List of colloquium tasks

1. Prove the property of cyclicity of the trace.

2. Show, that the HS inner product is invariant under the action of a unitary group.

3. Show, that the HS inner product of two density matrices given by their Bloch ball coordinates $\vec{r}_1 = [x_1, y_1, z_1]$,
   $\vec{r}_2 = [x_2, y_2, z_2]$ is $\frac{1}{2} + \frac{1}{2}\vec{r}_1 \cdot \vec{r}_2$.

4. Show, that a pair of projectors onto orthogonal subspaces is related to antipodal points on the Bloch sphere.

5. Show, that if a density matrix has two decompositions $\rho = \sum_i \alpha_i |\phi_i\rangle \langle\phi_i| = \sum_i \beta_i |\psi_i\rangle \langle\psi_i|$, then $\sqrt{\beta_i}\psi_i = \sum_j a_{ji}\sqrt{\alpha_i}\phi_j$, and the numbers a_{ji} are entries of a rectangular matrix A with the property$A \cdot A^\dagger = I$.

6. Find the image of the channel (bit flip channel): $\rho \mapsto p\rho + (1-p)\sigma_x\rho\sigma_x^\dagger$.

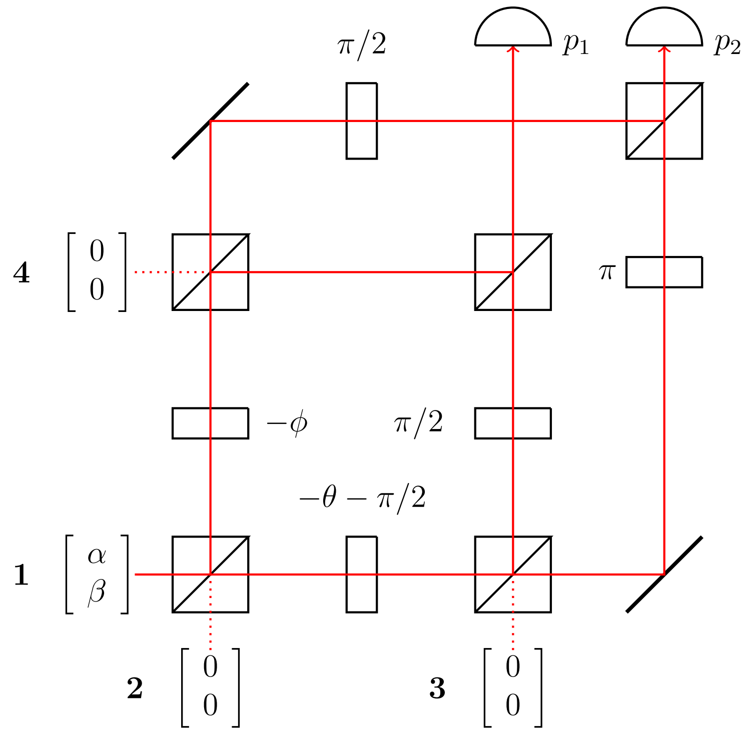
7. What is the Krauss representation of the channel, which scales the Bloch ball in directions x, y, leaving the direction z unchanged (phase flip channel)?

8. What is the Krauss representation of the channel, which scales the Bloch ball uniformly (depolarising channel)?

9. Derive the formula for the maximal probability of successful distinguishing of two pure states $|\Psi_1\rangle \langle\Psi_1|$, $|\Psi_2\rangle \langle\Psi_2|$, send with probabilities p_1, p_2.

10. A source produce two fixed, non-necessarily orthogonal pure qubit states $|\Psi_1\rangle \langle\Psi_1|$ and $|\Psi_2\rangle \langle\Psi_2|$. POVM has three results: 1, 2 i ? and has such a property, that if we transmit $|\Psi_1\rangle \langle\Psi_1|$, we can obtain only 1 or ?, and if $|\Psi_2\rangle \langle\Psi_2|$ is being send, then only 2 or ?. Find the matrices of effects for the POVM.

11. Analyse the system on the picture



Derive the formulas for state vectors on the inputs of detectors:

$$\Psi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad \Psi_1 = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{bmatrix} |\Psi\rangle\langle\Psi| \qquad \Psi_1 = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\phi \end{bmatrix} |\Psi\rangle\langle\Psi|$$

12. How using the above system and one-qubit gates one can construct any qubit POVM?

13. Show, that the CHSH inequality is not violated in separable states.

14. Assume, that:

$$p(ab|AB) = \left[\begin{array}{c|c|c|c} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \tag{1}$$

Show, that such a matrix of conditional probabilities is non-signaling. What is the value of the LHS of the CHSH inequality for such probabilities? How the above matrix should be modified to obtain a signaling distribution?

15. Let the sender and the receiver share a pair of qubits in the pure state $\alpha|00\rangle + \beta|11\rangle$, where $\alpha \geqslant \beta$. Construct a POVM, which with maximal probability will transform the teleported (unnormalised) state $\alpha\gamma|00\rangle + \beta\delta|11\rangle)$ to desired form, as in the ideal teleportation: $(\gamma|00\rangle + \delta|11\rangle)/sqrt2$. What is the value of the probability of success?

16. Show, that the positive partial transpose criterion detects all entangled pure states.

17. Find the subset of separable states in the simplex of states diagonal in the magical basis of two qubits.

List of exam questions

1. Define the convex set, what are the extreme points? What the Caratheodory's theorem says?

2. What are states and observables in classical and quantum probability calculus? How do we define the projective and the generalised measurement in the classical and the quantum probability calculus?

3. How do we obtain states of subsystems in the classical and quantum probability calculus?

4. What is a generator of evolution, equation of evolution and dynamical group in classical and quantum mechanics?

5. Show graphically a decomposition of a qubit density matrix into a combination of rank-one projectors. Which of decompositions is the spectral decomposition? Describe graphically a projective measurement of an observable of a fixed eigenbasis. What the probabilities of outcomes are related to? Describe graphically the uncertainty relation for qubit.

6. What is the Krauss representation of a quantum channel? How to check, whether two representations are related to the same channel?

7. What is the Schmidt decomposition of an element of tensor product of two Hilbert spaces? What is the maximal length of the decomposition?

8. How do we define separable and entangled states?

9. Show, that any channel can be written as$\rho \mapsto \mathrm{Tr}_1(U\eta \otimes \rho U^\dagger)$ for a unitary U.

10. What is the state of the system after POVM measurement? How do we realise a POVM measurement as a projective measurement on an ancilla system after a time of joint unitary evolution?

11. What is the probability of distinguishing of two states ρ_1 and ρ_2 send with probabilities p_1 and p_2?

12. Show the position of various polarisations on the Poincaré sphere.

13. Prove the non-clonning theorem. Which classical states can be cloned? How the cloning mashine acts on the rest of classical states?

14. Describe the key-sharing protocol BB84. How do we detect the eavesdropping in the protocol? What are the problems with the present implementations of the protocol?

15. Prove the CHSH inequality under the assumption of existence of probability space for the experiment.

16. Formulate the *non-signaling* condition. What is the maximal value, the LHS of the CHSH inequality can attain in quantum mechanics and in general non-signaling theory?

17. Describe the teleportation protocol. What is the probability of valid teleportation of qubit state, if we use a non-maximally entagled pair?

18. Describe the partial transposition criterion.