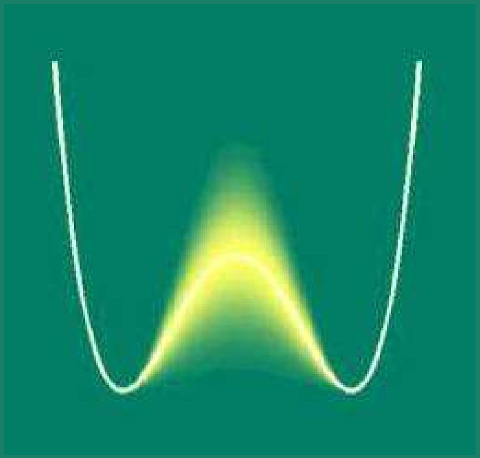
On the time scales in stochastic systems

Partial Noise–Averaging Method

in analysis of an escape over a fluctuating barrier



Jan Iwaniszewski

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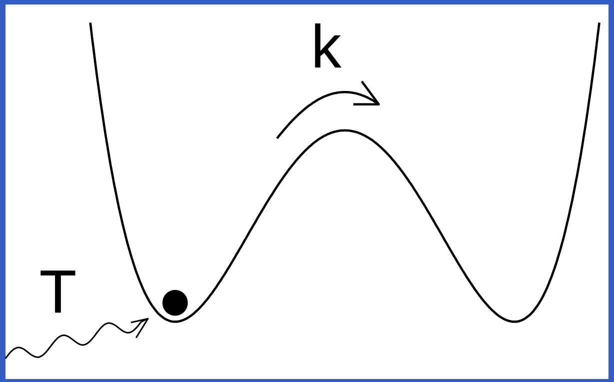
Partial noise-averaging method (PNAM)
applies for the whole range of time variability of the stochastic perturbation

Kramers problem – formulation

activation of an overdamped Brownian particle over a potential barrier



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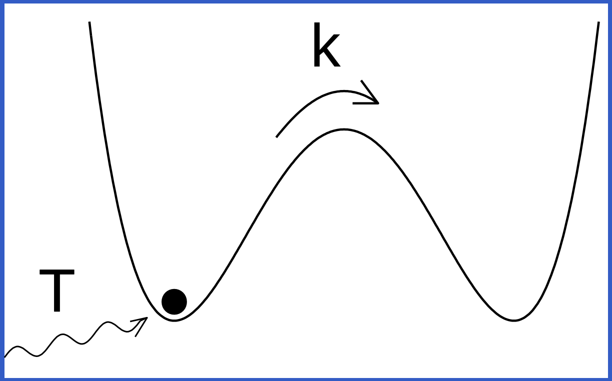

- Langevin equation

$$\frac{dx}{dt} = -U'(x) + \xi(t), \ \langle \xi(t) \rangle = 0, \ \langle \xi(t)\xi(t') \rangle = 2q\delta(t - t'), \ q = kT$$

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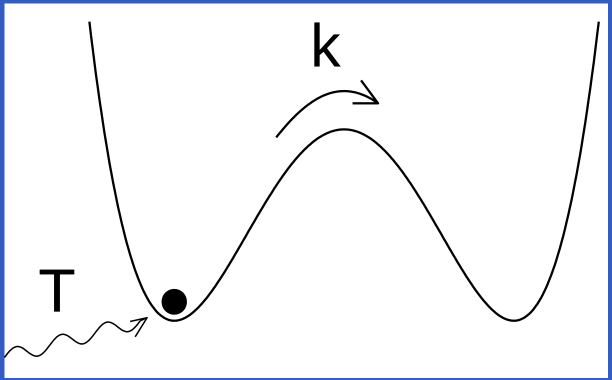

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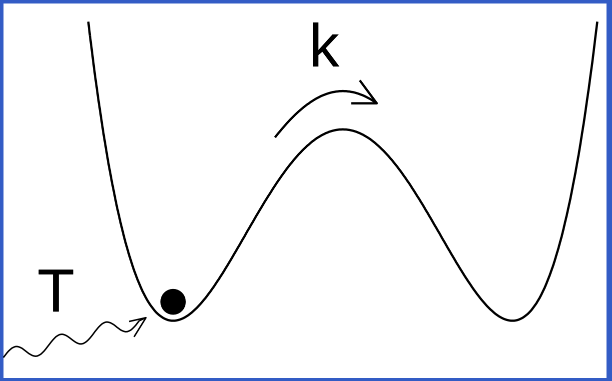

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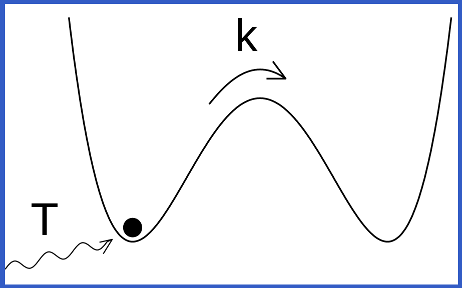
probability current

$$J(x,t) = -U'(x)P(x,t) - q\frac{\partial P(x,t)}{\partial x}$$

Kramers problem – escape time



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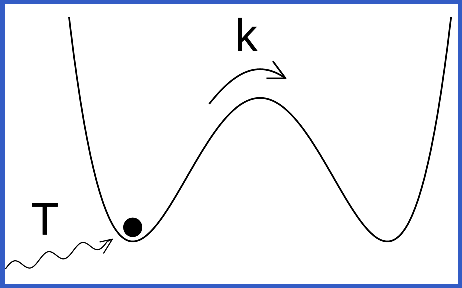

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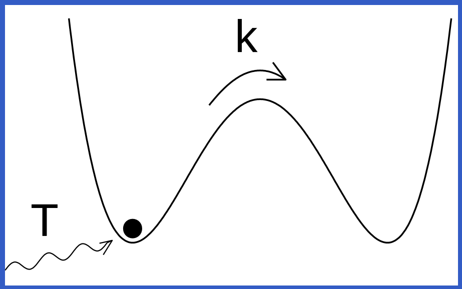

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mean escape time:

$$\mathcal{T} = \int_0^\infty dt \; t \, \mathbf{Q}(t) = \int_0^\infty dt \int_{-\infty}^{x_{max}} dx \; P(x,t)$$

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$t_r \sim \mathcal{O}(1)\ \{+\mathcal{O}(\ln q)\}$

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Kramers problem – approach

For weak noise approx. $t_r/T = \epsilon \ll 1$. Let $t_0 = t,\ t_1 = \epsilon t$, so $t = t(t_0, t_1)$ and

$$\frac{d}{dt} = \frac{\partial t_0}{\partial t}\frac{\partial}{\partial t_0} + \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1}$$

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stationary solution: $\lim_{t_0\to\infty} P(x,t_0;t_1) = \rho(t_1)\cdot N^{-1}\exp\left[-U(x)/q\right] \equiv \rho(t_1)P_{st}(x)$

quasiequilibrium

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quasiequilibrium $\Longrightarrow$ integrating $(*)$ over x from $-\infty$ to x_{thr} one gets

$$\epsilon\frac{d\rho(t_1)}{dt_1} = -J(x_{thr},t_1) \approx -k\rho(t_1)$$ flux-over-population method

kinetic approx. – the escape process is described by a single kinetic coefficient k
and $\mathcal{T} = k^{-1} = \int_0^\infty dt\ \rho(t)$

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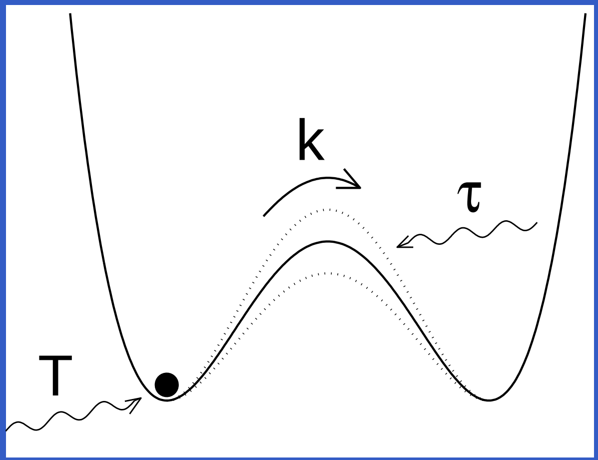
mean escape time: $\mathcal{T} = \int_0^\infty dt \int dy\; \rho(y,t) \qquad k(y) = \{\mathcal{T}(y)\}^{-1} \sim \epsilon$

Escape over a fluctuating barrier



Escape over a fluctuating barrier

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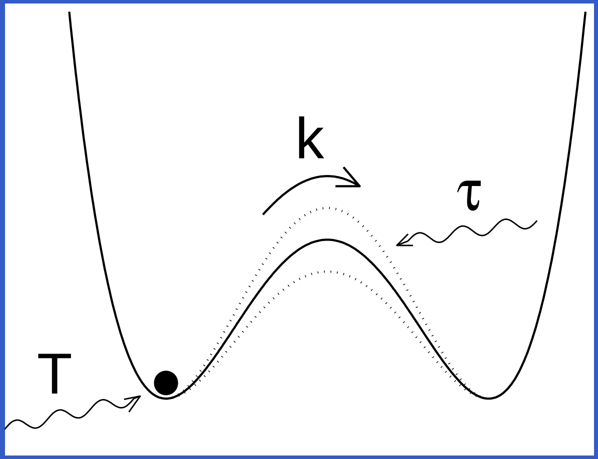

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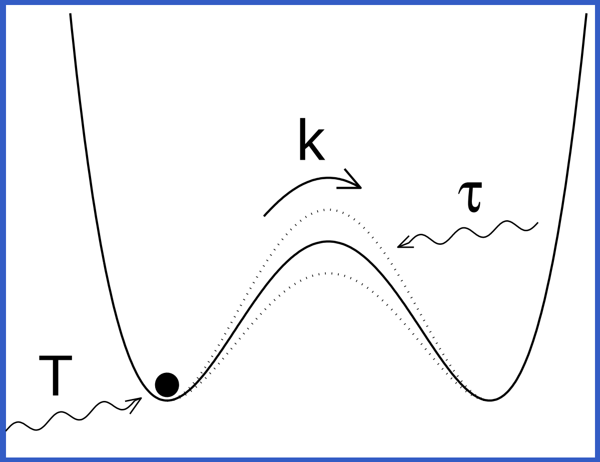

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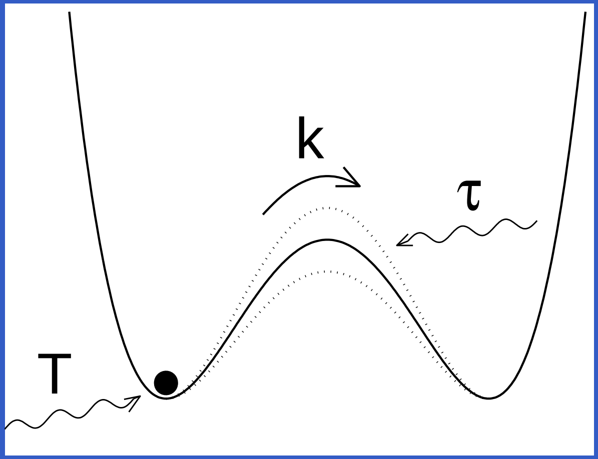

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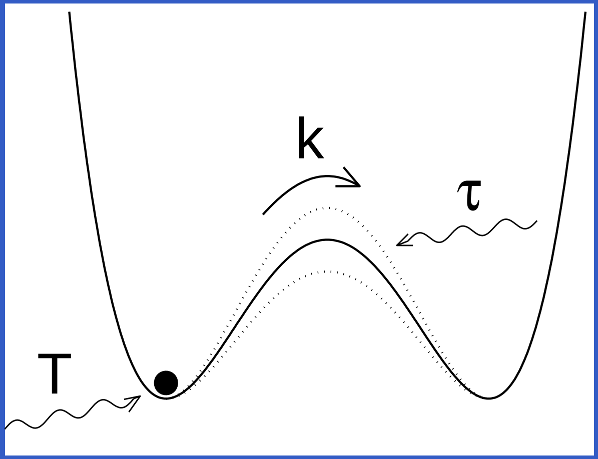

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2. Fluctuations of external parameters
   - dye laser pumped by an another laser
   - Supeconducting QUantum Interference Device acting under the external magnetic field
   - dynamics of populations of species in the presence of external random factors, e.g. climatic

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$D(\tau)$ – variance
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Problem: $\mathcal{T}(\tau)$ for any $\tau \in [0,\infty)$

Exponentially correlated noises

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stationary distribution	$W_0(z) = \begin{cases} \frac{1}{2} & \text{for } z = +\sqrt{D} \\ \frac{1}{2} & \text{for } z = -\sqrt{D} \end{cases}$	$W_0(z) = \dfrac{1}{\sqrt{2\pi D}} \exp(-z^2/2D)$

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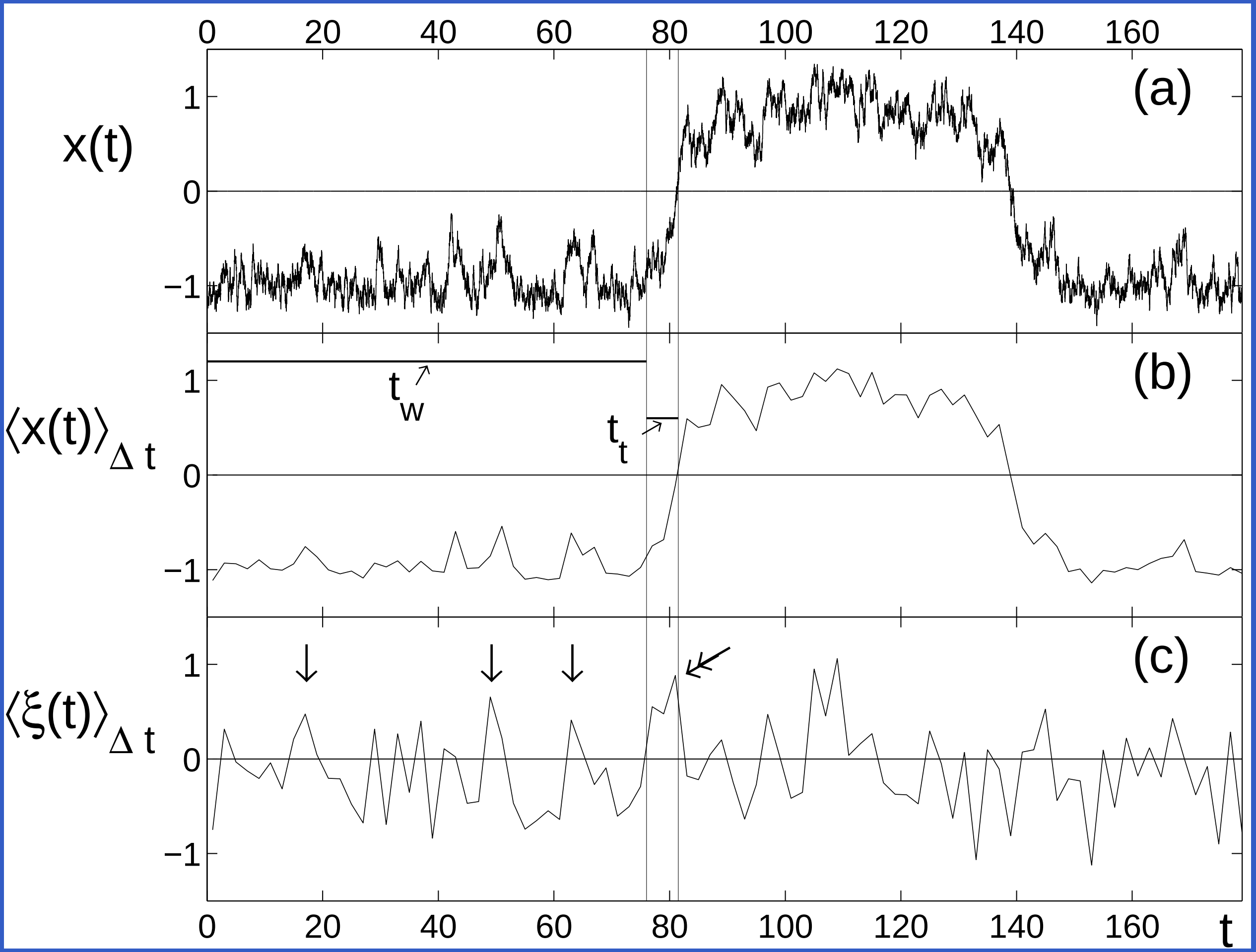
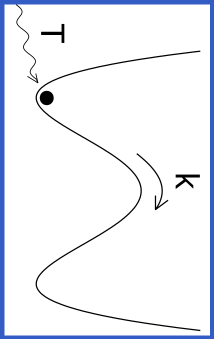
values	$\{-\sqrt{D}, +\sqrt{D}\}$	$(-\infty, \infty)$
differential equation	$\frac{dz_\pm}{dt} = -\gamma z_\pm + \gamma z_\mp$ $\gamma = 1/(2\tau)$	$\frac{dz}{dt} = -\frac{1}{\tau} z + \frac{\sqrt{2D}}{\tau}\eta(t)$ $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$
evolution operator	$\mathsf{L}_z = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$	$\mathsf{L}_z = \frac{\partial}{\partial z} z + D \frac{\partial^2}{\partial z^2}$
stationary distribution	$W_0(z) = \begin{cases} \frac{1}{2} & \text{for } z = +\sqrt{D} \\ \frac{1}{2} & \text{for } z = -\sqrt{D} \end{cases}$	$W_0(z) = \frac{1}{\sqrt{2\pi D}} \exp(-z^2/2D)$

limit	behaviour	survival
$\tau \to 0$	$C(t) \to Q\,\delta(t)$	$Q(\tau) \to$ const $\neq 0$
$\tau \to \infty$	$C(t) \to D$	$D(\tau) \to$ const $\neq 0$

Single escape event





$$\frac{dx}{dt} = -U'(x) + \xi(t)$$

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$$

$$\mathcal{T} = t_w + t_t$$

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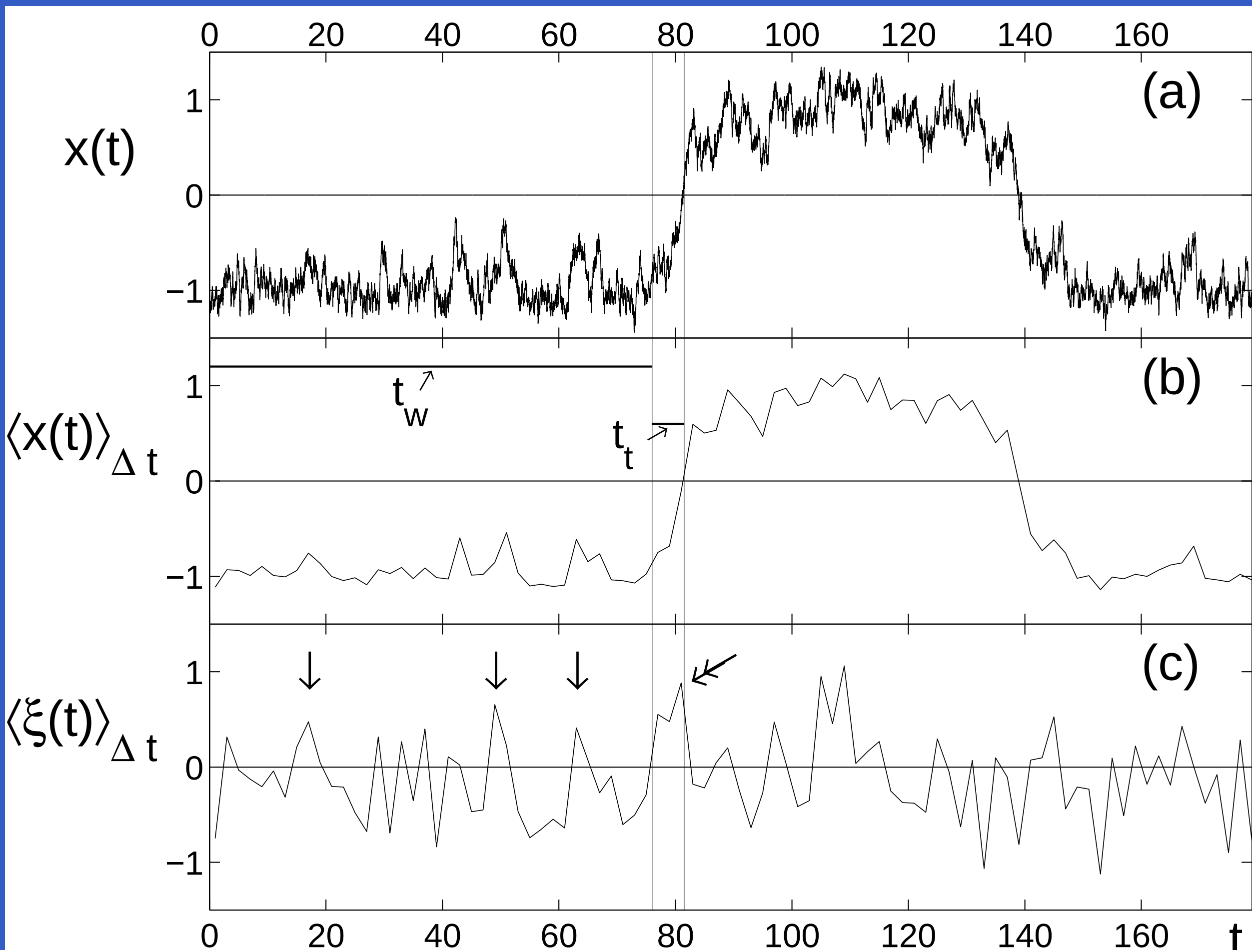

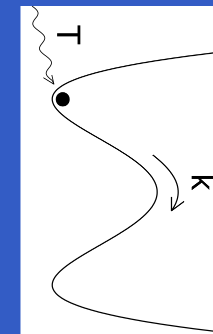

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If the potential variations influence the escape process

this happens mostly during the time interval t_t.

Idea of Partial Noise–Averaging Method

figs/ftraj.eps

$$z(t) = z_s(t) + z_f(t) \qquad \begin{cases} z_s(t) & \text{slow component – \textit{constant} within } t_t \\ z_f(t) & \text{fast component – \textit{white noise} within } t_t \end{cases}$$

$$z_s(t_0) \equiv \left\langle \frac{1}{t_t} \int_{t_0}^{t_0+t_t} ds\, z(s) \right\rangle$$

$\langle \ldots \rangle$– noise realisations

PNAM for OUN

$$z_s(t_0) = \frac{1}{\Delta}\left(1 - e^{-\Delta}\right)\, z_0 \qquad \text{where} \quad \Delta = t_t/\tau$$

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$z(t) = z_s(t) + z_f(t)$
independent gaussian noises with correlation time τ

$$\begin{cases} z_f & \text{exists for } \tau \lesssim t_t \\ z_s & \text{exists for } \tau \gtrsim t_t \end{cases}$$

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What is t_t?

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What is t_t? For equilibrium systems t_t=t_r, so one can estimate $t_r = \dfrac{(\Delta x)^2}{\Delta U}$, or take $t_r =$MFPT$(x_{thr} \rightarrow x_{min})$.

3D Fokker–Planck equation

$$\frac{\partial}{\partial t} P(x, y_f, y_s, t) = L(x, y_f, y_s) P(x, y_f, y_s, t)$$

normalized noise: $z(t) = \sqrt{D(\tau)}\, y(t)$

$$L(x, y_f, y_s) = \underbrace{L_0(x) + \sqrt{\frac{Q_f(\tau)}{\tau}}\, y_f L_b(x) + \sqrt{\frac{Q_s(\tau)}{\tau}}\, y_s L_b(x) + \frac{1}{\tau} L_{y_f}(y_f)} + \frac{1}{\tau} L_{y_s}(y_s)$$

$$\frac{\partial P}{\partial t_0} + \frac{1}{\tau}\frac{\partial P}{\partial t_1} = \left[\Lambda(x, y_f; y_s) + \frac{1}{\tau} L_{y_s}(y_s)\right] P$$

$$P \equiv P(x, y_f, y_s, t_0, t_1) \xrightarrow[t_0 \to \infty]{} \rho(y_s, t_1) P_{st}(x, y_f; y_s)$$

$$\Lambda(x, y_f; y_s) P_{st}(x, y_f; y_s) = 0 \text{ with } \mathcal{U}(x, y_s) \equiv U(x) + \sqrt{D_s(\tau)}\, y_s V(x)$$

$$\frac{1}{\tau}\frac{\partial}{\partial t_1} \varrho(y_s, t_1) = -\mathcal{J}(y_s, t_1; x_{thr}) + \frac{1}{\tau} L_{y_s}(y_s) \varrho(y_s, t_1) \approx \left[\frac{1}{\tau} L_{y_s}(y_s) - k(y_s)\right] \varrho(y_s, t)$$

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mean first passage time:

$$-1 \;=\; \mathcal{L}^{+}(x;y_s)\mathcal{T}(x;y_s)$$

$$\mathcal{T}(x;y_s) = \int_x^{x_{thr}} du\, \frac{1}{\sqrt{G(u;y_s)}} \frac{1}{\Psi(u;y_s)} \int_{-\infty}^{u} dv\, \frac{1}{\sqrt{G(v;y_s)}} \Psi(v;y_s)$$

$$\Psi(x;y_s) = \exp\left(- \int^x du\, \frac{\mathcal{U}'_{eff}(u;y_s)}{G(u;y_s)} \right)$$

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kinetic rate

$$k(y_s) \sim \{\mathcal{T}(x_{in};y_s)\}^{-1}$$

Kinetic approach

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- DN: exact (trivial) solution

$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho_+\\ \varrho_-\end{pmatrix} = \begin{pmatrix}-\gamma - k_+ & \gamma \\ \gamma & -\gamma - k_-\end{pmatrix}\begin{pmatrix}\varrho_+\\ \varrho_-\end{pmatrix}, \quad \gamma = \frac{1}{2\tau}$$

$$\mathcal{T}(\tau) = \frac{2\mathcal{T}_+\mathcal{T}_- + \tau(\mathcal{T}_+ + \mathcal{T}_-)}{\mathcal{T}_+ + \mathcal{T}_- + 2\tau}$$

Kinetic approach

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- OUN: PROBLEMS!!!

$$\frac{\partial}{\partial t}\varrho(y_s,t) = \left[\frac{1}{\tau}L_{y_s}(y_s) - k(y_s)\right]\varrho(y_s,t)$$

Solutions known only for limiting cases $\tau \to 0$ and $\tau \to \infty$.

WANTED

solution of the simplest case

$$\frac{\partial}{\partial t}\varrho(y,t) = \left[\frac{1}{\tau}\left(\frac{\partial}{\partial y}y + \frac{\partial^2}{\partial y^2}\right) - k_0\,\mathrm{e}^{-\Delta V y}\right]\varrho(y,t)$$

diffusion in the parabolic potential with exponentially distributed sink

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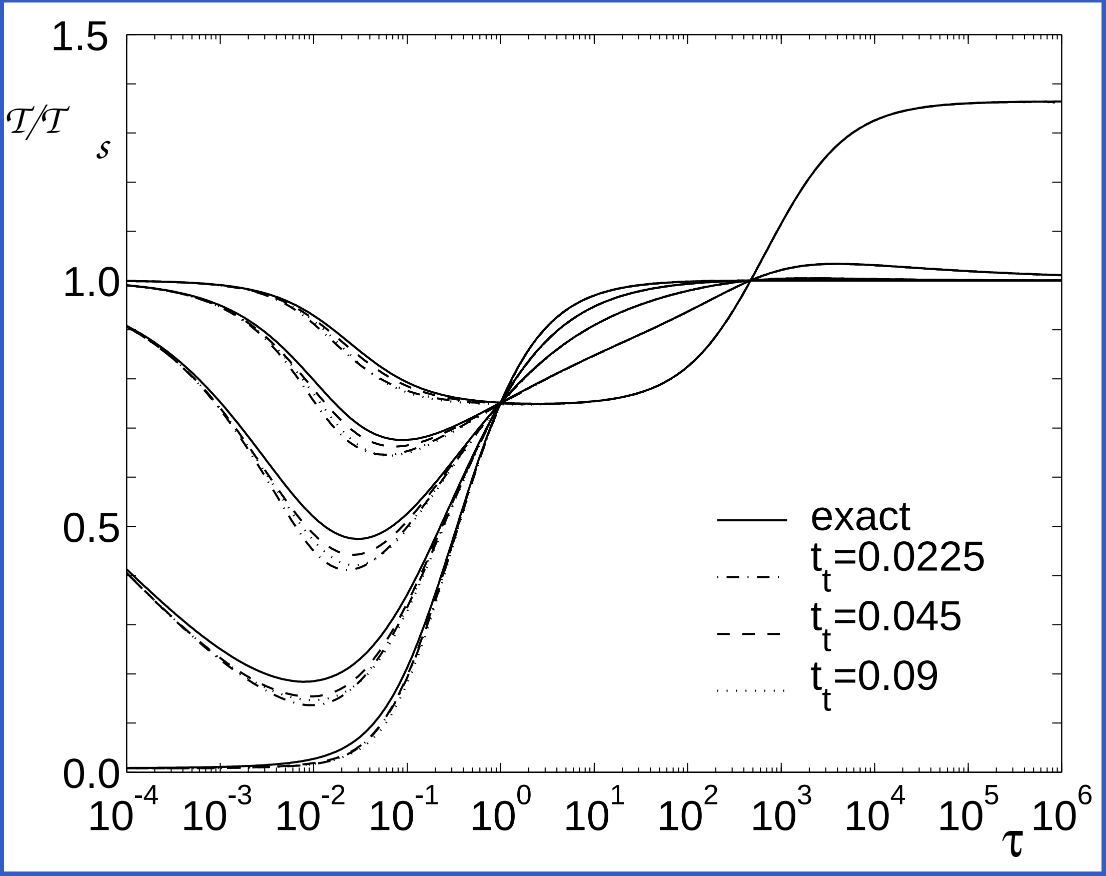
diffusion in the parabolic potential with exponentially distributed sink

REWARD

100's citations

PNAM at work I

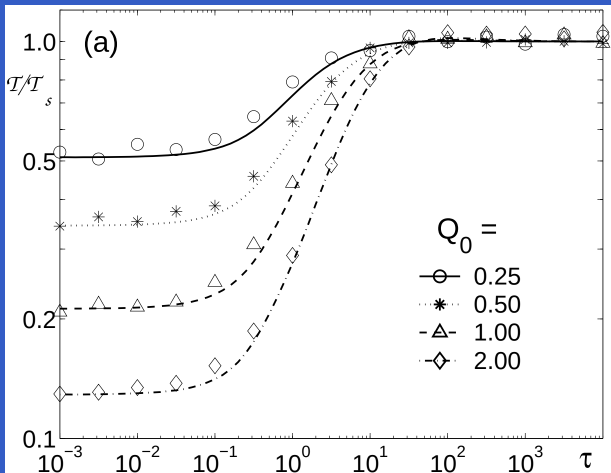
Doering–Gadoua problem — exact results for triangle barrier driven by DN

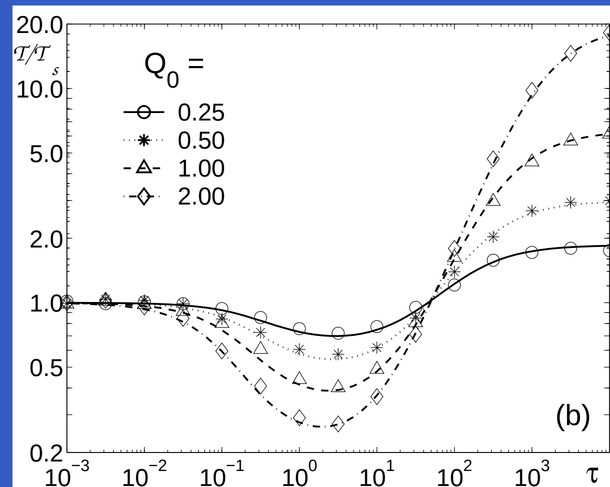


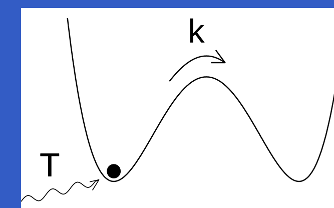
$$Q(\tau) = Q_0 \tau^{\alpha}\,, \quad 0 \leq \alpha \leq 1$$

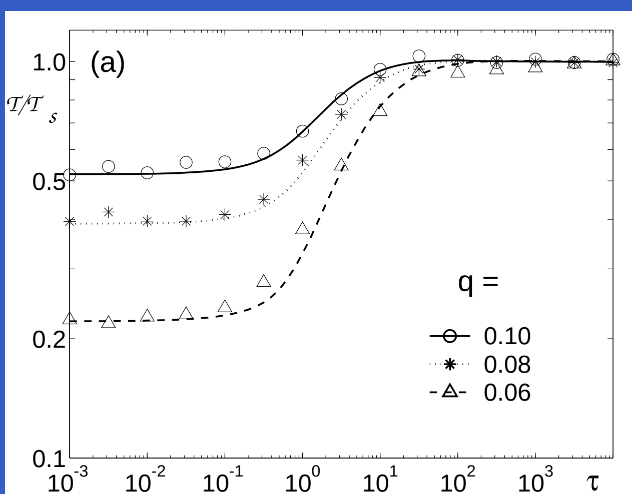
$$\alpha = 0, 0.25, 0.5, 0.75, 1.0$$

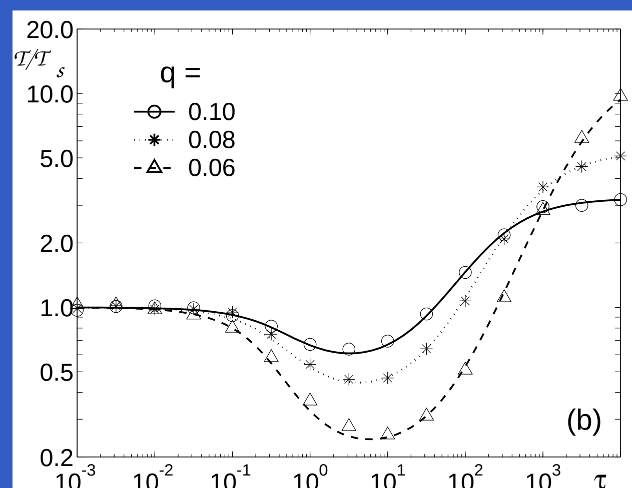
PNAM at work II

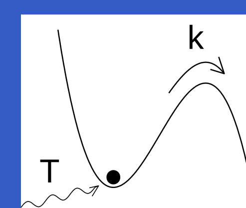




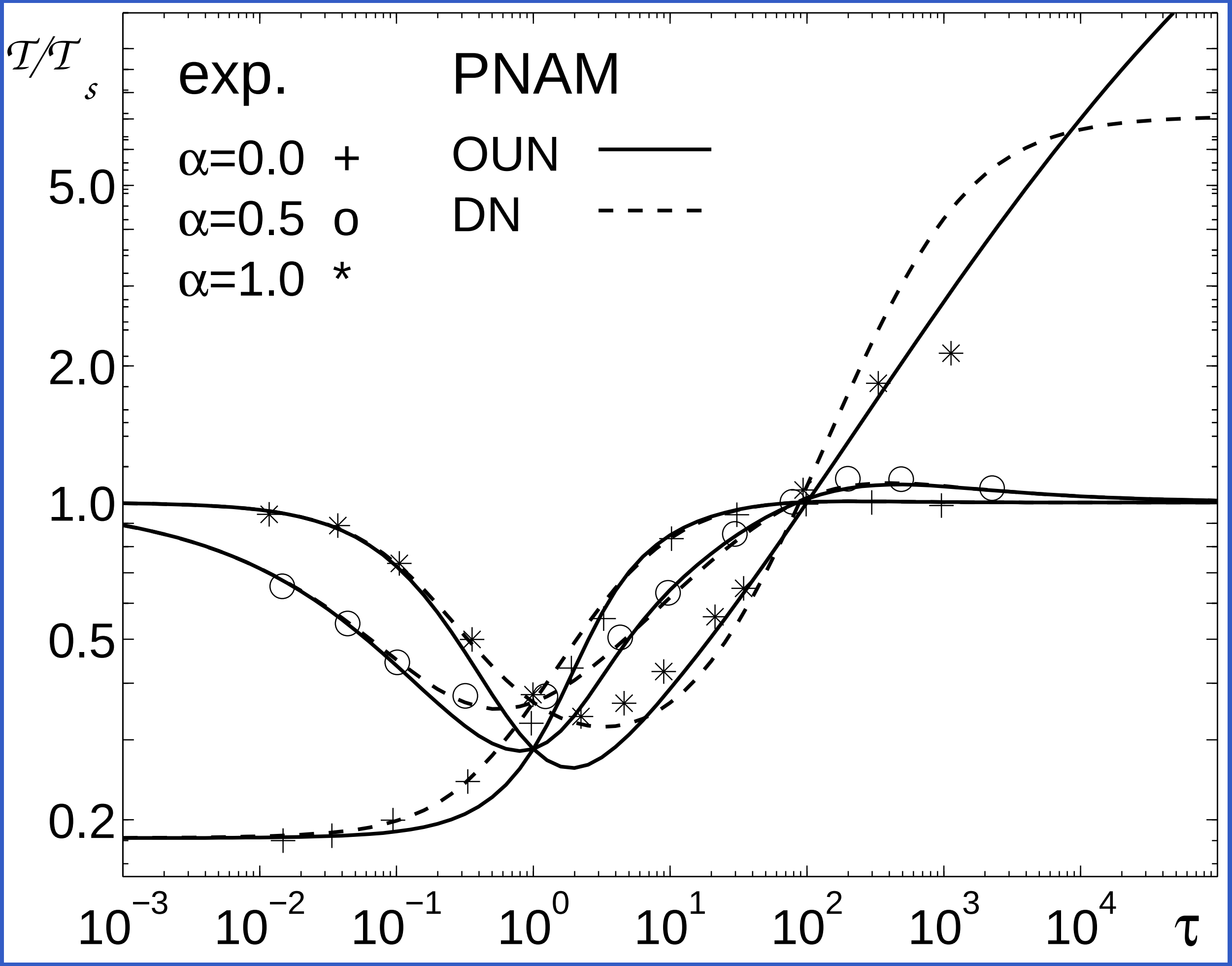


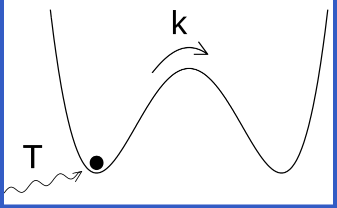




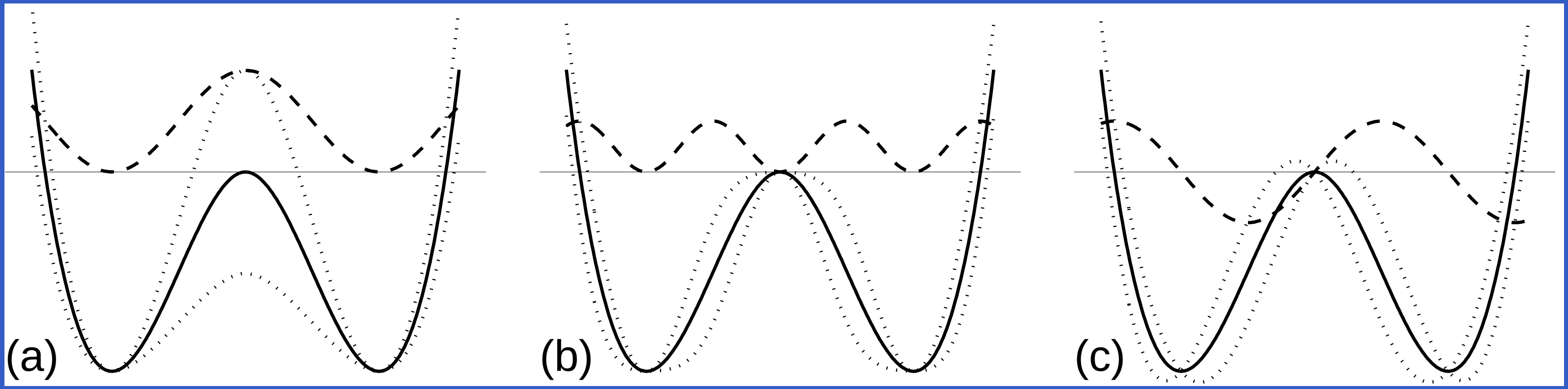


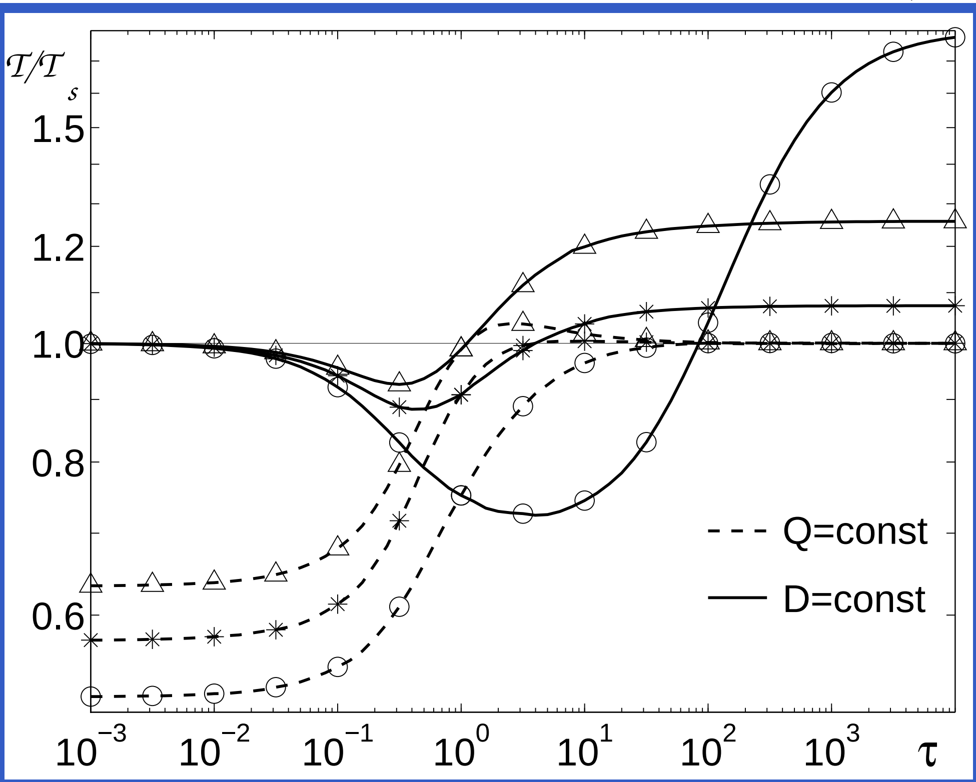
PNAM at work III





PNAM at work IV





What is in resonance?

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maxima	system dependent	(non-resonant) inhibition of activation
crossing points	$\tau \sim \mathcal{T}_s$	depend on the perturbation

Conclusions

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  1. consider all the processes
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THE END