The square of the Vandermonde determinant and its q-generalisation

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Abstract. The Vandermonde determinant plays a crucial role in the Quantum Hall Effect via Laughlin's wavefunction ansatz. Herein the properties of the square of the Vandermonde determinant as a symmetric function are explored in detail. Important properties satisfied by the coefficients arising in the expansion of the square of the Vandermonde determinant in terms of Schur functions are developed and generalised to q-dependent coefficients via the q-discriminant. Algorithms for the efficient calculation of the q-dependent coefficients as finite polynomials in q are developed. The properties, such as the factorisation of the q-dependent coefficients are exposed. Further light is shed upon the vanishing of certain expansion coefficients at $q = 1$. The q-generalisation of the sum rule for the squares of the coefficients is derived. A number of compelling conjectures are stated.

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1. Introduction

The Vandermonde determinant plays a crucial role in the Quantum Hall Effect via Laughlin's wavefunction ansatz[1] and in the description of One Component Plasmas (Tellez and Forrester[2]). This has resulted in considerable interest in the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions for N particles (Dunne[3], Di Francesco *et al*[4]). It is the *even* powers of the Vandermonde determinant that play the crucial role in determining the coefficients of the expansion of the Laughlin wavefunction as a linear combination of Slater determinantal wavefunctions. Indeed, the relevant coefficients are directly related to the signed integers that arise in the expansion of the even powers of the Vandermonde alternating function into Schur symmetric functions (Dunne[3], Di Francesco *et al*[4], Scharf *et al*[5]). The primary problem is to determine the signed integers for the second power, higher powers follow by application of the Littlewood-Richardson rule, see for example Macdonald[6]. Added interest in this problem is the realisation that the expansion of the even powers of the Vandermonde alternating function into Schur functions is directly related to the theory of Hankel's hyperdeterminants (Luque and Thibon[7]). Throughout we follow the standard combinatorial notation defined by Macdonald[6].

The Schur functions that arise in the expansion of the second power of the Vandermonde determinant are indexed by partitions, (λ), of the integer $n = N(N-1)$. Di Francesco *et al*[4] defined a class of *admissible partitions*, as those partitions of n thought to be associated with non-zero expansion coefficients, c^λ, and determined their number, $A(N)$, for all $N \leq 29$. They conjectured that these numbers would be the exact number of non-vanishing coefficients for every value of N *provided none of the coefficients accidentally vanished.* Scharf *et al*[5] developed algorithms for calculating the coefficients and computed them for all $N \leq 9$ and found departures from the conjectured numbers of non-vanishing coefficients for $N \geq 8$. Recently we have extended these calculations to $N = 10$.

In this paper we first recall some of the basic properties of the Laughlin wavefunction and the formal definition (Di Francesco *et al*[4] of admissible partitions. Having identified some of the important properties satisfied by the coefficients arising in the expansion of the square of the Vandermonde determinant in terms of Schur functions, their generalisation to q-dependent coefficients is introduced through a consideration of the $q-$discriminant. Algorithms for the evaluation of the $q-$dependent polynomial coefficients, $c_N^\lambda(q)$, arising from the $q-$discriminant are developed and applied. A further refinement of the algorithm greatly reduces the amount of overcounting leading to a substantive gain in calculation times for larger values of N. Properties of the polynomials $c_N^\lambda(q)$ are next considered with particular emphasis on their factorisation. This leads naturally to the consideration of explicit $N-$dependent results. Several specific results are given. Particular values of q gives further insight into the properties of the $c_N^\lambda(q)$ polynomials and clarifying, and extending, some of the earlier observations of Dunne for $q=1$. The $q-$polynomials associated with the vanishing of coefficients in the case of $q=1$ for $N=8,9$ are shown to all contain a factor of $(q-1)^4$. Finally, the $q-$extension of the remarkable sum rule derived by Di Francesco *et al*[4] for the sum of the squares of the coefficients of the square of the Vandermonde determinant is obtained. Evidence for the existence of a sum rule for the coefficients $c_N^\lambda(1)$ is given.

As a result of their ubiquitous nature in our q-dependent formulae, it is convenient to set out here the relevant notation for q-numbers. This is such that for any q and any positive integer m we have

$$[m]_q = \frac{1-q^m}{1-q} = (1+q+q^2+\cdots+q^{m-1}) \quad \text{so that} \quad \lim_{q\to 1}\,[m]_q = m. \tag{1.1}$$

In addition

$$\begin{aligned} [m]!_q &= [m]_q\ [m-1]_q\ [m-2]_q\ \cdots\ [1]_q \\ [m]!!_q &= [m]_q\ [m-2]_q\ [m-4]_q\ \cdots\ [m^{(2)}]_q; \\ [m]!!!_q &= [m]_q\ [m-3]_q\ [m-6]_q\ \cdots\ [m^{(3)}]_q, \end{aligned} \tag{1.2}$$

where $m^{(r)}$ is the residue of m mod r for any positive integer r. Of course, where appropriate q may be replaced by any positive power p of q to give

$$[m]_{q^p} = \frac{1-q^{mp}}{1-q^p} = (1+q^p+q^{2p}+\cdots+q^{(m-1)p}), \tag{1.3}$$

along with the obvious generalisations of the q-factorial formulae.

2. The Laughlin wavefunction and admissibility conditions

Laughlin[1] has described the fractional quantum Hall effect in terms of a wavefunction

$$\Psi^m_{Laughlin}(\mathbf{x}) = \left(\prod_{1\le i<j\le N} (x_i-x_j)^{2m+1}\right) \exp\left(-\tfrac{1}{2}\textstyle\sum_{i=1}^N |x_i|^2\right), \tag{2.1}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ and m and N are positive integers. The Vandermonde determinant, $V_N(\mathbf{x})$, is the alternating function of N variables $x_1,\ldots,x_N$ defined by

$$V_N(\mathbf{x}) = \left|x_i^{N-j}\right|_{1\le i,j\le N} = \prod_{1\le i<j\le N} (x_i-x_j). \tag{2.2}$$

In terms of the this function we have

$$\Psi^m_{Laughlin}(\mathbf{x}) = V_N^{2m}(\mathbf{x})\ \Psi^0_{Laughlin}(\mathbf{x}). \tag{2.3}$$

Since any even power of the alternating function $V_N(\mathbf{x})$ is a symmetric function of the variables $x_1,x_2,\ldots,x_N$ it follows that

$$\Psi^m_{Laughlin}(\mathbf{x})\Big/\Psi^0_{Laughlin}(\mathbf{x}) = V_N^{2m}(\mathbf{x}) = \sum_{\lambda\vdash n} c_N^{m;\lambda}\ s_\lambda(\mathbf{x}), \tag{2.4}$$

where $s_\lambda(\mathbf{x})$ is the Schur function of the variables $x_1, x_2, \ldots, x_N$, sometimes denoted more simply by $\{\lambda\}$, and the summation is carried out over all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of weight $|\lambda| = n = mN(N-1)$ and length $\ell(\lambda) = p$. The coefficients $c_N^{m;\lambda}$ appearing in (2.4) are all integers; positive, zero or negative. A necessary, but not sufficient, condition for $c_N^{m;\lambda}$ to be non-zero is that

$$N - 1 \leq \ell_\lambda \leq N. \tag{2.5}$$

In most of what follows we consider the case $m = 1$:

$$\Psi^1_{Laughlin}(\mathbf{x}) \Big/ \Psi^0_{Laughlin}(\mathbf{x}) = V_N^2(\mathbf{x}) = \sum_{\lambda \vdash n} c_N^\lambda \; s_\lambda(\mathbf{x}), \tag{2.6}$$

where it has been convenient to set $c_N^\lambda = c_N^{1;\lambda}$. In this case the partitions λ indexing the Schur functions are of weight $n = N(N-1)$. Moreover, with respect to the usual reverse lexicographic ordering of partitions, for a given N the partitions λ in (2.5) for which c_N^λ is non-vanishing are bounded by a highest partition $(2N-2, 2N-4, \ldots, 0)$ and a lowest partition $((N-1)^N)$.

It is of considerable interest to know more generally for what partitions λ the coefficients c_N^λ are non-vanishing. In this connection it is helpful to introduce, following Di Francesco *et al*[4] the notion of *admissible partitions*:

Definition 2.1 *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ *be a partition of weight* $|\lambda|$ *and length* $\ell(\lambda)$*, and let*

$$a_{N,k}(\lambda) = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) \quad \textit{for } k = 0, 1, \ldots, N-1. \tag{2.7}$$

Then λ *is said to be* N*-admissible if* $\ell(\lambda) \leq N$ *and*

$$\begin{aligned} &a_{N,k}(\lambda) \geq 0 \quad \textit{for } k = 0, 1, \ldots, N-2; \\ &a_{N,k}(\lambda) = 0 \quad \textit{for } k = N-1. \end{aligned} \tag{2.8}$$

The set of all N*-admissible partitions is denoted by* $\mathcal{A}_N$*.*

It should be noted that the condition $a_{N,N-1}(\lambda) = 0$ is just $|\lambda| = N(N-1)$. Using this in the condition $a_{N,N-2}(\lambda) \geq 0$ gives $|\lambda| - \lambda_1 - (N-2)(N-1) = 2(N-1) - \lambda_1 \geq 0$ so that $\lambda_1 \leq 2N-2$. The $k = 0$ and $k = 1$ conditions $a_{N,0}(\lambda) \geq 0$ and $a_{N,1}(\lambda) \geq 0$ give $\lambda_N \geq 0$ and $\lambda_N + \lambda_{N-1} \geq 2$, respectively. Together with the constraint $\ell(\lambda) \leq N$ these imply that $\ell(\lambda) = N$ or $N-1$, as in (2.5).

As mentioned earlier, Di Francesco *et al*[4] had conjectured that $c_N^\lambda \neq 0$ if and only if λ is N-admissible. This would imply that the number of non-vanishing coefficients c_N^λ appearing in the expansion (2.5) should be equal to the number, $A_N = \#\{\mathcal{A}_N\}$, of N-admissible partitions. However, it has been found (Scharf *et al*[5] that there exist N-admissible partitions λ such that $c_N^\lambda = 0$. In fact for $N = 8$, 9 and 10 the numbers of N-admissible partitions associated with *vanishing coefficients* are found to be

$$N = 8: \; 8, \qquad N = 9: \; 66, \qquad N = 10: \; 389. \tag{2.9}$$

In order to gain further insight into the occurrence of vanishing coefficients and to obtain additional information regarding the coefficients in general we turn shortly to the $q-$discriminant.

Before doing this we would just point out two important properties of c_N^λ that have been established (Di Francesco *et al*[4]. To this end it is helpful to introduce one more Definition and two Lemmas.

Definition 2.2 *For each N-admissible partition λ, the reverse partition $\lambda^{(r)} = (2N-2)^N/\lambda$ is the complement of λ in $(2N-2)^N$. That is, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$*

$$\lambda^{(r)} = (2N-2-\lambda_N, \ldots, 2N-2-\lambda_2, 2N-2-\lambda_1). \tag{2.10}$$

With this definition we have

Lemma 2.3 *If λ is N-admissible then so is $\lambda^{(r)}$.*

Proof If λ is N-admissible then $\ell(\lambda) \leq N$ and $\lambda_i \leq \lambda_1 \leq 2N-2$ for $i = 1, 2, \ldots, N$. Thus $\lambda^{(r)}$ is well defined and has length $\ell(\lambda^{(r)}) \leq N$. In addition $|\lambda^{(r)})| = N(2N-2) - |\lambda| = N(N-1)$ since $|\lambda| = N(N-1)$. It follows that $a_{N,N-1}(\lambda^{(r)}) = 0$, as required. Furthermore, for $m = 0, 1, \ldots, N-2$ we have

$$\begin{aligned} a_{N,m}(\lambda^{(r)}) &= \sum_{j=0}^{m} \lambda^{(r)}_{N-j} - m(m+1) = \sum_{j=0}^{m}(2N-2-\lambda_{j+1}) - m(m+1) \\ &= -\sum_{j=0}^{m} \lambda_{j+1} + (m+1)(2N-2-m) \\ &= \sum_{j=m+1}^{N-1} \lambda_{j+1} - N(N-1) + (m+1)(2N-2-m) \\ &= \sum_{i=0}^{k} \lambda_{N-i} - N(N-1) + (N-k-1)(N-k) \\ &= \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) = a_{N,k}(\lambda) \geq 0, \end{aligned} \tag{2.11}$$

where we have set $i = N-j-1$ and $k = N-m-2$, so that k takes the values $0, 1, \ldots, N-2$, as required to complete the proof. •

It has been observed (Dunne[3]) that the most striking property of the expansion coefficients c_N^λ of (2.6) is that they exhibit *reversal symmetry*, in the sense that

$$c_N^\lambda = c_N^{\lambda^{(r)}}. \tag{2.12}$$

This is the $s=1$ Property 0 (ii) of Di Francesco *et al*[4]

These authors also give as their Property 5 an important factorisation result linked to the vanishing of one of the admissibility parameters $a_{N,k}(\lambda)$. In this context it is important to note the following:

Lemma 2.4 *For any positive integers M and N, the partition λ is $(M+N)$-admissible with $a_{M+N,N-1}(\lambda)=0$ if and only if there exist partitions $\mu\in\mathcal{A}_M$ and $\nu\in\mathcal{A}_N$ such that $\lambda=((2N)^M+\mu,\nu)$. That is*

$$\lambda_i=\begin{cases}2N+\mu_i & \text{for } i=1,2,\ldots,M;\\ \nu_{i-M} & \text{for } i=M+1,M+2,\ldots,M+N.\end{cases} \tag{2.13}$$

Proof For any λ with $\ell(\lambda)\le M+N$ the last N parts of λ define a partition ν of length $\ell(\nu)\le N$ with $\nu_j=\lambda_{M+j}$ for $j=1,2,\ldots,N$, as in (2.13). The condition $a_{M+N,N-1}(\lambda)=0$ then implies that $|\nu|=N(N-1)$, using this in $a_{M+N,N-1}(\lambda)\ge 0$ gives $\lambda_M\ge 2N$. It follows that the first M parts of $\lambda-(2N)^M$ defines a partition μ with $\mu_i=\lambda_i-2N$ for $i=1,2,\ldots,M$, again as in (2.13). Conversely, if μ and ν are M-admissible and N-admissible, respectively, then $\ell(\mu)\le M$, $\ell(\nu)\le N$ and $\nu_1\le 2N-2$. Thus $\lambda=((2N)^M+\mu,\nu)$ is well defined and has length $\ell(\lambda)\le M+N$.

Moreover, for $k=0,1,\ldots,N-1$ we have

$$a_{N,k}(\nu)=\sum_{i=0}^{k}\nu_{N-i}-k(k+1)=\sum_{i=0}^{k}\lambda_{M+N-i}-k(k+1)=a_{M+N,k}(\lambda). \tag{2.14}$$

This implies that ν is N-admissible if and only if the first N of the $(M+N)$-admissibility conditions for λ are satisfied, along with the constraint $a_{M+N,N-1}(\lambda)=0$.

Finally, for $m=0,1,\ldots,M-1$ we have

$$\begin{aligned}a_{M,m}(\mu)&=\sum_{j=0}^{m}\mu_{M-j}-m(m+1)\\&=\sum_{j=0}^{m}\lambda_{M-j}-(m+1)(2N+m)\\&=\sum_{i=N}^{k}\lambda_{M+N-i}-k(k+1)+N(N-1)\\&=a_{M+N,k}(\lambda)-a_{M+N,N-1}(\lambda).\end{aligned} \tag{2.15}$$

where we have set $i=j+N$ and $k=m+N$, so that $k=N,N+1,\ldots,M+N-1$. This implies that μ is M-admissible if and only if the last M of the $(M+N)$-admissibility conditions for λ are replaced by $a_{M+N,k}(\lambda)\ge a_{M+N,N-1}(\lambda)$.

However combining (2.14) and (2.15) implies that $\lambda=((2N)^M+\mu,\nu)$ is $(M+N)$-admissible with $a_{M+N,N-1}(\lambda)=0$ if and only if μ and ν are M-admissible and N-admissible, respectively. •

The significance of this is that if λ is $(M+N)$-admissible with $a_{M+N,N-1}(\lambda)=0$, then from the $s=1$ case of Property 5 of Di Francesco *et al*[4] we have, in the notation of the Lemma,

$$c^{\lambda}_{M+N}=c^{\mu}_{M}\ c^{\nu}_{N}. \tag{2.16}$$

It will be shown later that both this result and the reversal symmetry may be generalised in a q-dependent way.

3. The q-generalisation of the square of the Vandermonde determinant

Given $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, the q-discriminant of $\mathbf{x}$ is defined to be

$$D_N(q;\mathbf{x}) = \prod_{1\le i\neq j\le N} (x_i - qx_j) = (-1)^{N(N-1)/2}\ R_N(q;\mathbf{x}) \tag{3.1}$$

where the quantity of particular interest here, $R_N(q;\mathbf{x})$, is given by

$$R_N(q;\mathbf{x}) = \prod_{1\le i<j\le N} (x_i - qx_j)(qx_i - x_j). \tag{3.2}$$

This is a q-generalisation of the square of the Vandermonde determinant in the sense that for $q = 1$ we have

$$R_N(1;\mathbf{x}) = \prod_{1\le i<j\le N} (x_i - x_j)^2 = V_N^2(\mathbf{x}). \tag{3.3}$$

It should be noted from the definition (3.2) that $R_N(q;\mathbf{x})$ is a polynomial in q of degree $q^{N(N-1)}$, and that for all q it is a symmetric function of the components of $\mathbf{x}$.

With the above notation and that of (2.5) we have

$$R_N(1;\mathbf{x}) = V_N^2(\mathbf{x}) = \sum_{\lambda} c_N^\lambda\ s_\lambda(\mathbf{x}). \tag{3.4}$$

The original problem was to evaluate and study the coefficients c_N^λ appearing here. However, one may be more ambitious, and gain additional insight, by seeking to evaluate the coefficients $c_N^\lambda(q)$ appearing in the expansion

$$R_N(q;\mathbf{x}) = \sum_{\lambda\vdash N(N-1)} c_N^\lambda(q)\ s_\lambda(\mathbf{x}), \tag{3.5}$$

and then to recover c_N^λ by setting $q = 1$.

First it should be noted (Macdonald[6]) that for any $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ we have

$$\prod_{1\le i,j\le N} (1 - qx_jy_i) = \sum_{\mu\subseteq N^N} (-q)^{|\mu|}\ s_\mu(\mathbf{x})\ s_{\mu'}(\mathbf{y}), \tag{3.6}$$

where μ' denotes the partition conjugate to μ and $\mu \subseteq N^N$ is equivalent to the restrictions $\ell(\mu') = \mu_1 \le N$ and $\ell(\mu) = \mu_1' \le N$. This identity with $\mathbf{y} = \overline{\mathbf{x}} = (x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1})$ may be exploited as follows to give

$$\begin{aligned} R_N(q;\mathbf{x}) &= (-1)^{N(N-1)/2} \prod_{1\le i,j\le N} (x_i - qx_j) \Big/ \prod_{1\le i\le N} (1-q)x_i \\ &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N}\ \frac{s_{N^N}(\mathbf{x})}{s_{1^N}(\mathbf{x})} \prod_{1\le i,j\le N} (1 - qx_jx_i^{-1}) \\ &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N}\ \frac{s_{N^N}(\mathbf{x})}{s_{1^N}(\mathbf{x})} \sum_{\mu\subseteq N^N} (-q)^{|\mu|}\ s_\mu(\mathbf{x})\ s_{\mu'}(\overline{\mathbf{x}}) \\ &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\mu\subseteq N^N} (-q)^{|\mu|}\ \frac{s_{N^N/\mu'}(\mathbf{x})\ s_\mu(\mathbf{x})}{s_{1^N}(\mathbf{x})}. \end{aligned} \tag{3.7}$$

In the above $s_{1^N}(\mathbf{x}) = x_1x_2\cdots x_N$ and $s_{N^N}(\mathbf{x}) = (x_1x_2\cdots x_N)^N$, and use has been made of the fact that $s_{N^N}(\mathbf{x})s_{\mu'}(\mathbf{x}) = s_{N^N/\mu'}(\mathbf{x})$, where / indicates the usual skew product of Schur functions (Macdonald[6]).

As far as the $\mathbf{x}$-dependence is concerned it should be noted that for any partition λ of length $\ell(\lambda) = N$ we have $s_\lambda(\mathbf{x}) = s_{1^N}(\mathbf{x})s_{\lambda/1^N}(\mathbf{x})$, where $\lambda/1^N$ is the partition obtained from λ by decreasing each of its N parts by 1. It follows, that the numerator of each summand of (3.7) contains a factor $s_{1^N}(\mathbf{x})$ since either $\ell(N^N/\mu') = N$ if $\ell(\mu) < N$ or $\ell(\mu) = N$. In fact taking the terms in pairs, one with $\ell(N^N/\mu') = N$ and one with $\ell(\mu) = N$, gives

$$R_N(q;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu\subseteq N^{N-1}} \left((-q)^{|\nu|} + (-q)^{N^2-|\nu|}\right)\ s_{(N-1)^N/\nu'}(\mathbf{x})\ s_\nu(\mathbf{x}), \quad (3.8)$$

where now $\nu \subseteq N^{N-1}$ is equivalent to the restrictions $\ell(\nu) = \nu'_1 \leq N-1$ and $\ell(\nu') = \nu_1 \leq N$. The product of Schur functions appearing here may be evaluated by means of the Littlewood-Richardson rule (Littlewood[8], Macdonald[6]) which determines the coefficients $c^\lambda_{\mu\,\nu}$ arising in the decomposition of the product

$$s_\mu(\mathbf{x})\ s_\nu(\mathbf{x}) = \sum_\lambda c^\lambda_{\mu\,\nu}\ s_\lambda(\mathbf{x}). \quad (3.9)$$

With this notation, it follows that

$$c^\lambda_N(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu\subseteq N^{N-1}} \left((-q)^{|\nu|} + (-q)^{N^2-|\nu|}\right) c^\lambda_{((N-1)^N/\nu')\,\nu}. \quad (3.10)$$

While (3.8) does not show directly that $R_N(q;\mathbf{x})$ is a polynomial in q, this fact does of course follow from the definition (3.3). As a result the coefficients $c^\lambda_N(q)$ appearing in (3.10) are also polynomials in q. One can say somewhat more since (3.3) can be recast in the form

$$R_N(q;\mathbf{x}) = q^{N(N-1)/2} \prod_{1\leq i<j\leq N} (x_i - qx_j)(x_i - q^{-1}x_j) = \sum_\lambda c^\lambda_N(q)\ s_\lambda(\mathbf{x}). \quad (3.11)$$

This implies that the polynomials $c^\lambda_N(q)$ must be symmetric in the sense that the coefficients of $q^{N(N-1)/2+k}$ and $q^{N(N-1)/2-k}$ are equal for all integers k. Equivalently,

$$c^\lambda_N(q) = q^{N(N-1)} c^\lambda_N(q^{-1}). \quad (3.12)$$

The case $q = 1$ is not the only case of interest. For $q = -1$ we have

$$R_N(-1;\mathbf{x}) = (-1)^{N(N-1)/2} \prod_{1\leq i<j\leq N} (x_i + x_j)^2 \quad (3.13)$$

and

$$R_N(-1;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{2^{N-1}} \sum_{\nu\subseteq(N^{N-1})} s_{(N-1)^N/\nu'}(\mathbf{x})\ s_\nu(\mathbf{x}), \quad (3.14)$$

from (3.3) and (3.8), respectively. Hence

$$c_N^\lambda(-1) = \frac{(-1)^{N(N-1)/2}}{2^{N-1}} \sum_{\nu \subseteq N^{N-1}} c^\lambda_{((N-1)^N/\nu')\,\nu}. \tag{3.15}$$

The significance of these $q=-1$ results lies in the fact that they are related to the Schur function product

$$s_\delta(\mathbf{x})\ s_\delta(\mathbf{x}) = \sum_\lambda c^\lambda_{\delta\,\delta}\ s_\lambda(\mathbf{x}), \tag{3.16}$$

where δ is the so-called staircase partition $\delta = (N-1, N-2, \ldots, 1, 0)$. This staircase partition is such that

$$s_\delta(\mathbf{x}) = \prod_{1\le i<j\le N} (x_i + x_j). \tag{3.17}$$

This formula may be derived from the fact that (Macdonald[6])

$$s_\lambda(\mathbf{x}) = \frac{a_{\lambda+\delta}(\mathbf{x})}{a_\delta(\mathbf{x})} = \frac{\left| x_i^{\lambda_j+N-j} \right|_{1\le i,j\le N}}{\left| x_i^{N-j} \right|_{1\le i,j\le N}}, \tag{3.18}$$

where the denominator $a_\delta(x)$ is nothing other than the Vandermonde determinant $V_N(\mathbf{x})$ which factorises as in (2.2). Hence

$$s_\delta(\mathbf{x}) = \frac{a_{2\delta}(\mathbf{x})}{a_\delta(\mathbf{x})} = \frac{\left| x_i^{2(N-j)} \right|_{1\le i,j\le N}}{\left| x_i^{N-j} \right|_{1\le i,j\le N}} = \prod_{1\le i<j\le N} \frac{(x_i^2 - x_j^2)}{(x_i - x_j)}, \tag{3.19}$$

which reduces to (3.17) as required. The outcome of all this is that

$$R_N(-1;\mathbf{x}) = (-1)^{N(N-1)/2} s_\delta(\mathbf{x})^2 = (-1)^{N(N-1)/2} \sum_\lambda c^\lambda_{\delta\,\delta}\ s_\lambda(\mathbf{x}). \tag{3.20}$$

Now we are in a position to use an important result due to Berenstein and Zelevinsky[9] that applies to the simple Lie algebra $sl(N)$ but which can be readily translated to the case of the reductive Lie algebra $gl(N)$ of interest here.

Theorem 3.1 *Let ρ be half the sum of the positive roots of $sl(N)$, that is $\rho = \delta - \frac{1}{2}(N-1)\eta$ where $\eta = (1,1,\ldots,1)$, and let $\Pi = \{\alpha_p = \epsilon_p - \epsilon_{p+1} \,|\, p = 1,2,\ldots N-1\}$ be the set of simple roots of $sl(N)$, where $\epsilon_i = (0,\ldots,0,1,0,\ldots,0)$ with the 1 appearing as the ith component for $i = 1,2,\ldots,N$. Let $\mathbb{Z}^+$ be the set of non-negative integers. Then the multiplicity $m^\kappa_{\rho\rho}$ of the irreducible representation V^κ in the decomposition of the tensor product $V^\rho \otimes V^\rho$ is such that $m^\kappa_{\rho\rho} > 0$ if and only if*

$$\kappa = 2\rho - \sum_{p=1}^{N-1} g_p\ \alpha_p \quad \textit{with } g_p \in \mathbb{Z}^+ \textit{ for all } \alpha_p \in \Pi. \tag{3.21}$$

In the context of $gl(N)$ this implies:

Corollary 3.2 *Let* $\delta = (N-1, N-2, \ldots, 1, 0)$ *then* $c^{\lambda}_{\delta\,\delta} > 0$ *if and only if* λ *is* N*-admissible.*

Proof. Let V be the defining N-dimensional irreducible representation of $gl(N)$. Then the irreducible constituents V^λ of $V^{\otimes N(N-1)}$ are specified by partitions λ of weight $|\lambda| = N(N-1)$ and length $\ell(\lambda) \le N$. They have character $s_\lambda(\mathbf{x})$ with x_i defined to be the formal exponential e^{ϵ_i} for $i = 1, 2, \ldots, N$. The passage from $gl(N)$ to $sl(N)$ is effected by setting $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_N = 0$, or equivalently $s_\eta(x) = x_1 x_2 \cdots x_N = 1$. Now let $\lambda = \kappa + (N-1)\eta$ and $\delta = \rho + \frac{1}{2}(N-1)\eta = (N-1, N-2, \ldots, 1, 0)$ so that the irreducible representations V^λ and V^δ of $gl(N)$ give on restriction to $sl(N)$ the irreducible representations V^κ and V^ρ, respectively. Then $c^{\lambda}_{\delta\,\delta} = m^{\kappa}_{\rho\,\rho}$, and it follows from Theorem 3.1 that $c^{\lambda}_{\delta\,\delta} > 0$ if and only if

$$\lambda = 2\delta - \sum_{p=1}^{N-1} g_p(\epsilon_p - \epsilon_{p+1}) \quad \text{with } g_p \in \mathbb{Z}^+ \text{ for all } p = 1, 2, \ldots N-1. \tag{3.22}$$

It only remains to show that these conditions (3.22) coincide with the N-admissibility conditions of Definition 2.1. It follows immediately from (3.22) that $\ell(\lambda) \le N$ and $|\lambda| = 2|\delta| = N(N-1)$. The former is required for N-admissibility and the latter is just the admissibility condition $a_{N,N-1}(\lambda) = 0$ of (2.8). In addition, taking the $(N-i)$th component of (3.22) gives

$$\lambda_{N-i} = 2i - g_{N-i} + g_{N-i-1} \quad \text{for } i = 0, 1 \ldots, N-1, \tag{3.23}$$

where it has been convenient to introduce $g_N = 0$. It follows from this that

$$a_{N,k}(\lambda) = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) = g_{N-k-1} \in \mathbb{Z}^+ \quad \text{for } k = 0, 1, \ldots, N-2. \tag{3.24}$$

These are nothing other than the remaining admissibility conditions of (2.8). Thus (3.22) coincides precisely with the N-admissibility conditions of Definition 2.1. •

This result, Corollary 3.2, has a wider significance in that it may be used to derive the following:

Proposition 3.3 *Let* $R_N(q;\mathbf{x})$ *be defined as in (3.3), then*

$$R_N(q;\mathbf{x}) = \sum_{\lambda \in \mathcal{A}_N} c^{\lambda}_{N}(q)\ s_\lambda(\mathbf{x}), \tag{3.25}$$

with $c^{\lambda}_{N}(q)$ *a non-zero polynomial in* q *for each* $\lambda \in \mathcal{A}_N$*.*

Proof We first deal with the case $\lambda \notin \mathcal{A}_N$. Comparing (3.14) and (3.20) reveals that

$$c^{\lambda}_{\delta\,\delta} = \frac{1}{2^{N-1}} \sum_{\nu \in \subseteq (N^{N-1})} c^{\lambda}_{((N-1)^N/\nu')\,\nu}. \tag{3.26}$$

Since there are no cancellations of any kind in this expansion it follows from Corollary 3.2 that for all $\lambda \notin \mathcal{A}_N$ and all $\nu \subseteq N^{N-1}$ we have

$$c^{\lambda}_{((N-1)^N/\nu')\,\nu} = 0. \tag{3.27}$$

This implies in (3.10) that $c^{\lambda}_N(q) = 0$ for all $\lambda \notin \mathcal{A}_N$. This means that, as required in (3.25), we can restrict the summation over λ in the expansion (3.5) to $\lambda \in \mathcal{A}_N$.

Moreover, it follows from (3.20) that

$$c^{\lambda}_N(-1;\mathbf{x}) = (-1)^{N(N-1)/2} c^{\lambda}_{\delta\,\delta}. \tag{3.28}$$

Corollary 3.2 therefore implies that

$$c^{\lambda}_N(-1,\mathbf{x}) \neq 0 \quad \text{if and only if } \lambda \in \mathcal{A}_N. \tag{3.29}$$

This shows that $c^{\lambda}_N(q)$ is a non-vanishing polynomial in q for all $\lambda \in \mathcal{A}_N$, as required.

Of course, as we have indicated in (2.8), $c^{\lambda}_N = c^{\lambda}_N(1) = 0$ cannot be excluded for all admissible λ, by virtue of possible cancellations in (3.10). Such cancellations simply indicate, as we will later exemplify, the presence of at least one factor $(q-1)$ in $c^{\lambda}_N(q)$.

4. An algorithm for the evaluation of $c^{\lambda}_N(q)$

While the formula (3.11) for $c^{\lambda}_N(q)$ is quite explicit it does not provide a very efficient way of calculating these polynomials. This is not only because its implementation requires the decomposition of *outer* products of Schur functions (corresponding to tensor products of irreducible representations of $gl(N)$), of the type $s_\mu(\mathbf{x})\ s_\nu(\mathbf{x})$ with $\mu = N^{N-1}/\nu'$ by means of the Littlewood-Richardson rule, but also because it necessarily involves a considerable degree of overcounting in the numerator so as to cancel the denominator factor $(1-q)^N$. An alternative formula has been provided elsewhere (Scharf *et al*[5]) but its implementation requires the decomposition of *inner* products of Schur functions (corresponding to tensor products of irreducible representations of S_{N^2}). albeit rather special inner products of the form $s_{N^N}(\mathbf{x}) * s_{a+1,1^b}(\mathbf{x})$ for all a and b such that $a+b+1 = N^2$. This becomes a formidable task for all but very small values of N.

However, in addition to these formulae there exists in the case $q = 1$ a recursive algorithm (Scharf *et al*[5]) for evaluating $c^{\lambda}_N = c^{\lambda}_N(1)$ which does not require any decomposition of either outer or inner products of Schur functions. This may be generalised to the case of $c^{\lambda}_N(q)$ as follows.

It is first necessary to introduce the linear operator Ω_N acting in the space P_N of functions $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_N)$ that are polynomial in the components of $\mathbf{x}$. The symmetric group S_N acts naturally on the components of $\mathbf{x}$ and is generated by the transpositions $\sigma_i = (i, i+1)$ for $i = 1, 2, \ldots, N-1$. Their action on $f(\mathbf{x})$ is defined by

$$\sigma_i f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_N) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_N). \tag{4.1}$$

Following Lascoux and Schutzenberger[10], Lascoux[11] and Macdonald[12], the isobaric divided difference operators π_i for $i = 1, 2, \ldots, N-1$ are then defined by

$$\pi_i f(x) = \frac{x_i f(\mathbf{x}) - x_{i+1}\sigma_i f(\mathbf{x})}{x_i - x_{i+1}}. \tag{4.2}$$

If the permutation $\omega_N = (N, N-1, \ldots, 1)$ has the reduced decomposition $\omega_N = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ as a word of minimal length r in the generators σ_i, then Ω_N is defined by

$$\Omega_N = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_N}. \tag{4.3}$$

This operator has a number of important properties (Lascoux and Schutzenbeger[10], Lascoux[11], Macdonald[12]). First for any $f(\mathbf{x})$ and $g(\mathbf{x})$ in P_N such that $f(\mathbf{x})$ is a symmetric function of the components of $\mathbf{x}$ we have

$$\Omega_N(f(\mathbf{x})) = f(\mathbf{x}) \quad \text{and} \quad \Omega_N(f(\mathbf{x})g(\mathbf{x})) = f(\mathbf{x})\Omega_N(g(\mathbf{x})). \tag{4.4}$$

Now consider any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with integer components α_i for $i = 1, 2, \ldots, N$ that are not necessarily weakly decreasing as required for a partition, and may even be negative. Then for $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ let $\mathbf{x}^\alpha$ be the monomial defined by $\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_N^{\alpha_N}$. For any such monomial $\mathbf{x}^\alpha$ we have

$$\Omega_N(\mathbf{x}^\alpha) = s_\alpha(\mathbf{x}), \tag{4.5}$$

where $s_\alpha(\mathbf{x})$ is to be defined as a ratio of determinants as in (3.18) with λ replaced by α. Since α is not necessarily a partition the right hand side of (4.5) may have to be standardised. From the determinantal definition (3.18) it can be seen that (Littlewood[8])

$$s_{\alpha_1,\ldots,\alpha_j,\alpha_{j+1},\ldots,\alpha_N}(\mathbf{x}) = -s_{\alpha_1,\ldots,\alpha_{j+1}-1,\alpha_j+1,\ldots,\alpha_N}(\mathbf{x}) \tag{4.6}$$

for all $j = 1, 2, \ldots, N-1$. The repeated application of (4.6) to $s_\alpha(\mathbf{x})$ will give either zero or $\pm s_\lambda(\mathbf{x})$ for some partition λ.

Finally, if we let $\mathbf{y} = (x_1, x_2, \ldots, x_{N-1})$ so that $\mathbf{x} = (\mathbf{y}, x_N)$, and let λ be a partition of length $\ell(\lambda) < N$ then

$$\Omega_N\left(s_\lambda(\mathbf{y})\ x_N^k\right) = s_{\lambda,k}(\mathbf{x}), \tag{4.7}$$

where $(\lambda, k) = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, k)$. Once again it may be necessary to standardise the right hand side of (4.7) through the repeated application of (4.6).

We may now exploit the operator Ω_N to give a simple derivation of two results linking monomial symmetric functions and Schur functions. For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ let $P(\mu)$ denote the set of all distinct permutations α of the parts of μ. Then the usual monomial symmetric function $m_\mu(\mathbf{x})$ of $\mathbf{x}$ (Macdonald[6]) is given by

$$m_\mu(\mathbf{x}) = \sum_{\alpha\in P(\mu)} \mathbf{x}^\alpha. \tag{4.8}$$

Since $m_\mu(\mathbf{x})$ is a symmetric function it follows from (4.4) and (4.5) that

$$m_\mu(\mathbf{x}) = \Omega_N(m_\mu(\mathbf{x})) = \Omega_N\left(\sum_{\alpha\in P(\mu)} \mathbf{x}^\alpha\right) = \sum_{\alpha\in P(\mu)} \Omega_n(\mathbf{x}^\alpha) = \sum_{\alpha\in P(\mu)} s_\alpha(\mathbf{x}), \tag{4.9}$$

where the final expression may be standardised through the use of (4.6). More generally, (Murnaghan[13]) we have

$$\begin{aligned} m_\mu(\mathbf{x}) s_\lambda(\mathbf{x}) &= m_\mu(\mathbf{x})\Omega_N(\mathbf{x}^\lambda) = \Omega_N(m_\mu(\mathbf{x})\mathbf{x}^\lambda) \\ &= \Omega_N\left(\sum_{\alpha\in P(\mu)} \mathbf{x}^{\lambda+\alpha}\right) = \sum_{\alpha\in P(\mu)} \Omega_N(\mathbf{x}^{\lambda+\alpha}) = \sum_{\alpha\in P(\mu)} s_{\lambda+\alpha}(\mathbf{x}), \end{aligned} \tag{4.10}$$

where once again the final expression may be standardised through the repeated use of (4.6).

Returning to the main problem, the evaluation of $R_N(q;\mathbf{x})$, it is clear from the definition (3.4) that

$$R_N(q;\mathbf{x}) = R_{N-1}(q;\mathbf{y})\; U_N(q;\mathbf{x}) \tag{4.11}$$

where

$$U_N(q;\mathbf{x}) = \prod_{1\le i\le N-1} (x_i - qx_N)(qx_i - x_N). \tag{4.12}$$

In (4.11) we can expand $R_{N-1}(q;\mathbf{y})$ and $U_N(q;\mathbf{x})$ in terms of Schur functions of $\mathbf{y}$ and monomials in $\mathbf{x}$, respectively. These expansions take the form:

$$R_{N-1}(q;\mathbf{y}) = \sum_\nu c^\nu_{N-1}(q)\; s_\nu(\mathbf{y}) \tag{4.13}$$

and

$$U_N(q;\mathbf{x}) = \sum_\alpha b^\alpha_N(q)\; \mathbf{x}^\alpha. \tag{4.14}$$

By setting $\alpha = (\beta, k)$ and using the fact that $\mathbf{x} = (\mathbf{y}, x_N)$ we then have,

$$U_N(q;\mathbf{x}) = \sum_{k,\beta} b^{\beta,k}_N(q)\; \mathbf{y}^\beta\; x_N^k. \tag{4.15}$$

However, from the definition (4.12), $U_N(q;\mathbf{x}) = U_N(q;\mathbf{y}, x_N)$ is a symmetric function in the components of $\mathbf{y}$. This implies that $b^{\beta,k}_N(q) = b^{\mu,k}_N(q)$ for all $\beta \in P(\mu)$, so that

$$U_N(q;\mathbf{x}) = \sum_{k,\mu} b^{\mu,k}_N(q)\; m_\mu(\mathbf{y})\; x_N^k. \tag{4.16}$$

It then follows that

$$R_N(q;\mathbf{x}) = \sum_{\nu,\mu,k} c^\nu_{N-1}(q)\; b^{\mu,k}_N(q)\; m_\mu(\mathbf{y})\; s_\nu(\mathbf{y})\; x_N^k. \tag{4.17}$$

Hence from (4.10) we have

$$R_N(q;\mathbf{x}) = \sum_{\nu,\mu,k,\beta\in P(\mu)} c^{\nu}_{N-1}(q)\ b^{\mu,k}_N(q)\ s_{\nu+\beta}(\mathbf{y})\ x_N^k$$
$$= \sum_{\nu,\beta,k} c^{\nu}_{N-1}(q)\ b^{\beta,k}_N(q)\ s_{\nu+\beta}(\mathbf{y})\ x_N^k. \tag{4.18}$$

But $R_N(q;\mathbf{x})$ is a symmetric function in the components of $\mathbf{x}$, so that

$$R_N(q;\mathbf{x}) = \Omega_N\left(R_N(q;\mathbf{x})\right) = \sum_{\nu,\beta,k} c^{\nu}_{N-1}(q)\ b^{\beta,k}_N(q)\ \Omega_N\left(s_{\nu+\beta}(\mathbf{y})\ x_N^k\right). \tag{4.19}$$

The identity (4.7) then implies that

$$R_N(q;\mathbf{x}) = \sum_{\nu,\beta,k} c^{\nu}_{N-1}(q)\ b^{\beta,k}_N(q)\ s_{\nu+\beta,k}(\mathbf{x})$$
$$= \sum_{\nu,\alpha} c^{\nu}_{N-1}(q)\ b^{\alpha}_N(q)\ s_{\nu+\alpha}(\mathbf{x}). \tag{4.20}$$

Comparison with the definition of the coefficients $c^{\lambda}_N(q)$ given in (3.5) then implies

Algorithm 4.1 *The polynomials $c^{\lambda}_N(q)$ defined in (3.5) may be determined recursively with respect to N from the identity*

$$c^{\lambda}_N(q) = \sum_{\nu,\alpha} \phi(\nu+\alpha,\lambda)\ c^{\nu}_{N-1}(q)\ b^{\alpha}_N(q), \tag{4.21}$$

where $c^{\nu}_{N-1}(q)$ and $b^{\alpha}_N(q)$ are defined in (4.13) and (4.14), respectively, and where $\phi(\nu+\alpha,\lambda)$ is ± 1 if $s_{\nu+\alpha}(\mathbf{x}) = \pm s_{\lambda}(\mathbf{x})$ under the repeated application of the standardisation rule (4.6), and is zero otherwise.

The significance of (4.21) is that it allows us to determine the expansion of $R_N(q;\mathbf{x})$ in terms of Schur functions $s_\lambda(\mathbf{x})$, and hence to evaluate the coefficients $c^{\lambda}_N(q)$, merely through the term by term addition of the labels ν of the Schur functions $s_\nu(\mathbf{y})$ that appear in the corresponding expansion of $R_{N-1}(q;\mathbf{y})$ to the weight α of the monomials $\mathbf{x}^\alpha$ appearing in the expansion of $U_N(q;\mathbf{x})$, followed by standardisation in accordance with (4.7) of $s_{\nu+\alpha}(\mathbf{x})$. It is notable that no products of symmetric functions are involved.

By way of illustration in the case $N=3$ we have

$$R_2(q;\mathbf{y}) = q\, s_2(\mathbf{y}) - (q^2+q+1)\, s_{1^2}(\mathbf{y}) \tag{4.22}$$

and

$$\begin{aligned} U_3(q;\mathbf{x}) &= (x_1-qx_3)(qx_1-x_3)(x_2-qx_3)(qx_2-x_3) \\ &= (qx_1^2-(q^2+1)x_1x_3+qx_3^2)(qx_2^2-(q^2+1)x_2x_3+qx_3^2) \\ &= q^2\, x_1^2x_2^2x_3^0 - q(q^2+1)x_1^2x_2^1x_3^1 - q(q^2+1)x_1^1x_2^2x_3^1 \\ &\quad + q^2\, x_1^2x_2^0x_3^2 + q^2\, x_1^0x_2^2x_3^2 + (q^2+1)^2x_1^1x_2^1x_3^2 \\ &\quad - q(q^2+1)x_1^1x_2^0x_3^3 - q(q^2+1)x_1^0x_2^1x_3^3 + q^2x_1^0x_2^0x_3^4. \end{aligned} \tag{4.23}$$

Combining these gives

$$\begin{aligned}
R_3(q;\mathbf{x}) = q^3\, & s_{420}(\mathbf{x}) - q^2(q^2+q+1)s_{330}(\mathbf{x}) \\
& - q^2(q^2+1)s_{411}(\mathbf{x}) + q(q^2+1)(q^2+q+1)s_{321}(\mathbf{x}) \\
& - q^2(q^2+1)s_{321}(\mathbf{x}) + q(q^2+1)(q^2+q+1)s_{231}(\mathbf{x}) \\
& + q^3\, s_{402}(\mathbf{x}) - q^2(q^2+q+1)s_{312}(\mathbf{x}) \\
& + q^3\, s_{222}(\mathbf{x}) - q^2(q^2+q+1)s_{132}(\mathbf{x}) \\
& + q(q^2+1)^2 s_{312}(\mathbf{x}) - (q^2+1)^2(q^2+q+1)s_{222}(\mathbf{x}) \\
& - q^2(q^2+1)s_{303}(\mathbf{x}) + q(q^2+1)(q^2+q+1)s_{213}(\mathbf{x}) \\
& - q^2(q^2+1)s_{213}(\mathbf{x}) + q(q^2+1)(q^2+q+1)s_{123}(\mathbf{x}) \\
& + q^3 s_{204}(\mathbf{x}) - q^2(q^2+q+1)s_{114}(\mathbf{x}).
\end{aligned} \tag{4.24}$$

By virtue of the standardisation rule (4.7) we have

$$\begin{aligned}
& s_{231}(\mathbf{x}) = s_{312}(\mathbf{x}) = s_{123}(\mathbf{x}) = s_{204}(\mathbf{x}) = 0,\ s_{402}(\mathbf{x}) = -s_{411}(\mathbf{x}), \\
& s_{213}(\mathbf{x}) = s_{132}(\mathbf{x}) = -s_{222}(\mathbf{x}),\ s_{114}(\mathbf{x}) = s_{222}(\mathbf{x}),\ s_{303}(\mathbf{x}) = -s_{321}(\mathbf{x}).
\end{aligned} \tag{4.25}$$

It then follows that

$$\begin{aligned}
R_3(q;\mathbf{x}) = q^3 & s_{42}(\mathbf{x}) \\
& - q^2(q^2+q+1)(s_{41^2}(\mathbf{x}) + s_{3^2}(\mathbf{x})) \\
& + q(q^2+q+1)(q^2+1)s_{321}(\mathbf{x}) \\
& - (q^2+q+1)(q^4+q^3+q^2+q+1)s_{2^3}(\mathbf{x}).
\end{aligned} \tag{4.26}$$

By proceeding in this way we can recursively calculate the polynomials $c_N^\lambda(q)$. Setting $q = 1$ then gives the coefficients c_N^λ appearing in (2.6). Explicit results for the polynomials $c_N^\lambda(q)$ for $N = 2, \ldots, 5$ are given below in Tables 4.1 to 4.4. In each case the first entry in square brackets is the value of the $q-$polynomial for $q = 1$, that is c_N^λ. The relevant Schur functions $s_\lambda(\mathbf{x})$ have for typographical convenience been denoted by $\{\lambda\}$ and are given to the right of the appropriate $q-$polynomial.

Table 4.1 N = 2

[1] $+q\ \{2\}$
[−3] $-(q^2+q+1)\ \{1^2\}$

Table 4.2 N = 3

[1] $+q^3\ \{42\}$
[−3] $-q^2(q^2+q+1)\ (\{41^2\}+\{3^2\})$
[6] $+q(q^2+1)(q^2+q+1)\ \{321\}$
[−15] $-(q^2+q+1)(q^4+q^3+q^2+q+1)\ \{2^3\}$

Table 4.3 N = 4

$[1]\quad +q^6\ \{642\}$

$[-3]\quad -q^5(q^2+q+1)\ (\{641^2\}+\{63^2\}+\{5^22\})$

$[6]\quad +q^4(q^2+1)(q^2+q+1)\ (\{6321\}+\{543\})$

$[9]\quad +q^4(q^2+q+1)^2\ \{5^21^2\}$

$[-15]\ -q^3(q^2+q+1)(q^4+q^3+q^2+q+1)\ (\{62^3\}+\{4^3\})$

$[-12]\ -q^3(q^2+1)^2(q^2+q+1)\ \{5421\}$

$[-9]\quad -q^3(q^2+q+1)(q^4+q^2+1)\ \{53^21\}$

$[-6]\quad -q^3(q^2+q+1)(q^4+1)\ \{4^22^2\}$

$[27]\quad +q^2(q^2+q+1)^2(q^4+q^2+1)\ (\{532^2\}+\{4^231\})$

$[-45]\ -q(q^2+q+1)(q^4+q^2+1)(q^4+q^3+q^2+q+1)\ \{43^22\}$

$[105]\ +(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)\ \{3^4\}$

Table 4.4 N = 5

$[1]\quad +q^{10}\ \{8642\}$

$[-3]\quad -q^9(q^2+q+1)\ (\{8641^2\}+\{863^2\}+\{85^22\}+\{7^242\})$

$[6]\quad +q^8(q^2+1)(q^2+q+1)\ (\{86321\}+\{8543\}+\{7652\})$

$[9]\quad +q^8(q^2+q+1)^2\ (\{85^21^2\}+\{7^241^2\}+\{7^23^2\})$

$[-12]\quad -q^7(q^2+1)^2(q^2+q+1)\ (\{85421\}+\{7643\})$

$[-9]\quad -q^7(q^2+q+1)(q^4+q^2+1)\ (\{853^21\}+\{75^23\})$

$[-6]\quad -q^7(q^2+q+1)(q^4+1)\ (\{84^22^2\}+\{6^24^2\}$

$[-15]\quad -q^7(q^2+q+1)(q^4+q^3+q^2+q+1)\ (\{862^3\}+\{84^3\}+\{6^32\})$

$[-18]\quad -q^7(q^2+1)(q^2+q+1)^2\ (\{7^2321\}+\{7651^2\})$

$[27]\quad +q^6(q^2+q+1)^2(q^4+q^2+1)\ (\{8532^2\}+\{84^231\}+\{754^2\}+\{6^253\})$

$[24]\quad +q^6(q^2+1)^3(q^2+q+1)\ \{76421\}$

$[18]\quad +q^6(q^2+1)(q^2+q+1)(q^4+q^2+1)\ (\{763^21\}+\{75^221\})$

$[45]\quad +q^6(q^2+q+1)^2(q^4+q^3+q^2+q+1)\ (\{7^22^3\}+\{6^31^2\})$

$[-45]\quad -q^5(q^2+q+1)(q^4+q^2+1)(q^4+q^3+q^2+q+1)\ (\{843^22\}+\{65^24\})$

$[-54]\quad -q^5(q^2+1)(q^2+q+1)^2(q^4+q^2+1)\ (\{7632^2\}+\{6^2521\})$

$[-36]\quad -q^5(q^2+1)^2(q^2+q+1)(q^4+q^2+1)\ \{75431\}$

$[-36]\quad -q^5(q^2+1)(q^2+q+1)^2(q^4+1)\ \{74^31\}$

$[-27]\quad -q^5(q^2+q+1)(q^4+q^2+1)^2\ (\{7542^2\}+\{6^2431\})$

$[-18]\quad -q^5(q^2+q+1)(q^4+q^2+1)(q^4+1)\ (\{6^23^22\}+\{65^22^2\})$

$[105]\quad +q^4(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$
$(\{83^4\}+\{5^4\})$

$[81]\quad +q^4(q^2-q+1)^2(q^2+q+1)^4\ (\{753^22\}+\{65^231\})$

[72] $+\,q^4(q^2+1)^2(q^2+q+1)^2(q^4+1)\ (\{74^232\}+\{654^21\})$
[111] $+\,q^4(q^2+q+1)(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1)$
$\{6^242^2\}$
[45] $+\,q^4(q^2+q+1)^2(q^8+q^6+q^4+q^2+1)\ (\{653^3\}+\{5^332\})$
[−180] $-\,q^3(q^2+1)(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)\ (\{743^3\}+\{5^341\})$
[−144] $-\,q^3(q^2+1)^3(q^2+q+1)^2(q^4+1)\ \{65432\}$
[−90] $-\,q^3(q^2+q+1)(q^4+q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)\ \{64^32\}$
[−75] $-\,q^3(q^2+q+1)(q^4+q^3+q^2+q+1)(q^8+q^6+q^4+q^2+1)\ \{5^243^2\}$
[270] $+\,q^2(q^2+q+1)^2(q^4+q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)\ (\{64^23^2\}+\{5^24^22\})$
[−420] $-\,q(q^2+1)(q^2+q+1)(q^4+1)(q^4+q^3+q^2+q+1)$
$\times\,(q^6+q^5+q^4+q^3+q^2+q+1)\ \{54^33\}$
[945] $+\,(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^3+1)$
$\times\,(q^6+q^5+q^4+q^3+q^2+q+1)\ \{4^5\}$

5. Refinement of the algorithm for calculating $c_N^\lambda(q)$

For $N \geq 4$ the Algorithm 4.1 based on (4.11) for the calculation of $c_N^\lambda(q)$ shows some remarkable properties if its implementation is carried out by simultaneously refining both $U_N(q;\mathbf{x})$ and $R_{N-1}(q;\mathbf{y})$. This is done by setting

$$U_N(q;\mathbf{x}) = \sum_{u=0}^{3} U_N^{(u)}(q;\mathbf{x}) \quad \text{with} \quad U_N^{(u)}(q;\mathbf{x}) = \sum_{\alpha:\alpha_1=u} b_N^\alpha(q)\mathbf{x}^\alpha, \tag{5.1}$$

for $u = 0, 1, 2$, and

$$R_{N-1}(q;\mathbf{y}) = \sum_{r=N-2}^{2N-4} R_{N-1}^{(r)}(q;\mathbf{y}) \quad \text{with} \quad R_{N-1}^{(r)}(q;\mathbf{y}) = \sum_{\nu:\nu_1=r} c_{N-1}^\nu(q)\ s_\nu(\mathbf{y}), \tag{5.2}$$

for $r = N-2, N-1, \ldots, 2N-4$. Then

$$W_N(q;\mathbf{x}) = \sum_{u=0}^{2} \sum_{r=N-2}^{2N-4} W_N^{(u,r)}(q;\mathbf{x}) \quad \text{with} \quad W_N^{(u,r)}(q;\mathbf{x}) = U_N^{(u)}(q;\mathbf{x})\ R_{N-1}^{(r)}(q;\mathbf{y}), \tag{5.3}$$

for $u = 0, 1, 2$ and $r = N-2, N-1, \ldots, 2N-4$.

Proceeding to calculate $R_N(q,\mathbf{y})$ in the above way for $N = 4$ leads to the following results, where $s_\lambda(\mathbf{y})$ has been denoted by $\{\lambda\}$.

Table 5.1 The symmetric functions $R_3^{(r)}(q;\mathbf{y})$.

$$R_3^{(4)}(q;\mathbf{y}) = q^3\{42\} - q^2(q^2+q+1)\{41^2\}$$

$$R_3^{(3)}(q;\mathbf{y}) = -q^2(q^2+q+1)\{3^2\} + q(q^2+q+1)(q^2+1)\{321\}$$

$$R_3^{(2)}(q;\mathbf{y}) = -(q^2+q+1)(q^4+q^3+q^2+q+1)\{2^3\}$$

Table 5.2 The multinomials $U_4^{(u)}(q;\mathbf{x})$.

$$\begin{aligned}
U_4^{(0)}(q;\mathbf{x}) = {} & q^3[(02^3)+(0204)+(0^224)+(0^36)] \\
& -q^2(q^2+1)[(0213)+(0123+(0105)+(0^215)] \\
& +q(q^2+1)^2(01^24)
\end{aligned}$$

$$\begin{aligned}
U_4^{(1)}(q;\mathbf{x}) = {} & -q^2(q^2+1)[(12^21)+(1203)+(1023)+(10^25)] \\
& +q(q^2+1)^2[(1212)+(1^22^2)+(1^204)+(1014)] \\
& -(q^2+1)^3(1^33)
\end{aligned}$$

$$\begin{aligned}
U_4^{(2)}(q;\mathbf{x}) = {} & q^3[(2^3)+(2^202)+(202^2)+(20^24)] \\
& -q^2(q^2+1)[(2^21^2)+(2121)+(2103)+(2013)] \\
& +q(q^2+1)^2(21^22)
\end{aligned}$$

Table 5.3 The products $W_4^{(u,r)}(q;\mathbf{x})=U_4^{(u)}(q;\mathbf{x})\ R_3^{(r)}(q;\mathbf{y})$.

$$W_4^{(0,4)}(q;\mathbf{x})=0$$

$$\begin{aligned}
W_4^{(0,3)}(q;\mathbf{x}) = {} & -q^5(q^2+q+1)\{4^231\} \\
& +q^4(q^2+q+1)^2\{4^22^2\} \\
& -q^2(q^2+q+1)^2(q^4+q^3+q^2+q+1)\{3^4\}
\end{aligned}$$

$$W_4^{(0,2)}(q;\mathbf{x})=-q^2(q^2+1)(q^2+q+1)(q^4+q^3+q^2+q+1)\{3^4\}$$

$$W_4^{(1,4)}(q;\mathbf{x})=0$$

$$W_4^{(1,3)}(q;\mathbf{x})=0$$

$$W_4^{(1,2)}(q;\mathbf{x})=(q^2+1)^2(q^2+q+1)^2(q^4+q^3+q^2+q+1)\{3^4\}$$

$$\begin{aligned}
W_4^{(2,4)}(q;\mathbf{x}) = {} & q^6\{642\} \\
& -q^5(q^2+q+1)\{641^2\}+\{63^2\} \\
& +q^4(q^2+1)(q^2+q+1)\{6321\} \\
& -q^3(q^2+q+1)(q^4+q^3+q^2+q+1)\{62^3\}
\end{aligned}$$

$$\begin{aligned}W_4^{(2,3)}(q;\mathbf{x}) = &-q^5(q^2+q+1)\{5^22\}\\ &+q^4(q^2+q+1)^2\{5^21^2\}\\ &+q^4(q^2+1)(q^2+q+1)\{543\}\\ &-q^3(q^2+1)^2(q^2+q+1)\{5421\}\\ &-q^3(q^2+q+1)(q^4+q^2+1)\{53^21\}\\ &+q^2(q^2+q+1)^2(q^4+q^2+1)\{532^2\}\end{aligned}$$

$$\begin{aligned}W_4^{(2,2)}(q;\mathbf{x}) = &-q^3(q^2+q+1)(q^4+q^3+q^2+q+1)(\{4^3\}+\{4^22^2\}\\ &+q^2(q^2+1)(q^2+q+1)(q^4+q^3+q^2+q+1)\{4^231\}\\ &-q(q^4+q^2+1)(q^2+q+1)(q^4+q^3+q^2+q+1)\{43^22\}\end{aligned}$$

Summing all these terms over $u=0,1,2$ and $r=2,3,4$ gives precisely the results displayed in Table 4.3.

It will be noted in Table 5.3 that a number of the terms $W_4^{(u,r)}(q;\mathbf{x})$ are identically zero. Indeed the main point of the above refinement is not just to break the calculation down into more manageable portions but to search for zeros of this kind. Our calculations of the separate terms $W_N^{(u,r)}(q;\mathbf{x})$ for $4\le N\le 9$ lead us to propose the following:

Conjecture 5.1 *For $N\ge 4$*

$$W_N^{(u,r)}(q;\mathbf{x}) = 0 \quad \textit{for} \quad \begin{cases} u=0, r>N-2+\left[\dfrac{N}{3}\right]; \\ u=1, r>N-2. \end{cases} \tag{5.4}$$

Thus for $u=1$ the only non-vanishing term is that for which $r=N-2$. In this case we have

$$R_{N-1}^{(N-2)}(q;\mathbf{y}) = (-1)^{(N-1)(N-2)/2}\ [2N-3]!!_q\ s_{(N-2)^{N-1}}(\mathbf{y}). \tag{5.5}$$

Then, for $4\le N\le 10$, the results of multiplication by $U_N^{(1)}(q;\mathbf{x})$ suggests:

Conjecture 5.2 *For $N\ge 4$, $W_N^{(1,N-2)}(q;\mathbf{x}) = w^{(1,N-2)}(q)\ s_{(N-1)^N}(\mathbf{x})$ with*

$$w^{(1,N-2)}(q) = \begin{cases} (-1)^{N/2}\ [2]_{q^2} \left[\dfrac{N}{2}\right]_{q^2} [N-1]_q\ [2N-3]!!_q & \text{for } N \text{ even;} \\ (-1)^{(N-1)/2}\ [2]_{q^2} \left[\dfrac{N-1}{2}\right]_{q^2} [N]_q\ [2N-3]!!_q & \text{for } N \text{ odd.} \end{cases} \tag{5.6}$$

Similar calculations in the case $u=0$ suggest

Conjecture 5.3 *For $N\ge 4$, $W_N^{(0,N-2)}(q;\mathbf{x}) = w^{(0,N-2)}(q)\ s_{(N-1)^N}(\mathbf{x})$ with*

$$w^{(0,N-2)}(q) = \begin{cases} (-1)^{(N+2)/2}\ q^2 \left[\dfrac{N}{2}\right]_{q^2} [N-3]_q\ [2N-3]!!_q & \text{for } N \text{ even;} \\ (-1)^{(N+1)/2}\ q^2 \left[\dfrac{N-3}{2}\right]_{q^2} [N]_q\ [2N-3]!!_q & \text{for } N \text{ odd.} \end{cases} \tag{5.7}$$

The case $u = 2$ on the other hand is more complicated and we are not able to suggest any general form for $W_N^{(2,r)}(q;\mathbf{x})$ although the standardisation rules are such that all the surviving Schur functions are necessarily of the form $s_\lambda(\mathbf{x})$ with $\lambda = (r+2,\nu)$ for some partition ν of the appropriate weight. Furthermore, if r takes its maximum value, $2N-4$, then $r+2 = 2N-2$ and we must have $\lambda = (2N-2,\nu)$ with $\nu \in \mathcal{A}_{N-1}$. To be more precise, using the notation of (4.14) and (4.16),

$$\begin{aligned} W_N^{(2,2N-4)}(q;\mathbf{x}) &= \sum_{\gamma,\mu\in\mathcal{A}_{N-1}} c_{N-1}^{(2N-4,\mu)}(q)\ b_N^{(2,\gamma)}(q)\ s_{(2N-2,\mu+\gamma)}(\mathbf{x}) \\ &= \sum_{\gamma,\mu\in\mathcal{A}_{N-1}} q^{N-1}\ c_{N-2}^{\mu}(q)\ b_{N-1}^{\gamma}(q)\ s_{(2N-2,\mu+\gamma)}(\mathbf{x}) \\ &= \sum_{\nu\in\mathcal{A}_{N-1}} q^{N-1}\ c_{N-1}^{\nu}(q)\ s_{(2N-2,\nu)}(\mathbf{x}), \end{aligned} \tag{5.8}$$

where use has been made of both Property 6.2 (see Section 6) and (4.21) with the standardisation rule (4.7) necessarily leaving the first part, $2N-2$, of the relevant partitions fixed. A further application of Property 6.2 then gives

$$W_N^{(2,2N-4)}(q;\mathbf{x}) = \sum_{\nu\in\mathcal{A}_{N-1}} c_N^{(2N-2,\nu)}(q)\ s_{(2N-2,\nu)}(\mathbf{x}); \tag{5.9}$$

These results are well illustrated in Table 5.3 for the case $N = 4$ where it can be seen that the relevant coefficients are just q^3 times those appearing in Table 4.2, and coincide with those appearing in Table 4.3. This is necessary since in our refinement of the calculation of $R_N(q;\mathbf{x})$ the terms $s_{(2N-2,\nu)}(\mathbf{x})$ can only arise from $W_N^{(2,2N-4)}(q;\mathbf{x})$.

6. Properties of the polynomials $c_N^\lambda(q)$

On the basis of explicit calculations of $c_N^\lambda(q)$ up to $N = 9$, we were led first to conjecture and then prove the following factorisation property, which is the q-dependent generalisation of Property 5 given by Di Francesco *et al*[4] in the $q = 1$ case.

Property 6.1 *Let $\lambda \in \mathcal{A}_{M+N}$ be such that $a_{M+N,N-1}(\lambda) = 0$, so that from Lemma 2.4 $\lambda = ((2N)^M + \mu,\nu)$ with $\mu \in \mathcal{A}_M$ $\nu \in \mathcal{A}_N$. Then*

$$c_{M+N}^\lambda(q) = q^{MN}\ c_M^\mu(q)\ c_N^\nu(q) \quad \textit{with } \mu =\in \mathcal{A}_M \textit{ and } \nu \in \mathcal{A}_N. \tag{6.1}$$

Proof Let $\mathbf{x} = (\mathbf{y},\mathbf{z})$ with $\mathbf{y} = (y_1,y_2,\ldots,y_M)$ and $\mathbf{z} = (z_1,z_2,\ldots,z_N)$. Then from (3.3) and (3.5) we have

$$R_{M+N}(q;\mathbf{x}) = R_M(q;\mathbf{y})\ R_N(q;\mathbf{z}) \prod_{i=0}^{M}\prod_{j=0}^{N}(y_i - qz_j)(qy_i - z_j)$$

$$
\begin{aligned}
&= \sum_{\mu\in\mathcal{A}_M} c_M^\mu(q)\ s_\mu(\mathbf{y}) \sum_{\nu\in\mathcal{A}_N} c_N^\nu(q)\ s_\nu(\mathbf{z})\ \left(\prod_{i=0}^{M}(qy_i^2)^N + \cdots + \prod_{j=0}^{N}(qz_j^2)^M\right) \\
&= \sum_{\mu\in\mathcal{A}_M} c_M^\mu(q)\ s_\mu(\mathbf{y}) \sum_{\nu\in\mathcal{A}_N} c_N^\nu(q)\ s_\nu(\mathbf{z})\ \left(q^{MN}\ s_{(2N)^M}(\mathbf{y}) + \cdots + q^{MN}\ s_{(2M)^N}(\mathbf{z})\right) \\
&= \sum_{\mu\in\mathcal{A}_M}\sum_{\nu\in\mathcal{A}_N} \Big(q^{MN}\ c_M^\mu(q)\ c_N^\nu(q)\ s_{(2n)^M+\mu}(\mathbf{y})\ s_\nu(\mathbf{z}) \\
&\qquad\qquad + \cdots + q^{MN}\ c_M^\mu(q)\ c_N^\nu(q)\ s_\mu(\mathbf{y})\ s_{(2M)^N+\nu}(\mathbf{z})\Big).
\end{aligned} \tag{6.2}
$$

The final $+\cdots+$ indicates a linear combination of terms that are necessarily symmetric in the components of $\mathbf{y}$ and in those of $\mathbf{z}$. They are of the form $p_N^{\xi,\zeta}(q)\ s_\xi(\mathbf{y})\ s_\zeta(\mathbf{z})$ with $p_N^{\xi,\zeta}(q)$ a polynomial in q, $|\mu| < |\xi| < 2MN + |\mu|$ and $|\nu| < |\zeta| < 2MN + |\nu|$. This implies that for fixed μ and ν the two terms displayed in (6.2) are of a different weight in the components of $\mathbf{y}$ and $\mathbf{z}$ from all the others. However

$$
\begin{aligned}
R_{M+N}(q;\mathbf{y},\mathbf{z}) &= \sum_{\lambda\in\mathcal{A}_{M+N}} c_{M+N}^\lambda(q)\ s_\lambda(\mathbf{y},\mathbf{z}) \\
&= \sum_{\lambda\in\mathcal{A}_{M+N}} c_{M+N}^\lambda(q) \sum_{\sigma,\tau} c_{\sigma\,\tau}^\lambda\ s_\sigma(\mathbf{y})\ s_\tau(\mathbf{z}).
\end{aligned} \tag{6.3}
$$

For fixed $\mu\in\mathcal{A}_M$ and $\nu\in\mathcal{A}_N$, comparing (6.2) and (6.3) gives

$$
\sum_{\lambda\in\mathcal{A}_{M+N}} c_{M+N}^\lambda(q)\ c_{((2N)^M+\mu)\,\nu}^\lambda = q^{2MN}\ c_M^\mu(q)\ c_N^\nu(q), \tag{6.4}
$$

and

$$
\sum_{\lambda\in\mathcal{A}_{M+N}} c_{M+N}^\lambda(q)\ c_{\mu\,((2M)^N+\nu)}^\lambda = q^{2MN}\ c_M^\mu(q)\ c_N^\nu(q). \tag{6.5}
$$

However, the summation over $\lambda\in\mathcal{A}_{M+N}$ in each of the cases (6.4) and (6.5) reduces to a single term. The two cases are entirely analogous, so it suffices to consider (6.4). Since ν is N-admissible with $|\nu| = N(N-1)$ we have

$$
a_{M+N,N-1}(\lambda) = \sum_{i=0}\lambda_{M+N-1} - N(N-1) = \sum_{i=0}\lambda_{M+N-1} - |\nu|. \tag{6.6}
$$

It follows that if λ is to be $(M+N)$-admissible, then all the boxes of F^ν, when added to $F^{(2N)^M+\mu}$ to form F^λ in accordance with the Littlewood-Richardson rule, must be added below the Mth row. This can be done in one and only one way, namely by simply adjoining F^ν to the bottom of $F^{(2N)^M+\mu}$ to give $F^{((2N)^M+\mu,\nu)}$. The corresponding Littlewood-Richardson coefficient is 1. Thus

$$
c_{((2N)M+\mu)\,\nu}^\lambda = \begin{cases} 1 & \text{if } \lambda = ((2N)^M+\mu,\nu); \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}
$$

Using this in (6.5) gives (6.1), thereby completing the proof of Property 6.1. •

Two special cases are of particular interest. First, setting $M=1$ forces $\mu=(0)$ by virtue of the admissibility conditions. Then for consistency with (6.2) we must have $R_1(q;y_1)=1$, so that $c_1^{(0)}(q)=1$. Hence we obtain:

Property 6.2 *Let* $\nu \in \mathcal{A}_N$*, then*

$$c_{N+1}^{2N,\nu}(q) = q^N c_N^{\nu}(q). \tag{6.8}$$

This is illustrated, for example, by

$$\begin{aligned} c_4^{6321}(q) &= q^4\ c_1^0(q)\ c_3^{321}(q) \\ &= q^3 \cdot 1 \cdot c_3^{321}(q) \\ &= q^3 \cdot 1 \cdot q(q^2+1)(q^2+q+1) \\ &= q^4(q^2+1)(q^2+q+1). \end{aligned} \tag{6.9}$$

Similarly, setting $N=1$ gives:

Property 6.3 *Let* $\mu \in \mathcal{A}_M$*, then*

$$c_{M+1}^{2^M+\mu}(q) = q^M c_M^{\mu}(q). \tag{6.10}$$

For example we have

$$\begin{aligned} c_5^{75440}(q) &= q^4\ c_1^0(q)\ c_4^{5322}(q) \\ &= q^4 \cdot q^2(q^2-q+1)(q^2+q+1)^3 \\ &= q^6(q^2-q+1)(q^2+q+1)^3. \end{aligned} \tag{6.11}$$

It might also be pointed out that the recursive use of either Property 6.2 or Property 6.3 gives a rather easy way to rederive the result (4.27), $c_N^{2\delta}(q) = q^{(N(N-1)/2}$.

As we have already noted, if λ is N-admissible then $\ell(\lambda)$ is either N or $N-1$. Moreover if $\ell(\lambda)=N-1$ then $\lambda_{N-1}\geq 2$ and $\lambda=(2^{N-1}+\mu)$ for some $\mu\in\mathcal{A}_{N-1}$. It then follows from Property 6.3 that

$$c_N^{\lambda}(q) = q^{N-1} c_{N-1}^{\lambda/2^{N-1}}(q) \quad \text{if } \ell(\lambda)=N-1. \tag{6.12}$$

Thus Property 6.3 allows $c_N^{\lambda}(q)$ to be written down immediately in terms of some $c_{N-1}^{\mu}(q)$ if $\ell(\lambda)=N-1$, leaving only those λ to be dealt with for which $\ell(\lambda)=N$.

The cases $M=2$, 3 and 4 of Property 6.1, with N arbitrary, generalise Dunne's observation (Dunne[3]) in the $q=1$ case that, with respect to the reverse lexicographic ordering of partitions, certain consecutive c_{M+N}^{λ} may be obtained from a corresponding sequence c_N^{μ} through multiplication by factors 1, -3, 6 and -12. The q-dependent generalisations of these factors are just q^p, $q^p(q^2+q+1)$, $q^p(q^2+q+1)(q^2+1)$ and $q^p(q^2+q+1)(q^2+1)^2$, respectively, for some appropriate power p of q.

For example, for $\lambda=(9,9,6,3,2,1)$ with $M=2$ and $N=4$, in the notation of Conjecture 6.2, we have $\mu=(1,1)$ and $\nu=(6,3,2,1)$ so that

$$\begin{aligned} c_6^{9^26321}(q) &= q^8\ c_2^{1^2}(q)\ c_4^{6321}(q) \\ &= q^8 \cdot (-1)(q^2+q+1) \cdot c_4^{6321}(q) \\ &= q^8 \cdot (-1)(q^2+q+1) \cdot q^4(q^2+1)(q^2+q+1) \\ &= -q^{12}(q^2+1)(q^2+q+1)^2. \end{aligned} \tag{6.13}$$

The second line of this gives $c_6^{9^2 6321} = -3c_4^{6321}$ in the case $q = 1$. This is a result of the type given by Dunne[3].

Property 6.1 itself implies much more than this. It may be used to express the polynomials $c_N^\lambda(q)$ for arbitrary N-admissible λ in terms of a multiplicative basis of polynomials $c_M^\mu(q)$ where μ is characterised by the fact that $a_{M,k}(\mu) > 0$ for all $k = 0, 1, \ldots, M-2$. In fact the recursive use of Property 6.1 leads immediately to

Corollary 6.4 *Let* λ *be* N*-admissible with*

$$a_{N,k}(\lambda) \begin{cases} = 0 & \text{for } k = k_i \text{ for } i = 1, \ldots, z; \\ \geq 0 & \text{otherwise,} \end{cases} \tag{6.14}$$

with $z \geq 1$ *and* $N - 1 = k_1 > k_2 > \cdots > k_z > k_{z+1} = -1$. *Let* $M_i = k_i - k_{i+1}$ *for* $i = 1, 2, \ldots, z$. *Then* $M_1 + M_2 + \cdots + M_z = N$ *and*

$$\lambda = (((2M_z)^{M_1+M_2+\cdots+M_{z-1}} + \cdots + ((2M_3)^{M_1+M_2} + ((2M_2)^{M_1} + \mu(1)), \mu(2)), \ldots), \mu(z)), \tag{6.15}$$

where for $i = 1, 2, \ldots, z$, $\mu(i)$ *is an* M_i*-admissible partition with* $a_{M_i,m}(\mu(i)) > 0$ *for* $m = 0, 1, \ldots, M_i - 2$, *and*

$$c_N^\lambda(q) = q^{\sum_{1 \leq i < j \leq z} M_i M_j} \prod_{i=1}^{z} c_M^{\mu(i)}(q). \tag{6.16}$$

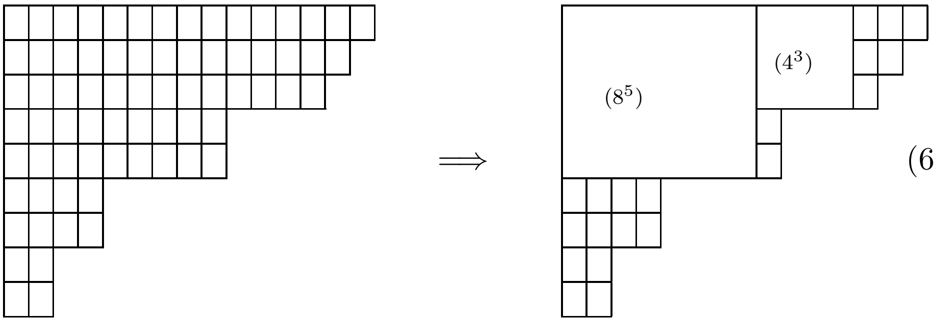
This rather formidable looking q-dependent factorisation property is illustrated as follows in the case $N = 9$ for the 9-admissible partition $\lambda = (15, 14, 13, 9, 9, 4, 4, 2, 2)$. The relevant data are shown in Table 6.1

Table 6.1 Factorisation data for the case $N = 9$, $\lambda = (15, 14, 13, 9, 9, 4, 4, 2, 2)$.

k	8	7	6	5	4	3	2	1	0
λ_{N-k}	15	14	13	9	9	4	4	2	2
$\sum_{i=0}^{k} \lambda_{N-i}$	72	57	43	30	21	12	8	4	2
$k(k+1)$	72	56	42	30	20	12	6	2	0
$a_{N,k}(\lambda)$	0	1	1	0	1	0	2	2	2

From this data it can be seen that in the notation of Corollary 6.4 we have $z = 3$, $k_1 = 8$, $k_2 = 5$, $k_3 = 3$, $M_1 = 3$, $M_2 = 2$, $M_3 = 4$, $\mu(1) = (321)$, $\mu(2) = (11)$ and $\mu(3) = (4422)$. All this has the rather simple diagrammatic realization given below:

$$(15, 14, 13, 9, 9, 4, 4, 2, 2) \qquad \Longrightarrow \qquad ((8^5 + (4^3 + 321), 11), 4422)$$



(6.17)

It then follows from Corollary 6.4 that

$$
\begin{aligned}
&c_9^{15,14,13,9,9,4,4,2,2}(q) = q^{3\cdot 2+3\cdot 4+2\cdot 4}\ c_3^{3,2,1}(q)\ c_2^{1,1}(q)\ c_4^{4,4,2,2}(q)\\
&= q^{26}\ q(q^2+1)(q^2+q+1)\cdot(-1)(q^2+q+1)\cdot(-1)q^3(q^2+q+1)(q^4+1)\\
&= q^{30}\ (q^2+1)\ (q^2+q+1)^3\ (q^4+1). \qquad (6.18)
\end{aligned}
$$

Just as the factorisation property has been extended from the $q = 1$ case to arbitrary values of q, the same can be done in respect of the reversal symmetry property (2.12). Indeed we have:

Property 6.5 *Let λ be N-admissible then $\lambda^{(r)} = ((2N-2)^N)/\lambda$ is also N-admissible and*

$$c_N^{\lambda^{(r)}}(q) = c_N^{\lambda}(q). \qquad (6.19)$$

Proof We have already noted in Section 2 that if λ is N-admissible then so is $\lambda^{(r)}$. In addition, following a q-dependent version of an argument given by Dunne[3], we have

$$
\begin{aligned}
R_N(q;\mathbf{x}) &= \prod_{1\le i<j\le N} (x_i - qx_j)(qx_i - x_j)\\
&= \prod_{1\le i<j\le N} q^2x_i^2x_j^2\ (q^{-1}x_j^{-1} - x_i^{-1})(x_j^{-1} - q^{-1}x_i^{-1})\\
&= q^{N(N-1)} s_{(2N-2)^N}(\mathbf{x})\ R_N(q^{-1};\overline{\mathbf{x}})\\
&= q^{N(N-1)} s_{(2N-2)^N}(\mathbf{x})\ \sum_{\mu} c_N^{\mu}(q^{-1})\ s_{\mu}(\overline{\mathbf{x}})\\
&= \sum_{\mu} q^{N(N-1)}\ c_N^{\mu}(q^{-1})\ s_{(2N-2)^N/\mu}(\mathbf{x})\\
&= \sum_{\lambda} q^{N(N-1)}\ c_N^{\lambda^{(r)}}(q^{-1})\ s_{\lambda}(\mathbf{x})\\
&= \sum_{\lambda} c_N^{\lambda^{(r)}}(q)\ s_{\lambda}(\mathbf{x}), \qquad (6.20)
\end{aligned}
$$

where the notation is such that $\overline{\mathbf{x}} = (x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1})$, and μ has been set equal to $\lambda^{(r)}$, with $\mu^{(r)} = \lambda$. Then in the final step use has been made of (3.12) with λ replaced by $\lambda^{(r)}$. Comparison of (6.20) with the usual expansion (3.5) of $R_N(q;\mathbf{x})$ then gives (6.19) as required. •

It is not difficult to see that Property 6.1 is consistent with the reversal symmetry Property 6.5 in that the application of (6.1) to $c^{\lambda^{(r)}}_{M+N}(q)$ gives

$$c^{\lambda^{(r)}}_{M+N}(q) = q^{MN}\ c^{\mu^{(r)}}_{M}(q)\ c^{\nu^{(r)}}_{N}(q) = q^{MN}\ c^{\mu}_{M}(q)\ c^{\nu}_{N}(q) = c^{\lambda}_{M+N}(q). \tag{6.21}$$

While the reversal symmetry property applies to all N-admissible partitions λ, Property 1 of Di Francesco *et al*[4] applies specifically to those N-admissible partitions λ for which the conjugate partition λ' is also N-admissible. In such a case Di Francesco *et al* noted that $c^{\lambda'}_N \equiv (-1)^N c^{\lambda}_N \bmod 2N$. Before establishing the corresponding q-dependent result, the following result should be noted.

Lemma 6.6 *The partitions λ and λ' are both N-admissible if and only if $\lambda \subset N^N$ and $|\lambda| = N(N-1)$, that is $\lambda = N^N/\zeta$ where ζ is a partition of weight $|\zeta| = N$.*

Proof If λ and λ' are N-admissible then both $\ell(\lambda) \le N$ and $\ell(\lambda') \le N$ so that $\lambda \subseteq N^N$. In addition we must have $|\lambda| = |\lambda'| = N(N-1)$ so that $\lambda \subset N^N$ and there exists ζ of weight $|\zeta| = N$ such that $\lambda = N^N/\zeta$.

Conversely for any $\lambda = N^N/\zeta$ with ζ a partition of weight $|\zeta| = N$ we have $\ell(\lambda) \le N$ and $|\lambda| = N(N-1)$, two of the conditions for N-admissibility. In addition we have, for $k = 0, 1, \ldots, N-1$

$$\begin{aligned} a_{N,k}(\lambda) &= \sum_{i=0}^{k} \lambda_{N-i} - k(k+1) = \sum_{i=0}^{k} (N - \zeta_{i+1}) - k(k+1) \\ &= k(N-1-k) + \left(N - \sum_{i=0}^{k} \zeta_{i+1}\right). \end{aligned} \tag{6.22}$$

Since $k \le N-1$ and $\sum_{i=0}^{k} \zeta_{i+1} \le |\zeta| = N$, with equalities in each case if $k = N-1$, we have $a_{N,k} \ge 0$ for $k = 0, 1, \ldots, N-2$ and $a_{N,k} = 0$ for $k = N-1$. Thus λ is admissible. The same argument applies to λ' with ζ replaced by ζ'. •

Now we are in a position to establish the following q-dependent version of Di Francesco *et al*'s Property 1:

Property 6.7 *If λ and λ' are both N-admissible then*

$$c^{\lambda}_N(q) \equiv (-q)^N\ c^{\lambda}_N(q)\ \bmod [2N]_q. \tag{6.23}$$

Proof ¿From (3.10) we have, on replacing λ by λ',

$$c^{\lambda'}_N(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu \subseteq N^{N-1}} \left((-q)^{|\nu|} + (-q)^{N^2-|\nu|}\right) c^{\lambda'}_{((N-1)^N/\nu')\,\nu'}. \tag{6.24}$$

However, the conjugacy operation is such that the Littlewood-Richardson coefficients satisfy, $c^{\lambda}_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}$, so that

$$c^{\lambda'}_N(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu \subseteq N^{N-1}} \left((-q)^{|\nu|} + (-q)^{N^2-|\nu|}\right) c^{\lambda}_{(N^{N-1}/\nu)\,\nu}. \tag{6.25}$$

If we now set $\mu = N^{N-1}/\nu$ so that $\nu = N^{N-1}/\mu$ and $|\mu| = |\nu| + N(N-1)$, we have

$$c_N^{\lambda'}(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\mu \subseteq N^{N-1}} \left((-q)^{N(N-1)-|\mu|} + (-q)^{N+|\mu|}\right) c^{\lambda}_{\mu,((N-1)^N/\mu')}. \quad (6.26)$$

Using this and (3.20) with ν replaced by ν we find

$$c_N^{\lambda'}(q) - (-q)^N\, c_N^{\lambda}(q) = \frac{1-q^{2N}}{1-q}\, \frac{(-1)^{N(N-1)/2}}{(1-q)^{N-1}} \sum_{\mu \subseteq N^{N-1}} (-q)^{N(N-1)-|\mu|} c^{\lambda}_{((N-1)^N/\mu')\,\mu}. \quad (6.27)$$

Since the right hand side vanishes if $[2N]_q = 1+q+q^2+\cdots+q^{2N-1} = (1-q^{2N})/(1-q)$ this completes our proof. •

This information about $c_N^{\lambda}(q)$ and $c_N^{\lambda'}(q)$ can be considerably strengthened. This can be done by exploiting, as pointed out by Scharf *et al*[5], the connection with the graded decomposition of the exterior algebra of $gl(N)$. If the components x_i of $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ are viewed as eigenvalues of $gl(N)$, then the graded decomposition of the exterior powers of the adjoint representation of $gl(N)$ takes the form

$$\prod_{1 \le i,j \le N} (1 + q\, x_i\, x_j^{-1}) = \sum_{L=0}^{N-1} \sum_{\zeta \vdash LN} e_N^{\zeta}(q)\, \frac{s_{\zeta}(\mathbf{x})}{s_{L^N}(\mathbf{x})}, \quad (6.28)$$

where L describes what has been called the layer of each term (Stembridge[15]) and the expansion coefficients $e_N^{\zeta}(q)$ are polynomials in q. Rewriting $R_N(q;\mathbf{x})$ we find

$$R_N(q;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{L=0}^{N-1} \sum_{\zeta \vdash LN} e_N^{\zeta}(-q)\; s_{((N-L-1)^N+\zeta)}(\mathbf{x}). \quad (6.29)$$

It necessarily follows that for any $\lambda \in \mathcal{A}_N$ we can write $\lambda = (N-L-1)^N + \zeta$ for some $L \in \{0, 1, \ldots, N-1\}$ and some ζ of weight $|\zeta| = LN$. Then

$$c_N^{\lambda}(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N}\; e_N^{\zeta}(-q). \quad (6.30)$$

This formula is useful only in as far as there exists information on $e_N^{\zeta}(-q)$. Fortunately, Stembridge[15] has provided an explicit formula for these coefficients in the $L = 1$, layer one case.

Proposition 6.8 *Let ζ have weight $|\zeta| = N$ then*

$$e_N^{\zeta}(q) = \prod_{k=1}^{N} (1-q^{2k}) \prod_{(i,j)\in F^{\zeta}} \frac{q^{2i-1} + q^{2j-2}}{1 - q^{2h(i,j)}}, \quad (6.31)$$

where $(i,j) \in F^{\zeta}$ specifies a box in the ith row and jth column of the Young diagram F^{ζ} defined by the partition ζ, and $h(i,j)$ is the corresponding hook length $h(i,j) = \zeta_i - j + \zeta'_j - i + 1$, see for example Macdonald[6].

This has an immediate Corollary, namely:

Corollary 6.9 *Let both λ and λ' be N-admissible, with $\lambda = N^N/\zeta$ for ζ a partition of weight N which in Frobenius notation (Macdonald[6]) takes the form $\zeta = (a_1, a_2, \ldots, a_r \,|\, b_1, b_2, \ldots, b_r)$. Then*

$$c_N^\lambda(q) = (-1)^{N(N-1)/2}\ f^\zeta(q^2)\ g^\zeta(q), \tag{6.32}$$

where

$$f^\zeta(q^2) = [N]!_{q^2} \Big/ \prod_{(i,j)\in F^\zeta} [h(i,j)]_{q^2} \tag{6.33}$$

and

$$g^\zeta(q) = \prod_{k=1}^{r} (-q)^{a_k}\ q^{2(a_k+b_k+1)(k-1)}\ [2a_k-1]!!_q\ [2b_k+1]!!_q\,. \tag{6.34}$$

Proof Under the given hypothesis $\lambda = (N^N/\zeta)$, we have $\lambda^{(r)} = ((2N-2)/\lambda) = ((N-2)^N + \zeta)$. Hence from Property 6.5, (6.30) and (6.31)

$$c_N^\lambda(q) = (-1)^{N(N-1)/2} \frac{\prod_{k=1}^N 1-q^{2k}}{\prod_{(i,j)\in F^\zeta} 1-q^{2h(i,j)}} \prod_{(i,j)\in F^\zeta} \frac{-q^{2i-1}+q^{2j-2}}{1-q}. \tag{6.35}$$

The first quotient is just $f^\zeta(q^2)$ as given by (6.33). This is just the q^2 form of the familiar hook length formula for the dimension, f^ζ, of the irreducible representation of the symmetric group S_N specified by the partition ζ. The final product serves to define $g^\zeta(q)$. This can be written in many equivalent ways, but that offered in (6.34) is arrived at by writing each factor $(-q^x+q^y)/(1-q)$ as $-q^x(1-q^{y-x})/(1-q) = -q^x[y-x]_q$ or $q^y(1-q^{x-y})/(1-q) = q^y[x-y]_q$ according as $x<y$ or $x>y$. •

Finally in this section, we offer a q-dependent generalisation of yet another remarkable formula due to Di Francesco *et al*[4] in the case $q=1$, namely their Property 7. We believe that the appropriate generalisation takes the form:

Conjecture 6.10 *For any fixed partition μ of weight $|\mu|$*

$$\sum_{\substack{2N-2\ge\alpha_1\ge\alpha_2\ge -N+2\\ \alpha_1+\alpha_2=N(N-1)-|\mu|}} q^{2N-2-\alpha_1}\ [\alpha_1-\alpha_2+1]_q\ c_N^{(\alpha_1,\alpha_2,\mu)}(q) = 0, \tag{6.36}$$

where for any $\alpha = (\alpha_1, \alpha_2, \mu)$

$$c_N^\alpha(q) = \begin{cases} \pm c_N^\lambda(q) & \text{if } s_\alpha(\mathbf{x}) = \pm s_\lambda(\mathbf{x}) \text{ with } \lambda\in\mathcal{A}_N; \\ 0 & \text{otherwise.} \end{cases} \tag{6.37}$$

The evidence for this is rather compelling, but as yet we can offer no proof.

7. Explicit N-dependent formulae for $c_N(\lambda)$

While many of the results of Section 4 and other calculations appear to conform to no discernible pattern, some of them do suggest the possibility of writing down explicit N-dependent formulae. The simplest such case is afforded by $\lambda = 2\delta = (2N-2, 2N-4, \ldots, 2, 0)$ for which we have

$$c_N^{(2N-2,2N-4,\ldots,2,0)}(q) = q^{N(N-1)/2}. \tag{7.1}$$

This may be established from the definition (3.3) which is such that

$$R_N(q;\mathbf{x}) = \prod_{1\le i<j\le N} \left(q\, x_i^2\right) + \cdots = q^{N(N-1)/2} x_1^{2N-2} x_2^{2N-4} \cdots x_{N-1}^2 + \cdots, \tag{7.2}$$

where the term that has been singled out is the unique highest weight term This is sufficient to prove (7.1).

At the other end of the lists we can use (3.7) to identify $c_N^\lambda(q)$ for $\lambda = (N-1)^N$. Because of the necessity of dividing by $s_{1^N}(\mathbf{x})$, the required polynomial is the coefficient of $s_{N^N}(\mathbf{x})$ arising from the sum over products of the form $s_{N^N/\mu'}(\mathbf{x})\; s_\mu(\mathbf{x})$. Each such product contributes a term $s_{N^N}(\mathbf{x})$ if and only if μ is a self-conjugate partition, that is $\mu' = \mu$. Hence

$$\begin{aligned} c_N^{(N-1)^N}(q) &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\mu'=\mu\subseteq N^N} (-q)^{|\mu|} \\ &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \prod_{k=0}^{N-1} (1-q^{2k+1}) \\ &= (-1)^{N(N-1)/2}\; [2N-1]!!_q. \end{aligned} \tag{7.3}$$

Similarly,

$$c_N^{N^{N-1}}(q) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\mu\subseteq N^N} (-q)^{|\mu|} c^{((N+1)^{N-1},1)}_{(N^N/\mu')\,\mu}. \tag{7.4}$$

There are just two possibilities $\ell(\mu) = N-1$ and $\mu = (1^{N-1}+\nu)$ with $\nu' = \nu \subseteq (N-1)^{N-1}$, and $\ell(\mu) = N$ with $\mu = (1^N+\nu)$ with $\nu' = \nu \subseteq (N-1)^N$. This gives

$$\begin{aligned} c_N^{N^{N-1}}(q) &= \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \left((-q)^{N-1} + (-q)^N\right) \sum_{\nu'=\nu\subseteq(N-1)^{N-1}} (-q)^{|\nu|} \\ &= \frac{(-1)^{N(N-1)/2}}{(1-q)^{N-1}}\; (-q)^{N-1} \prod_{k=0}^{N-2} (1-q^{2k+1}) \\ &= (-1)^{(N-1)(N-2)/2}\; q^{N-1}\; [2N-3]!!_q \end{aligned} \tag{7.5}$$

Both Dunne[3] and Di Francesco *et al*[4] have given a number of other specific formulae. One of the most interesting can be generalised to the q-dependent case as follows. It concerns the case $\lambda = 2\delta - \alpha$ where $\alpha = \epsilon_k - \epsilon_{k+m}$, with $m \geq 1$, is any positive root of $gl(N)$. Since every such positive root can be expressed as a linear combination of simple roots α_p with non-negative integer coefficients g_p, it follows from the argument used in Corollary 3.2 that every such λ is N-admissible. In fact we find

$$c_N^{2\delta-\epsilon_k+\epsilon_{k+m}}(q) = (-1)^m \; q^{N(N-1)/2-m} \; (q^2+1)^{m-1} \; (q^2+q+1). \tag{7.6}$$

This is illustrated for example in Table 4.4 with $N = 5$ and $\lambda = (7,6,4,2,1) = (8,6,4,2,0) - (1,0,0,0,-1) = 2\delta - \epsilon_1 + \epsilon_5$, for which we have $m = 4$ and $c_5^{76421} = q^6(q^2+1)^3(q^2+q+1)$.

One way to derive (7.6) is to note that the use of (3.18) in (3.25) gives

$$R_N^\lambda(q;\mathbf{x}) \; a_\delta(\mathbf{x}) = \sum_{\lambda\in\mathcal{A}_N} c_N^\lambda(q) \; a_{\lambda+\delta}(\mathbf{x}) = \sum_{\lambda\in\mathcal{A}_N} c_N^\lambda(q) \left(\mathbf{x}^{\lambda+\delta} + \cdots \right) \tag{7.7}$$

where $\cdots$ indicates terms on the Weyl orbit of $\lambda+\delta$ obtained by permuting its components, and $\lambda+\delta$ is the unique term on this orbit in the dominant sector. It follows that

$$\begin{aligned} c_N^\lambda(q) &= R_N^\lambda(q;\mathbf{x}) \; a_\delta(\mathbf{x}) \Big|_{\mathbf{x}^{\lambda+\delta}} \\ &= \prod_{1\leq i<j\leq N} (x_i - qx_j)(qx_i - x_j)(x_i - x_j) \Big|_{\mathbf{x}^{\lambda+\delta}} \\ &= q^{N(N-1)/2} \prod_{1\leq i<j\leq N} (x_i^3 - (q{+}1{+}q^{-1})x_i^2x_j + (q{+}1{+}q^{-1})x_ix_j^2 + x_j^3) \Big|_{\mathbf{x}^{\lambda+\delta}}, \end{aligned} \tag{7.8}$$

where $\cdots\Big|_{\mathbf{x}^{\lambda+\delta}}$ indicates that the polynomial $c_N^\lambda(q)$ is to be obtained by taking the coefficient of $\mathbf{x}^{\lambda+\delta}$ in the relevant expansion. In the case $\lambda = 2\delta - \epsilon_k + \epsilon_{k+m}$ we need to select from the above factors the terms in x_i^3 for all $i < k$, and in x_i^3 for all $i = k$ except for one term of the form $-(q{+}1{+}q^{-1})\; x_i^2x_j$ with $i = k < j \leq k+m$. This leads to

$$c_N^{2\delta-\epsilon_k+\epsilon_{k+m}}(q) = q^{N(N-1)/2} \; d_m(q), \tag{7.9}$$

where

$$d_m(q) = -(q+1+q^{-1}) \sum_{l=0}^{m-1} d_l(q) \quad \text{with} \quad d_0(q) = 1. \tag{7.10}$$

The solution to this recurrence relation,

$$d_m(q) = (-1)^m \; (q+1+q^{-1}) \; (q+q^{-1})^{m-1}, \tag{7.11}$$

leads directly to the result (7.6), which is the q-dependent generalisation of formula (61) of Dunne[3] and equation (D.10) of Di Francesco *et al*[4].

Trying to extend this type of analysis to cases of the form $\lambda = 2\delta - 2\epsilon_k + 2\epsilon_{k+m}$ with $m \geq 2$ is more complicated. In fact the formula (D.11) offered by Di Francesco *et al*[4] does not apply to all N and m, nor does there appear to be any simple q-dependent formula valid for all m. In the q-dependent case we find from our calculations of $c_N^\lambda(q)$ the following formulae:

$$\begin{aligned}
c_N^{2\delta-2\epsilon_k+2\epsilon_{k+2}}(q) &= -q^{N(N-1)/2-3}\ [3]_q\ (q^4+q^3+q^2+q+1);\\
c_N^{2\delta-2\epsilon_k+2\epsilon_{k+3}}(q) &= -q^{N(N-1)/2-3}\ [3]_q\ (q^4+1);\\
c_N^{2\delta-2\epsilon_k+2\epsilon_{k+4}}(q) &= +q^{N(N-1)/2-6}\ [3]_q\ (q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1);\\
c_N^{2\delta-2\epsilon_k+2\epsilon_{k+5}}(q) &= -q^{N(N-1)/2-7}\ [3]_q\ (q^{12}+2q^{10}+6q^8+5q^6+6q^4+2q^2+1).
\end{aligned} \tag{7.12}$$

The third of these results is illustrated for $N = 5$ in Table 4.4 by the case $\lambda = (6,6,4,2,2)$ for which $m = 4$. For $q = 1$ the first three of the results (7.12) are in agreement with the formula (D.11) of Di Francesco *et al*[4]. However, the fourth result shows that (D.11) does not extend to the case $m = 5$. The rather formidable and varied nature of the q-dependent factors displayed in (7.12) appears to preclude the derivation of any q-dependent formula appropriate to all N and m.

By the same token the pattern of results indicated by Dunne[3] in his formulae (60) for the cases $\lambda = (N-1)\eta + n(\epsilon_1 - \epsilon_N)$ with $\eta = (1,1,\ldots,1)$ does not extend to all N and n. For example, for $N = 7$ we find

$$\begin{aligned}
c_7^{666666}(q) &= -[13]!!_q;\\
c_7^{766665}(q) &= +[11]!!_q\ [6]_{q^2};\\
c_7^{866664}(q) &= +[9]!!_q\ [1]!!_q\ [5]_{q^2}\ [3]_{q^4};\\
c_7^{966663}(q) &= +[7]!!_q\ [3]!!_q\ [5]_{q^2}\ [2]_{q^4}\ [2]_{q^6};\\
c_7^{10\,66662}(q) &= +[5]!!_q\ [5]!!_q\ [5]_{q^2}\ [3]_{q^4};\\
c_7^{11\,66661}(q) &= +[3]!!_q\ [7]!!_q\ [6]_{q^2};\\
c_7^{12\,66660}(q) &= +[1]!!_q\ [9]!!_q\ .
\end{aligned} \tag{7.13}$$

The results for $N = 8$ are even more complicated with factors that cannot be expressed in the form $[m]_{q^p}$ for any m or p. Only the first two and last two expressions in (7.39) generalise for all sufficiently large N for the relevant q-numbers to be well defined:

$$\begin{aligned}
c_N^{(N-1)^N}(q) &= (-1)^{[N/2]}\ [2N-1]!!_q;\\
c_N^{N,(N-1)^{N-2},N-2}(q) &= -(-1)^{[N/2]}\ [2N-3]!!_q\ [N-1]_{q^2};\\
\\
c_N^{2N-3,(N-1)^{N-2},1}(q) &= -(-1)^{[N/2]}\ [3]!!_q\ [2N-7]!!_q\ [N-1]_{q^2};\\
c_N^{2N-2,(N-1)^{N-2},0}(q) &= -(-1)^{N/2]}\ [2N-5]!!_q\ .
\end{aligned} \tag{7.14}$$

8. Specific values of q

In this section, we turn to specific values of q. First setting $q=0$ gives

$$R_N(0;\mathbf{x}) = \prod_{1\le i<j\le N} (-x_ix_j) = (-1)^{N(N-1)/2} s_{(N-1)^N}(\mathbf{x}). \tag{8.1}$$

Thus

$$c_N^\lambda(0) = \begin{cases} (-1)^{N(N-1)/2} & \text{if } \lambda=(N-1)^N; \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

This implies that for all $\lambda \in \mathcal{A}_N$ other than $\lambda=(N-1)^N$ the polynomial $c_N^\lambda(q)$ contains a factor q^p with $p\ge 1$.

Setting $q=\pm i$ gives

$$R_N(\pm i;\mathbf{x}) = \prod_{1\le j<k\le N} (\pm i)(x_j^2+x_k^2) = (\pm i)^{N(N-1)/2} s_\delta(\mathbf{x}^2). \tag{8.3}$$

where $\mathbf{x}^2=(x_1^2,x_2^2,\ldots,x_N^2)$, so that

$$s_\delta(\mathbf{x}^2) = s_\delta\otimes p_2(\mathbf{x}) = p_2\otimes s_\delta(\mathbf{x}) = (s_2-s_{1^2})\otimes s_\delta(\mathbf{x}), \tag{8.4}$$

where $p_2(\mathbf{x}) = x_1^2+x_2^2+\cdots+x_N^2$ is a power sum symmetric function and $\otimes$ signifies the operation of plethysm (Macdonald[6]). Since this plethysm is just $s_\delta(\mathbf{x}^2)$ it may be evaluated by expanding $s_\delta(\mathbf{x})$ as a sum of monomials of length $\le N$, doubling their parts, and then expanding the resulting monomials as a sum of Schur functions $s_\lambda(\mathbf{x})$ with $\ell(\lambda)\le N$. This method may be used to establish, for example, the results of Table 8.1. Once again the relevant Schur functions $s_\lambda(\mathbf{x})$ have for typographical convenience been denoted by $\{\lambda\}$.

Table 8.1 The expansion of $R_N(\pm i;\mathbf{x})$ for $N=2$, 3 and 4

N	$R_N(\pm i;\mathbf{x})$
2	$\pm\, i(\{2\}-\{1^2\})$
3	$\mp\, i(\{42\}-\{41^2\}-\{3^2\}+\{2^3\})$
4	$-\,(\{642\}-\{641^2\}-\{63^2\}+\{62^3\}-\{5^22\}+\{5^21^2\}+\{53^21\}$ $-\{532^2\}+\{4^3\}-\{4^231\}+2\{4^22^2\}-\{43^22\}+\{3^4\})$

More interestingly, setting $q=\omega$ with ω a primitive cube root unity satisfying $\omega^2+\omega+1=0$ we have

$$\begin{aligned} R_N(\omega;\mathbf{x}) &= \prod_{1\le i<j\le N} (x_i-\omega x_j)(\omega x_i - x_j) \\ &= \prod_{1\le i<j\le N} (\omega x_i^2 - (\omega^2+1)x_ix_j + \omega x_j) \\ &= \prod_{1\le i<j\le N} \omega\ (x_i^2+x_ix_j+x_j^2) \\ &= \omega^{N(N-1)/2} \prod_{1\le i<j\le N} \frac{x_i^3-x_j^3}{x_i-x_j} \\ &= \omega^{N(N-1)/2}\ s_{2\delta}(\mathbf{x}), \end{aligned} \tag{8.5}$$

where use has been made of (3.18). This implies that

$$c_N^\lambda(\omega) = \begin{cases} \omega^{N(N-1)/2} & \text{if } \lambda = 2\delta; \\ 0 & \text{otherwise.} \end{cases} \tag{8.6}$$

This implies the following:

Property 8.1 *Each polynomial $c_N^\lambda(q)$ contains a factor (q^2+q+1) for all $\lambda \in \mathcal{A}_N$ except $\lambda = 2\delta = (2N-2, 2N-4, \ldots, 2, 0)$.*

This is the origin of the observation (Dunne[3]) in the case $q=1$ that c_N^λ is divisible by 3 for all λ except $\lambda = 2\delta$.

9. Vanishing coefficients in the case $q = 1$

At first sight it appears that $c_N^\lambda = c_N^\lambda(1) \neq 0$ for any $\lambda \in \mathcal{A}_N$. A study of the cases $N = 2, 3, \ldots, 7$ supports this. Indeed for $N = 2, 3, \ldots, 6$ we find that the expansions of the polynomials in the form

$$c_N^\lambda(q) = \sum_p n_p \, q^p \tag{9.1}$$

are such that for each N-admissible λ the non-vanishing coefficients n_p are integers, all of the same sign. The first exception to this occurs for $N = 7$. For example, we find for $\lambda = (9,8,8,7,4,4,2)$ that

$$\begin{aligned} c_7^{98^274^22}(q) = {} & q^{33} + q^{32} + 4q^{31} + q^{30} + 8q^{29} + 15q^{27} - q^{26} + 25q^{25} + 2q^{24} \\ & + 38q^{23} + 6q^{22} + 43q^{21} + 6q^{20} + 38q^{19} + 2q^{18} + 25q^{17} - q^{16} \\ & + 15q^{15} + 8q^{13} + q^{12} + 4q^{11} + q^{10} + q^9 \end{aligned} \tag{9.2}$$

The fact that $n_{26} = n_{16} = -1$, while all the other coefficients are positive, is the first indication that for $q = 1$ the coefficients c_N^λ might vanish for some N-admissible λ. As can be seen this does not happen in the case (9.2), and it turns out that for $N = 7$ it never happens. For $N = 7$ there are 15 admissible partitions λ such that the coefficients n_p are positive for some values of p and negative for other values, giving rise to a total of seven distinct polynomials of the type (9.1) having this property. In the case $\lambda = (10,9,7,6,6,3,1)$ it is found that n_p varies from -10 to $+26$. However none of the seven polynomials vanishes at $q = 1$, that is when factorised they contain no factor $(q-1)$. Typically, we find for (9.2) the factorisation

$$\begin{aligned} c_7^{98^274^22} = {} & q^9(q^2+q+1)^3(q^4+q^2+1)^2 \\ & \times (q^{10} - 2q^9 + 2q^8 - 2q^7 + 3q^6 - 3q^5 + 3q^4 - 2q^3 + 2q^2 - 2q + 1). \end{aligned} \tag{9.3}$$

On the other hand, turning to $N = 8$ we find that there are eight N-admissible partitions λ such that $c_N^\lambda = 0$. They occur as four pairs of partitions λ and $\lambda^{(r)}$ related by the reversal symmetry (2.11). They are

$$\lambda = (13\ 11\ 985^2 41), \qquad \lambda^{(r)} = (13\ 10\ 9^2 6531) \tag{9.4a}$$
$$\lambda = (13\ 11\ 9854^2 2), \qquad \lambda^{(r)} = (12\ 10^2\ 96531) \tag{9.4b}$$
$$\lambda = (13\ 11\ 976541), \qquad \lambda^{(r)} = (13\ 10\ 987531) \tag{9.4c}$$
$$\lambda = (12\ 11\ 97^2 4^2 2), \qquad \lambda^{(r)} = (12\ 10^2\ 7^2 532) \tag{9.4d}$$

The corresponding four q-polynomials $c_8^\lambda(q)$ take the form

$$c_8^{13\ 11\ 985^2 41} = -q^{17}(q^2+1)^2(q^2+q+1)^3(q^4+q^2+1)^2(q-1)^4; \tag{9.5a}$$
$$c_8^{13\ 11\ 9854^2 2} = q^{16}(q^2+1)(q^2+q+1)^3(q^4+q^2+1)^3(q-1)^4; \tag{9.5b}$$
$$c_8^{13\ 11\ 976541} = q^{16}(q^2+1)^3(q^2+q+1)^3(q^4+q^2+1)^2(q-1)^4; \tag{9.5c}$$
$$\begin{aligned} c_8^{12\ 11\ 97^2 4^2 2} &= q^{14}(q^2+q+1)^3(q^4+q^2+1)^2(q-1)^4 \\ &\times (q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1). \end{aligned} \tag{9.5d}$$

In each of the above cases the factor $(q-1)$ occurs, so that $c_8^\lambda = 0$, as claimed. It is notable that in each case the power of $(q-1)$ is 4. There are no factors of $(q+1)$ so that as required by Corollary 3.2 and (3.28), these polynomials do not vanish for $q=-1$. The values of $c_8^\lambda(-1) = c_{\delta\,\delta}^\lambda$ are given in the right most position below:

$$\lambda = (13\ 11\ 985^2 41\}, \qquad \lambda^{(r)}(13\ 10\ 9^2 6531) \quad (576) \tag{9.6a}$$
$$\lambda = (13\ 11\ 9854^2 2\}, \qquad \lambda^{(r)}(12\ 10^2\ 96531) \quad (864) \tag{9.6b}$$
$$\lambda = (13\ 11\ 976541\}, \qquad \lambda^{(r)}(13\ 10\ 987531) \quad (1152) \tag{9.6c}$$
$$\lambda = (12\ 11\ 97^2 4^2 2\}, \qquad \lambda^{(r)}(12\ 10^2\ 7^2 532) \quad (1872) \tag{9.6d}$$

As indicated earlier, extending the analysis to $N=9$ we find that there are 66 different 9-admissible partitions λ such that $c_9^\lambda = 0$, while for $N=10$ there are 389 different 10-admissible partitions λ such that $c_{10}^\lambda = 0$. Remarkably, as in the case of $N=8$, all the $N=9$ polynomials vanishing at $q=1$ contain a factor of $(q-1)^4$.

10. Sum rules

In their seminal work on the $q=1$ case Di Francesco *et al*[4] established a remarkable set of sum rules for the coefficients appearing in the Schur function expansion of even powers of the Vandermonde determinant. In the case of the square of the Vandermonde determinant their result takes the form

$$\sum_\lambda (c_N^\lambda)^2 = \frac{(3N)!}{(3!)^N\ N!}. \tag{10.1}$$

Once again there exists a q-dependent form of this result, namely
Property 10.1 *For all $N \geq 2$*

$$\sum_\lambda c_N^\lambda(q)\ c_N^\lambda(q^2) = \frac{[3N]!_q}{([3]!_q)^N\ [N]!_{q^3}} = \frac{[3N-1]!!!_q\ [3N-2]!!!_q}{([2]_q)^N} \tag{10.2}$$

Proof Consider the product

$$
\begin{aligned}
&V_N(\mathbf{x})\ R_N(q;\mathbf{x})\ R_N(q^2;\mathbf{x})\\
&\quad=\sum_{\mu,\nu} c^\mu_N(q)\ c^\nu_N(q^2)\ s_\mu(\mathbf{x})\ s_\nu(\mathbf{x})\ a_\delta(x)\\
&\quad=\sum_{\mu,\nu,\lambda} c^\mu_N(q)\ c^\nu_N(q^2)\ c^\lambda_{\mu\,\nu}\ a_{\lambda+\delta}(\mathbf{x}).
\end{aligned}
\tag{10.3}
$$

where, as in (3.18), $a_\delta(\mathbf{x}) = \left|x_i^{N-j}\right|$ and, $a_{\lambda+\delta}(\mathbf{x}) = \left|x_i^{\lambda_j+N-j}\right|$, while $c^\lambda_{\mu\,\nu}$ is the Littlewood-Richardson coefficient defined in (3.9). In this expansion the coefficient of $\mathbf{x}^{\lambda+\delta}$ is given by

$$
V_N(\mathbf{x})\ R_N(q;\mathbf{x})\ R_N(q^2;\mathbf{x})\ \Big|_{\mathbf{x}^{\lambda+\delta}} = \sum_{\mu,\nu,\lambda} c^\mu_N(q)\ c^\nu_N(q^2)\ c^\lambda_{\mu\,\nu}. \tag{10.4}
$$

However, for $\lambda = ((2N-2)^N)$ we have $c^\lambda_{\mu\,\nu} = 0$ unless $\nu = \mu^{(r)}$, in which case its value is 1. Recalling reversal symmetry Property 6.5, and it follows that

$$
\begin{aligned}
&\sum_\mu c^\mu_N(q)\ c^\mu_N(q^2) = V_N(\mathbf{x})\ R_N(q;\mathbf{x})\ R_N(q^2;\mathbf{x})\ \Big|_{\mathbf{x}^{(2N-2)^N+\delta}}\\
&= \prod_{1\le i<j\le N} (x_i-x_j)(x_i-qx_j)(qx_i-x_j)(x_i-q^2x_j)(q^2x_i-x_j)\Big|_{\mathbf{x}^{(2N-2)^N+\delta}}\\
&= \prod_{1\le i<j\le N} \left(1-\frac{x_j}{x_i}\right)\left(1-\frac{qx_j}{x_i}\right)\left(1-\frac{q^2x_j}{x_i}\right)\left(1-\frac{qx_i}{x_j}\right)\left(1-\frac{q^2x_i}{x_j}\right)\Big|_{\mathbf{x}^0},
\end{aligned}
\tag{10.5}
$$

where $\mathbf{x}^0 = 1$. However, we have at our disposal the following constant term identity due to Bressoud and Goulden[14]:
Theorem 10.2 *For $i = 1, 2, \ldots, N$ let a_i be positive integers, then*

$$
\begin{aligned}
&\prod_{1\le i<j\le N} \left(1-\frac{x_j}{x_i}\right)\left(1-\frac{qx_j}{x_i}\right)\cdots\left(1-\frac{q^{a_i-1}x_j}{x_i}\right)\left(1-\frac{qx_i}{x_j}\right)\cdots\left(1-\frac{q^{a_j-1}x_i}{x_j}\right)\Big|_{\mathbf{x}^0}\\
&\qquad= \frac{[a_1+a_2+\cdots+a_N]!_q}{[a_1]!_q[a_2]!_q\cdots[a_N]!_q}\ \prod_{i=1}^N \frac{1-q^{a_i}}{1-q^{a_i+a_{i+1}+\cdots+a_N}}.
\end{aligned}
\tag{10.6}
$$

Setting $a_i = 3$ for all $i = 1, 2, \ldots, N$ gives the first form offered for the required sum in (10.2) since $(1-q^3)/(1-q^{3m}) = 1/[m]_{q^3}$. The second form just follows by noting that $[3N]!!!_q/(([3]!_q)^N\ [N]!_{q^3}) = 1/([2]_q)^N$. •

More generally, Theorem 10.2 implies in exactly the same way as before

$$
V_N(\mathbf{x})\ \prod_{k=1}^{p-1} R_N(q^k;\mathbf{x})\ \Big|_{\mathbf{x}^{k(N-1)+\delta}} = (-1)^{(p-1)N(N-1)/2}\frac{[pN]!_q}{([p]!_q)^N\ [N]!_{q^p}}. \tag{10.7}
$$

However this only gives

$$\sum_{\lambda,\mu,\ldots,\nu} c_N^\lambda(q)\ c_N^\mu(q^2)\ \cdots\ c_N^\nu(q^{p-1})\ c_{\lambda\,\mu\,\cdots\,\nu}^{M^N} = (-1)^{(p-1)N(N-1)/2} \frac{[pN]!_q}{([p]!_q)^N\ [N]!_{q^p}} \tag{10.8}$$

with $M = (p-1)(N-1)$, where the generalised Littlewood-Richardson coefficient $c_{\lambda\,\mu\,\cdots\,\nu}^{M^N}$ appearing here is the multiplicity of the Schur function $s_{M^N}(\mathbf{x}) = \mathbf{x}^{(M,M,\ldots,M)}$ in the product $s_\lambda(\mathbf{x})\, s_\mu(\mathbf{x})\ \cdots\ s_\nu(\mathbf{x})$. It does not give what one might have hoped for, namely an expression for

$$\sum_\lambda c_N^\lambda(q)\ c_N^\lambda(q^2)\ \cdots\ c_N^\lambda(q^{p-1}). \tag{10.9}$$

On the other hand by setting

$$\prod_{k=1}^m R_N(q^k;\mathbf{x}) = \sum_\lambda\ d_N^{m;\lambda}(q)\ s_\lambda(\mathbf{x}) \tag{10.10}$$

and

$$\prod_{k=m+1}^{2m}\ R_N(q^k;\mathbf{x}) = \sum_\lambda e_N^{m;\lambda}(q)\ s_\lambda(\mathbf{x}), \tag{10.11}$$

we have two different q-generalisations, $d_N^{m;\lambda}(q)$ and $e_N^{m;\lambda}(q)$, of the coefficients $c_N^{m;\lambda}$ appearing in the expansion (2.9) of the $2m$th power of the Vandermonde determinant. Extending the reversal symmetry argument to these cases it then follows from (10.7) with $p = 2m+1$ that

$$\sum_\lambda d_N^{m;\lambda}(q)\ e_N^{m;\lambda}(q)\ \ = \frac{[(2m+1)N]!_q}{([2m+1]!_q)^N\ [N]!_{q^{2m+1}}}. \tag{10.12}$$

Setting $q = 1$ then gives the remarkable result (1.4) of Di Francesco *et al*[4]

$$\sum_\lambda (c_N^{m;\lambda})^2 = \frac{((2m+1)N)!}{((2m+1)!)^N\ N!}. \tag{10.13}$$

Turning to the simpler case $p = 2$ in (10.8) gives

$$\sum_\lambda c_N^\lambda(q) c_\lambda^{M^N} = (-1)^{N(N-1)/2} \frac{[2N]!_q}{([2]!_q)^N\ [N]!_{q^2}} = [2N-1]!!_q, \tag{10.14}$$

where now $M = N-1$ and $c_\lambda^{M^N} = 1$ if $\lambda = (N-1)^N$ and is zero otherwise. Hence

$$c_N^{(N-1)^N}(q) = (-1)^{N(N-1)/2}\ [2N-1]!!_q, \tag{10.15}$$

precisely as in (7.3).

Thus our analysis fails to give an expression for what might be thought of as the simplest sum of all, namely

$$C_N(q) = \sum_\lambda c_N^\lambda(q). \tag{10.16}$$

With this notation, and letting $C_N = C_N(q)$ we find the data shown in Table 10.1

Table 10.1 Values of $C_N(q)$ and $C_N = C_N(1)$ for $N \leq 6$

N	C_N	$C_N(q)$
2	-2	$-(q^2+1)$
3	-14	$-(q^6+q^5+4q^4+2q^3+4q^2+q+1)$
4	$+70$	$+(q^{12}+2q^{11}+6q^{10}+4q^9+11q^8+4q^7$
		$+14q^6+4q^5+11q^4+4q^3+6q^2+2q+1)$
5	$+910$	$+(q^{20}+3q^{19}+9q^{18}+13q^{17}+30q^{16}+31q^{15}+69q^{14}$
		$+52q^{13}+112q^{12}+68q^{11}+134q^{10}+68q^9+112q^8+52q^7$
		$+69q^6+31q^5+30q^4+13q^3+9q^2+3q+1)$
6	-7280	$-(q^{30}+4q^{29}+13q^{28}+26q^{27}+56q^{26}+78q^{25}+146q^{24}$
		$+146q^{23}+293q^{22}+210q^{21}+509q^{20}+242q^{19}+732q^{18}$
		$+220q^{17}+866q^{16}+196q^{15}+866q^{14}+220q^{13}+732q^{12}$
		$+242q^{11}+509q^{10}+210q^9+293q^8+146q^7+146q^6$
		$+78q^5+56q^4+26q^3+13q^2+4q+1)$

In the case $q = 1$ the results have been extended up to $N = 10$ as indicated in Table 10.2.

Table 10.2 Values of $C_N = C_N(1)$ for $N \leq 10$

N	$A_N = \#\{\mathcal{A}_N\}$	$C_N = \sum_\lambda c_N^\lambda$	$\lvert C\rvert_N = \sum_\lambda \lvert c_N^\lambda\rvert$
2	2	-2	4
3	5	-14	28
4	16	$+70$	292
5	59	$+910$	4102
6	247	-7280	73444
7	1111	-138320	1605838
8	5294	$+1521520$	41603200
9	26310	$+38038000$	1247676262
10	135281	-532532000	42551137984

This data on C_N is entirely consistent with the recursive formula

$$\frac{C_N}{C_{N-1}} = (-1)^{N+1} \begin{cases} \frac{3N-2}{2} & N \text{ even} \\ 3N-2 & N \text{ odd} \end{cases} \tag{10.17}$$

and thereby leads to the following:

Conjecture 10.3 *The sum C_N of the coefficients appearing in the Schur function expansion of the square of the Vandermonde determinant $V_N(\mathbf{x})$ is given by*

$$C_N = (-1)^{N(N-1)/2} \frac{(3N-2)!!!}{2^{[N/2]}} \tag{10.18}$$

In general $C_N(q)$ does not factorise nicely over the integers and $C_N(q)/C_{N-1}(q)$ is not a polynomial in q. Thus the most obvious q-dependent generalisation of (10.18) cannot be valid. A possible remedy is to introduce a weighting $w_N^\lambda(q)$ such that $w_N^\lambda(1) = 1$ for all N and all λ and

$$C_{N,w}(q) = \sum_\lambda w_N^\lambda(q)\; c_N^\lambda(q) = (-1)^{N(N-1)/2} \frac{[3N-2)]!!!_q}{([2]_q)^{[N/2]}}. \tag{10.19}$$

Although one can indeed fit the data for each N by some choice of $w_N^\lambda(q)$ for various λ there appears to be no acceptable rationale for its dependence on λ. The existence of an appropriate form of $w_N^\lambda(q)$ therefore remains an open problem, as indeed does that of either proving Conjecture 10.3 or finding a value of N for which it breaks down.

11. Concluding remarks

Our original objective was to shed light upon the vanishing of certain of the coefficients, $c_N^\lambda(1)$, in the expansion of the square of the Vandermonde determinant into Schur functions. This led us to reconsider the concept of admissibility, originally formulated by Di Francesco *et al* [4], and to explore the q-generalisation of the square of the Vandermonde determinant. In that process it was necessary to sharpen the algorithms for evaluating the q-dependent coefficients, $c_N^\lambda(q)$, for arbitrary values of q and to study their dependence on q, N and λ. The calculation of complete data for $N \leq 9$ for arbitrary q and for $N = 10$ with $q = 1$ allowed us to test a number of hypotheses and stimulated various conjectures, most of which we have been able to prove here.

To be more precise, we have determined q-dependent generalisations of all eight Properties 0-7 established in the $q = 1$ case by Di Francesco *et al* [4]. These generalisations have all been proved, save that of Property 7 which is embodied in our Conjecture 6.10. The proven generalisations include the factorisation Property 6.1 and its Corollary 6.4 which allows all polynomials $c_N^\lambda(q)$ to be expressed in terms of a multiplicative basis, of polynomials $c_M^\mu(q)$ for which the admissibility coefficients $a_{M,k}(\mu)$ are positive for all $k = 0, 1, \ldots, M-2$.

In addition, having made the connection through the $q = -1$ case with the work of Berenstein and Zelevinsky[9] we have proved in Proposition 3.3 that the admissibility conditions on λ are necessary and sufficient for $c_N^\lambda(q)$ to be non-vanishing. The fact, reported previously (Scharf *et al*[5]), that for $N \geq 8$ there exist some N-admissible partitions λ such that $c_N^\lambda = c_N^\lambda(1) = 0$ then has to be interpreted as accidental in the sense that for such λ it just so happens that the polynomial $c_N^\lambda(q)$ contains a factor $(q-1)$. This has been exhibited explicitly in Section 9.

Consideration of the q-dependent case also links the problem of the square of the Vandermonde determinant with that of the graded decomposition of the exterior algebra of $gl(N)$. This allowed us to lean on the work of Stembridge[15] to greatly strengthen the observations in the $q = 1$ case made by Di Francesco *et al*[4] regarding those partitions λ for which both λ and its conjugate λ' are N-admissible. The outcome is the explicit formula of Corollary 6.9 for $c_N^\lambda(q)$.

While many of the q-dependent calculations lead to formulae such as (7.13) in which the $q = 1$ results are generalised merely by replacing integer factors by q-numbers, it is notable that the q-numbers themselves may sometimes be $[m]_{q^2}$ or $[m]_{q^3}$ rather than just $[m]_q$. This is particularly striking in the case of the remarkable new sum rule, Property 10.1, which is the q-dependent generalisation of the formula of Di Francesco *et al*[4] for the sum of the squares of the expansion coefficients.

Unfortunately, although we have provided a conjecture, namely Conjecture 10.3, regarding the apparently simpler sum of the coefficients themselves, its proof or disproof remains an open problem. The fact that we have verified it to be true for all $N \leq 10$ might be construed as compelling evidence for its validity for all N. This, along with similarly compelling evidence for the validity of the conjectures of Section 5 regarding our refinement of the algorithm for calculating $c_N^\lambda(q)$, especially Conjecture 5.1, hints at the richness of the field and the possibility that much remains to be uncovered.

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