S = \left[\begin{array}{cccc:ccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c:c} \text{parity part of } C & I_m \end{array}\right]$$

$$S\bar{y}' = \begin{bmatrix} y_1 + y_2 + y_4 & + y_5 \\ y_1 + y_3 + y_4 & + y_6 \\ y_2 + y_3 + y_4 & + y_7 \end{bmatrix} \underset{\substack{\uparrow \\ \text{no error} \\ \bar{y}' = \bar{y}}}{=} \begin{bmatrix} x_1 + x_2 + x_4 & + y_5 \\ x_1 + x_3 + x_4 & + y_6 \\ x_2 + x_3 + x_4 & + y_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Suppose $\bar{e} = \bar{e}_2$

$$\bar{y}' = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_7 \end{bmatrix}}_{\bar{y}} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\bar{e}_2} \longrightarrow \bar{S}\bar{y}' = S\bar{y} + S\bar{e}_2 = \bar{0} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

But the result is the same also for

$\bar{e} = \bar{e}_1 + \bar{e}_3$

$\bar{e} = \bar{e}_4 + \bar{e}_6$

$\bar{e} = \bar{e}_5 + \bar{e}_7$

....

In general

$$\bar{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_7 \end{bmatrix} \qquad S\bar{e} = \begin{bmatrix} e_1 + e_2 + e_4 + e_5 \\ e_1 + e_3 + e_4 + e_6 \\ e_2 + e_3 + e_4 + e_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This is a system of 3 linear equations in 7 variables. Solving it for any chosen 3 variables, (e.g. e_1, e_2, e_3) leaves 4 as independent ones. (e.g. e_4, e_5, e_6, e_7)

So in principle there are 2^4 error patterns which yield the syndrome $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The most likely $\bar{e}$ is the one with smallest number of 1-s.

In general:

$$C = \begin{array}{c} k \\ \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & A & \end{array}\right] \end{array} \begin{array}{l} \\ k \\ \\ m \end{array} \qquad S = \begin{array}{cc} k & m \\ \left[\begin{array}{c} \\ A \\ \\ \end{array}\right. & \left.\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}\right] \end{array} m$$

Any specific syndrome vector $S\bar{y}' = \bar{u}$ leads to m equations in $k+m$ variables – hence the possible solution space is k-dimensional (hence contains 2^k possible error patterns)