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Plethysm and the Non-Compact Groups $Sp(2n, R)$

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Summary

- **Why $Sp(2n, R)$?**

- **The $Sp(2n, R) \to U(n)$ Decomposition**

- **Plethysms in $Sp(2n, R)$**

- **Some New $S-$function Identities**

- **Stability Properties of Kronecker Products and Plethysms**

- **Concluding Remarks**

Why $Sp(2n, R)$?

- **Physics Motivation - $Sp(2n, R)$ is the dynamical group of the $n-$dimensional isotropic harmonic oscillator.**

- **The infinite-dimensional fundamental unitary irreps $< s; (0) >$ and $< s; (1) >$ are of particular significance as they are spanned by the infinite set of single particle states of *even* and *odd* parity respectively.**

- **For $N-$particle systems we need to resolve symmetrised $N-$th powers of the two fundamental irreps. i.e, Plethysms $s_\lambda(< s; (0) >)$ and $s_\lambda(< s; (1) >)$ where $\lambda \vdash N$.**

- **Arbitrary positive discrete harmonic series irreps of $Sp(2n, R)$ will be labelled as $< \frac{k}{2}; (\lambda) >$ or equivalently as $< s\kappa; (\lambda) >$ where κ and s are respectively the integer and residue parts of $\frac{k}{2}$.**

The $Sp(2n,R) \to U(n)$ Decomposition

- **Under the restriction $Sp(2n,R) \to U(n)$ a given irrep of $Sp(2n,R)$ decomposes into an infinite set of finite dimensional irreps of the unitary group $U(n)$. In the case of the two fundamental irreps of $Sp(2n,R)$ we have**

$$(< s; (0) > \to \varepsilon^{\frac{1}{2}} M_+ \tag{1}$$

$$(< s; (1) > \to \varepsilon^{\frac{1}{2}} M_- \tag{2}$$

where M_+ and M_- are the *even* and *odd* weight $S-$functions $\{m\}$ appearing in the infinite series

$$M = \sum_{m=0}^{\infty} \{m\} \tag{3}$$

- **Notice that the expansion is essentially stable with respect to n.**

- **In general one has**

$$< \frac{k}{2}; (\lambda) > \rightarrow \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^k \cdot D_N\}\}_N \qquad (4)$$

where $N = min(n, k)$**,** D **is the infinite** $S-$**function series**

$$D = \sum_{\delta} \{\delta\} \qquad (5)$$

where the δ **are partitions involving only** *even* **parts. The subscript** N **means that all terms involving partitions into more than** N **parts are to be discarded. The first** $\cdot$ **indicates a product in** $U(n)$ **and the second** $\cdot$ **a product in** $U(N)$**.**

- $\{\lambda_s\}^k$ **is a** *signed sequence* **of terms** $\pm\{\rho\}$ **such that** $\pm\{\rho\}$ **is equivalent to** $\{\lambda\}$ **under the modification rules of the orthogonal group** $O(k)$**.**

- **Notice that Eq. (4) is stable for** $n \geq k$**. Sometimes it is** *prematurely stable* **for smaller values of** n**.**

- **Example, the terms to weight 16 for the decomposition of the irrep** $< s1; (21) >$ **of** $Sp(6,R)$ **to** $U(3)$ **are:-**

$$
\begin{array}{llll}
\{s1;21\} & +\ \{s1;31^2\} & +\ \{s1;32\} & +\ \{s1;3^21\} \\
+\ \{s1;41\} & +\ \{s1;421\} & +\ \{s1;43\} & +\ \{s1;432\} \\
+\ \{s1;51^2\} & +\ \{s1;52\} & +\ 2\{s1;531\} & +\ \{s1;53^2\} \\
+\ \{s1;54\} & +\ \{s1;542\} & +\ \{s1;5^21\} & +\ \{s1;5^23\} \\
+\ \{s1;61\} & +\ \{s1;621\} & +\ \{s1;63\} & +\ \{s1;632\} \\
+\ \{s1;641\} & +\ \{s1;643\} & +\ \{s1;65\} & +\ \{s1;652\} \\
+\ \{s1;654\} & +\ \{s1;71^2\} & +\ \{s1;72\} & +\ 2\{s1;731\} \\
+\ \{s1;73^2\} & +\ \{s1;74\} & +\ \{s1;742\} & +\ 2\{s1;751\} \\
+\ 2\{s1;753\} & +\ \{s1;76\} & +\ \{s1;762\} & +\ \{s1;7^21\} \\
+\ \{s1;81\} & +\ \{s1;821\} & +\ \{s1;83\} & +\ \{s1;832\} \\
+\ \{s1;841\} & +\ \{s1;843\} & +\ \{s1;85\} & +\ \{s1;852\} \\
+\ \{s1;861\} & +\ \{s1;87\} & +\ \{s1;91^2\} & +\ \{s1;92\} \\
+\ 2\{s1;931\} & +\ \{s1;93^2\} & +\ \{s1;94\} & +\ \{s1;942\} \\
+\ 2\{s1;951\} & +\ \{s1;96\} & +\ \{s1;10\ 1\} & +\ \{s1;10\ 21\} \\
+\ \{s1;10\ 3\} & +\ \{s1;10\ 32\} & +\ \{s1;10\ 41\} & +\ \{s1;10\ 5\} \\
+\ \{s1;11\ 1^2\} & +\ \{s1;11\ 2\} & +\ 2\{s1;11\ 31\} & +\ \{s1;11\ 4\} \\
+\ \{s1;12\ 1\} & +\ \{s1;12\ 21\} & +\ \{s1;12\ 3\} & +\ \{s1;13\ 1^2\} \\
+\ \{s1;13\ 2\} & +\ \{s1;14\ 1\} & &
\end{array}
$$

Plethysms in $Sp(2n,R)$

- **We are primarily interested in plethysms of the form $\{\lambda\}(< s;(0) >)$ and $\{\lambda\}(< s;(1) >)$. These plethysms involve infinite sets of $Sp(2n,R)$ irreps. No general procedure seems to be known. We can evaluate the terms, up to a given weight by first decomposing the $Sp(2n,R)$ into $U(n)$ irreps, performing the plethysm at the $U(n)$ level and then inverting to get irreps of $Sp(2n,R)$. This has been done for all $\lambda \vdash 4$ and in some cases to $\lambda \vdash 6$. Remarkably, one finds that generally**

$$\{2\}(< s;(0) >) = \sum_{i=0}^{\infty} < 1;(0+4i) >$$

$$\{1^2\}(< s;(0) >) = \sum_{i=0}^{\infty} < 1;(2+4i) >$$

$$\{2\}(< s;(1) >) = \sum_{i=0}^{\infty} < 1;(2+4i) >$$

$$\{1^2\}(< s;(1) >) =< 1;(1^2) > + \sum_{i=0}^{\infty} < 1;(4+4i) >$$

  This result implies that the following $S-$function identity must hold

$$\{1^2\}(M_+) \equiv \{2\}(M_-) \tag{6}$$

  as indeed may be shown to be the case.

- **In precisely the same manner one finds**

$$\{1^2\}(L_+) \equiv \{2\}(L_-) \tag{7}$$

where L_+ and L_- are respectively the positive and neagative terms of the series

$$L = \sum_{m=0}^{\infty} (-1)^m \{1^m\} \tag{8}$$

Still further identities arise for the infinite $S-$function series defined by

$$\begin{aligned} A_\pm &= L_\pm(\{1^2\}) \qquad & B_\pm &= M_\pm(\{1^2\}) \\ C_\pm &= L_\pm(\{2\}) \qquad & D_\pm &= M_\pm(\{2\}) \end{aligned} \tag{9}$$

Use of the associativity property of plethysms leads directly to

$$\{1^2\}(Z_+) \equiv \{2\}(Z_-) \tag{10}$$

for $Z = A, B, C, D$. Furthermore,

$$\{2\}(Z) = ZZ_+ \quad \textbf{and} \quad \{1^2\}(Z) = ZZ_- \tag{11}$$

These identities appear to be unknown.

- **The study of plethysms within the group $Sp(2n,R)$ leads to still further identities involving infinite series of $S-$functions. The observation that**

$$\{21^2\}(< s;(0) >) \equiv \{31\}(< s;(1) >) \tag{12}$$

leads to the remarkable $S-$function identity

$$\{21^2\}(M_+) \equiv \{31\}(M_-) \tag{13}$$

which generalises to

$$\{\sigma\}(\{1^2\}(M_+)) \equiv \{\sigma\}(\{2\}(M_-)) \tag{14}$$

Again these identities extend to the series Z defined earlier.

Stability of Kronecker Products and Plethysms

- **A given plethysm, Kronecker product or decomposition will be said to be *stable* if at the stable value of $n = n_s$ there is a one-to-one mapping between the resultant list of irreps obtained at the stable value n_s and those obtained for all values of $n > n_s$.**

- **The $Sp(2n, R)$ Kronecker product**

$$< \frac{k}{2}(\lambda) > \times < \frac{\ell}{2}(\nu) > = < \frac{(k+\ell)}{2}(\{\lambda_s\}^k \cdot \{\nu_s\}^\ell \cdot D)\}_{k+\ell,n} > \tag{15}$$

is certainly stable for all $n \geq (k+\ell)$. We say *certainly* because in some cases *premature stability* may occur for values of $n < (k+\ell)$.

- **One observes that the third-order plethysms for the two fundamental irreps stabilise at $n = 3$ which is consistent with the stabilisation of the products $< s;(0) > \times < 1;(\mu) >$ and $< s;(1) > \times < 1;(\mu) >$ at $n = 3$ and for similar reasons stabilisation of the $N-$th order plethysms must occur at $n = N$ as observed, Again, premature stabilisation for individual plethysms may occur for $n < N$. Thus for $N = 3$ all the plethysms stabilise at $n = 2$ except for $\{1^3\}(< s;(1) >)$ which stabilises at $n = 3$. Stabilisation for arbitrary N occurs at $n = N - 1$ except for $\{1^N\}(< s;(1) >)$ which stabilises at $n = N$.**

Plethysm Conjugacy?

- **Below we give two short examples of plethysms with terms kept to weight 10**

$\{4\}(< s;(0) >) =$

$$
\begin{array}{lll}
< 2;(0) > & + \ < 2;(4) > & + \ < 2;(4^2) > \\
+ \ < 2;(6) > & + \ < 2;(62) > & + \ < 2;(73) > \\
+\ 2 < 2;(8) > & + \ < 2;(91) > & + \ < 2;(10\ ) >
\end{array}
$$

$\{1^4\}(< s;(1) >) =$

$$
\begin{array}{lll}
< 2;(1^4) > & + \ < 2;(41^2) > & + \ < 2;(4^2) > \\
+ \ < 2;(61^2) > & + \ < 2;(62) > & + \ < 2;(73) > \\
+\ 2 < 2;(81^2) > & + \ < 2;(91) > &
\end{array}
$$

- **Looking at the above results one cannot be but struck by the apparent simple mapping between them. Indeed looking at much more extensive tabulations one observes that the terms in $\{\lambda\}(< s;(0) >)$ are simply related to those in $\{\tilde{\lambda}\}(< s;(1) >)$ by a one-to-one mapping subject to the following simple rules:-**

$\lambda \vdash 2$	$(0) \to (1^2)$		
$\lambda \vdash 3$	$(0) \to (1^3)$	$(a) \to (a1)$	$(a1) \to (a)$
$\lambda \vdash 4$	$(0) \to (1^4)$	$(a) \to (a1^2)$	$(a1^2) \to (a)$
$\lambda \vdash 5$	$(0) \to (1^5)$	$(a) \to (a1^3)$	$(a1^3) \to (a)$
		$(ab) \to (ab1)$	$(ab1) \to (ab)$
$\lambda \vdash 6$	$(0) \to (1^6)$	$(a) \to (a1^4)$	$(a1^4) \to (a)$
		$(ab) \to (ab1^2)$	$(ab1^2) \to (ab)$

Concluding Remarks

- The study of plethysms for the non-compact group $Sp(2n, R)$ throws up many surprises that could be of interest to combinatorialists. As the group is associated with infinite dimensional irreps it is not surprising that infinite series of $S-$functions arise.

- The subject is wide open and barely explored. The conjugacy relations just noticed hint at further structure to be discovered.

- Tables of the relevant plethysms are located at http://www.phys.uni.torun.pl/~bgw/

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