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Colour–Spin Matrix Elements for Multiquark Systems

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Abstract

Tensor operator methods have been developed for calculating the matrix elements of the two-particle colour–spin operator that arises in the calculation of the quark–gluon interaction in the MIT bag model treatment of the S-wave colour singlet states of multiquark hadrons. A group classification scheme for multiquark states which distinguishes the nonstrange and strange quarks, and thus avoids the occurrence of hidden strangeness $s\bar{s}$ pairs, is constructed. This scheme has the added advantage of avoiding any need to approximate the strangeness dependence of the relevant interaction integrals. The colour–spin matrix elements for all the $q^4\bar{q}$ colour singlet states and for the strangeness -2 states of q^6 are given by way of examples. A number of checking procedures have been developed to ensure the correctness of the calculated matrix elements.

Introduction

The MIT bag model of hadrons (Chodos *et al.* 1974*a*, 1974*b*; De Grand *et al.* 1975; De Grand and Jaffe 1976) provides a practical method for obtaining a phenomenological description of hadrons within the framework of quantum chromodynamics (QCD). It is generally believed, but by no means proven, that the only observable states are those corresponding to colour singlets under the colour group $\mathrm{SU}_3^{\mathrm{C}}$. Following upon the original work of Jaffe (1977*a*, 1977*b*, 1977*c*) there has been great interest in calculating the masses and decay channels for multiquark hadrons such as q^N ($N = 0 \pmod 3$) (Jaffe 1977*c*; Aerts *et al.* 1978; Wybourne 1978), $q^2\bar{q}^2$ (Jaffe 1978) and $q^4\bar{q}$ (Strottman 1978, 1979).

An essential part of all these calculations has been the evaluation of the matrix elements of the two-particle colour–spin operator representing the colour–magnetic interaction energy associated with the exchange of coloured gluons, namely

$$H_g = -(\alpha_c/R)\sum_{i<j} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j M(m_i R, m_j R), \tag{1}$$

where R is the bag radius, α_c the quark–gluon coupling constant, $M(m_i, R, m_j R)$ a radial integral defined in terms of the quark masses and bag radius (Jaffe 1977*b*), and $\boldsymbol{\lambda}_i$ the colour of the ith quark with $\boldsymbol{\sigma}_i$ its spin. In this paper we limit our attention to the case of two ordinary quark flavours ($o \equiv$ u, d) and one strange flavour (s) and assume u and d have the same mass. For reasons of brevity it is convenient to make the replacement

$$ab \equiv M(m_a R, m_b R) \tag{2}$$

and to introduce the colour–spin operator

$$\Delta_{\mathrm{g}}^{ab} = -\sum_{i<j}^{a,b} \boldsymbol{\sigma}_i . \boldsymbol{\sigma}_j \boldsymbol{\lambda}_i . \boldsymbol{\lambda}_j , \tag{3}$$

where we understand the summation over i and j is restricted to quarks of flavour a and b only. Equation (1) now becomes

$$H_{\mathrm{g}} = (\alpha_{\mathrm{c}}/R) \sum \Delta_{\mathrm{g}}^{ab}\, ab , \tag{4}$$

where the summation is over all pairs of flavoured quarks q_α that arise in a given multiquark configuration, say $q_\alpha^r q_\beta^s \ldots \bar{q}_\varepsilon^x \bar{q}_\zeta^y \ldots$. Our primary object now becomes to classify the states of a given multiquark configuration and then to evaluate all the matrix elements of all the relevant Δ_{g}^{ab} operators.

If we choose our classification of the multiquark configurations appropriately then we can avoid the need to make the Jaffe (1977*a*) approximation of replacing the integrals $M(m_i R, m_j R)$ for an N-particle state having n_s strange quarks by $M((n_s/N)m_s R, (n_s/N)m_s R)$ and also avoid the difficulties of hidden strangeness $\mathrm{s\bar{s}}$ pairs.

In this paper we shall first discuss the classification of the multiquark states and then represent the colour–spin operator in terms of the generators of various $\mathrm{SU}_6 \supset \mathrm{SU}_2 \otimes \mathrm{SU}_3$ group structures. The relevant $3jm$ and $6j$ symbols can then be constructed and the Wigner–Eckart theorem used to evaluate the matrix elements of the group generators and hence those of the colour–spin operators. Explicit colour–spin matrices are constructed for the colour singlet states of $q^4\bar{q}$ and the strangeness -2 states of q^6. Finally, we establish a number of checking procedures for verifying the correctness of the matrix elements. These matrices are currently being used to make mass calculations for $q^4\bar{q}$ and q^6 avoiding the usual Jaffe (1977*a*) approximation.

Classification of Multiquark States

The complete set of states formed from x quarks coming in three flavours (u, d, s) and three colours will span the 1^x irreducible representation (irrep) of U_{18}. This irrep may be decomposed into those of the direct product group $\mathrm{SU}_{12} \otimes \mathrm{U}_6$ with the nonstrange quarks (u, d) transforming under SU_{12} and the strange quarks (s) under U_6. The states formed by the u, d quarks may be further described by the subgroup chain

$$\mathrm{SU}_{12} \rightarrow \mathrm{SU}_2^I \otimes (\mathrm{SU}_6^{CS} \supset \mathrm{SU}_2^S \otimes \mathrm{SU}_3^C) , \tag{5}$$

while the states formed by the strange quarks can be associated with the subgroup chain

$$\mathrm{U}_6 \rightarrow \mathrm{U}_1^{\mathscr{S}} \otimes (\mathrm{SU}_6^{CS} \supset \mathrm{SU}_2^S \otimes \mathrm{SU}_3^C) . \tag{6}$$

In the decomposition (5) the group SU_2^I is the isospin group and in (6) $\mathrm{U}_1^{\mathscr{S}}$ generates the strangeness quantum number $\mathscr{S}$. The group SU_6^{CS} is the usual colour–spin group with SU_2^S being the spin group and SU_3^C the group of colour. The states for a system of x antiquarks may be described by an equivalent group–subgroup structure except that each irrep is replaced by its conjugate irrep. Thus the set of antiquark states belong to the 1^{x*} irrep of U_{18}.

Table 1. Colour–Spin states for $q^4\bar{q}$ and $q_a^4 q_b^2$ quark configurations

See the text for an explanation of the symbolism used

(a) Generic configurations of $q^4\bar{q}$

State	Quantum numbers	State	Quantum numbers
Configuration $q_a^4\,\bar{q}_b$			
a_1	$\lvert\,21^{2}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	d	$\lvert\,2^{2}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
a_2	$\lvert\,21^{2}\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	e	$\lvert\,2^{2}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
b_1	$\lvert\,21^{2}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	f	$\lvert\,1^{4}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
b_2	$\lvert\,21^{2}\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	g	$\lvert\,1^{4}\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
c	$\lvert\,21^{2}\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{6}0\rangle$		
Configuration $q_a^3\,q_b\,\bar{q}_c$			
a_1	$\lvert(21\,{}^{4}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	c	$\lvert(21\,{}^{4}21,\ 1\,{}^{2}1)\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{6}0\rangle$
a_2	$\lvert(21\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	d_1	$\lvert(1^{3}\,{}^{4}0,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
a_3	$\lvert(21\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	d_2	$\lvert(1^{3}\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
a_4	$\lvert(21\,{}^{2}0,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	d_3	$\lvert(1^{3}\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
a_5	$\lvert(21\,{}^{2}0,\ 1\,{}^{2}1)\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	e_1	$\lvert(1^{3}\,{}^{4}0,\ 1\,{}^{2}1)\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
b_1	$\lvert(21\,{}^{4}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	e_2	$\lvert(1^{3}\,{}^{4}0,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
b_2	$\lvert(21\,{}^{4}21,\ 1\,{}^{2}1)\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	e_3	$\lvert(1^{3}\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
b_3	$\lvert(21\,{}^{2}21,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	f	$\lvert(1^{3}\,{}^{4}0,\ 1\,{}^{2}1)\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{6}0\rangle$
b_4	$\lvert(21\,{}^{2}0,\ 1\,{}^{2}1)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$		
Configuration $q_a^2\,q_b^2\,\bar{q}_c$			
g_1	$\lvert(2\,{}^{3}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	k_1	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
g_2	$\lvert(2\,{}^{3}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	k_2	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
g_3	$\lvert(2\,{}^{1}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	k_3	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{1}2)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
g_4	$\lvert(2\,{}^{1}1^{2},\ 1^{2}\,{}^{1}2)\,{}^{1}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$	k_4	$\lvert(1^{2}\,{}^{1}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{2}0\rangle$
h_1	$\lvert(2\,{}^{3}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	l_1	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
h_2	$\lvert(2\,{}^{3}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	l_2	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
h_3	$\lvert(2\,{}^{1}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$	l_3	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{1}2)\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
i	$\lvert(2\,{}^{3}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{6}0\rangle$	l_4	$\lvert(1^{2}\,{}^{1}2,\ 1^{2}\,{}^{3}1^{2})\,{}^{3}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{4}0\rangle$
		m	$\lvert(1^{2}\,{}^{3}1^{2},\ 1^{2}\,{}^{3}1^{2})\,{}^{5}1,\quad 1^{5}\,{}^{2}1^{2};\ {}^{6}0\rangle$

(b) Generic configurations of $q_a^4\,q_b^2$

State	Quantum numbers	State	Quantum numbers
a_1	$\lvert\,21^{2}\,{}^{5}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{3}0\rangle$	f	$\lvert\,2^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{3}0\rangle$
a_2	$\lvert\,21^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{3}0\rangle$	g_1	$\lvert\,2^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{5}0\rangle$
a_3	$\lvert\,21^{2}\,{}^{1}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{3}0\rangle$	g_2	$\lvert\,2^{2}\,{}^{5}2^{2},\quad 1^{2}\,{}^{1}2;\ {}^{5}0\rangle$
a_4	$\lvert\,21^{2}\,{}^{3}2^{2},\quad 1^{2}\,{}^{1}2;\ {}^{3}0\rangle$	h_1	$\lvert\,1^{4}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{1}0\rangle$
b	$\lvert\,21^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{1}0\rangle$	h_2	$\lvert\,1^{4}\,{}^{1}2^{2},\quad 1^{2}\,{}^{1}2;\ {}^{1}0\rangle$
c_1	$\lvert\,21^{2}\,{}^{5}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{5}0\rangle$	i	$\lvert\,1^{4}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{3}0\rangle$
c_2	$\lvert\,21^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{5}0\rangle$	j	$\lvert\,1^{4}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{5}0\rangle$
d	$\lvert\,21^{2}\,{}^{5}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{7}0\rangle$	k_1	$\lvert\,1^{4}\,{}^{3}1,\quad 2\,{}^{1}1^{2};\ {}^{3}0\rangle$
e_1	$\lvert\,2^{2}\,{}^{3}1,\quad 1^{2}\,{}^{3}1^{2};\ {}^{1}0\rangle$	k_2	$\lvert\,1^{4}\,{}^{1}2^{2},\quad 2\,{}^{3}2;\ {}^{3}0\rangle$
e_2	$\lvert\,2^{2}\,{}^{1}2^{2},\quad 1^{2}\,{}^{1}2;\ {}^{1}0\rangle$		

The multiquark states formed from x quarks may be designated by the typical ket vectors

$$q_o^k I\,\lambda_o^{CS} S_o\,\mu_o^C q_s^{x-k}\mathscr{S}\,\lambda_s^{CS} S_s\,\mu_s^C;\ S_{os}\,\mu_{os}^C\,i\rangle, \tag{7}$$

where the nonstrange ($o \equiv$ u, d) and strange (s) quarks are clearly distinguished and i stands for all other quantum numbers required such as, for example, isospin components I_z. Each set of kets for a given (k, x) will be associated with a particular quark configuration $q_o^k q_s^{x-k}$.

The states formed from a set of y antiquarks may be designated by a set of kets analogous to those given in (7). In the case of a configuration of the type $q_o^k q_s^{x-k} \bar{q}_o^l \bar{q}_s^{y-l}$ the multiquark states may be formed by coupling the quark and antiquark kets together to yield colour singlet states. In this case the irreps μ_{os}^C and $\mu_{\overline{os}}^C$ must be conjugate irreps in order to yield a colour singlet. If we are concerned with just quarks then μ_o^C and μ_s^C must be conjugate irreps with μ_{os}^C the identity irrep.

Tables of the possible states of multiquark systems can become very voluminous. For example in $q^4\bar{q}$ we must consider the states associated with the 10 distinct configurations $(q_o^4 \bar{q}_o, q_o^4 \bar{q}_s, q_s^4 \bar{q}_o, q_s^4 \bar{q}_s)$, $(q_o^3 q_s \bar{q}_o, q_o^3 q_s \bar{q}_s, q_s^3 q_o \bar{q}_o, q_s^3 q_o \bar{q}_s)$ and $(q_o^2 q_s^2 \bar{q}_o, q_o^2 q_s^2 \bar{q}_s)$. It is useful to represent each set of configurations by a generic configuration. Thus for $q^4\bar{q}$ it suffices to consider the three generic configurations $q_a^4 \bar{q}_b$, $q_a^3 q_b \bar{q}_c$ and $q_a^2 q_b^2 \bar{q}_c$ and to list just the colour–spin quantum numbers, omitting the specification of the quark and antiquark flavour quantum numbers, such as I and $\mathscr{S}$, which can be easily determined for each specific configuration.

Table 2. Isospin and colour–spin content of SU_{12} irreps

SU_{12}	I	λ	SU_{12}	I	λ	SU_{12}	I	λ
0	0	0	1^3	$\frac{1}{2}$	21	1^5	$\frac{1}{2}$	2^21
				$\frac{3}{2}$	1^3		$\frac{3}{2}$	21^3
1	$\frac{1}{2}$	1					$\frac{5}{2}$	1^5
			1^4	0	2^2			
1^2	0	2		1	21^2	1^6	0	2^3
	1	1^2		2	1^4		1	2^21^2
							2	21^4
							3	0

The colour–spin quantum numbers associated with the generic configurations arising in $q^4\bar{q}$ and for $q_a^4 q_b^2$ are given in Tables 1*a* and 1*b* respectively. Note that the spin multiplicity $2S+1$ is given as a left superscript attached to its associated SU_3^C irrep label. The coupling of the colour–spin quantum numbers to produce colour singlets is given in an obvious way: thus the ket $|21^2\,{}^31, 1^5\,{}^21^2; {}^20\rangle$ of $q_a^4\bar{q}_b$ corresponds to the coupling $|\lambda_a^{CS}\,{}^{2S_a+1}\mu_a^C, \lambda_b^{CS}\,{}^{2S_b+1}\mu_b^C; {}^{2S_{ab}+1}\mu_{ab}^C\rangle$, where $\lambda_a^{CS} \equiv 21^2$, ${}^{2S_a+1}\mu_a^C \equiv {}^31$ etc.

To assist in the determination of the isospin I and strangeness $\mathscr{S}$ to be associated with a specific configuration we note that under $U_{18} \to SU_{12} \otimes U_6$ we have

$$1^n \to \sum_{x=0} 1^{n-x} \times 1^x, \tag{8}$$

where x ranges from 0 to the lesser of n and 6. The isospin and colour–spin representations that arise in the reduction of the 1^k irrep of SU_{12} into those of $SU_2^I \otimes SU_6^{CS}$ are given by (see Wybourne 1970, 1978)

$$1^k \to \sum_{\xi} \tilde{\xi}\,\xi, \tag{9}$$

where ξ is an irrep of SU_6^{CS} involving a partition of k into not more than six parts and $\tilde{\xi}$ is an irrep of SU_2^I involving not more than two parts (ξ_1, ξ_2) and conjugate to

ξ with

$$I = \tfrac{1}{2}(\xi_1 - \xi_2). \tag{10}$$

The isospin and colour–spin content of the antisymmetric irreps of SU_{12} are given in Table 2. The decomposition of the irrep 1^x of U_6 into irreps of $U_1^{\mathscr{S}} \otimes SU_6^{CS}$ is simply $1^x \to x\,.\,1^x$ with $\mathscr{S} = -x$.

Generators of $SU_6^{CS} \supset SU_2^S \otimes SU_3^C$ and Colour–Spin Operator

The generators of SU_6^{CS} belong to the 35-dimensional irrep 21^4 of SU_6. Under $SU_6^{CS} \to SU_2^S \otimes SU_3^C$ we have (Wybourne 1970)

$$21^4 \to {}^3 21 + {}^3 0 + {}^1 21 . \tag{11}$$

The three spin generators $\boldsymbol{\sigma}$ transform as ${}^3 0$ and generate the Lie algebra associated with the spin group SU_2^S, while the eight colour generators $\boldsymbol{\lambda}$ transform as ${}^1 21$ and generate SU_3^C.

The colour–spin operator involves the scalar product of the 24 operators $\boldsymbol{\sigma\lambda}$ which collectively transform under $SU_6^{CS} \supset SU_2^S \otimes SU_3^C$ as the $21^4\,{}^3 21$ irrep. Since the colour–spin interaction can be expressed entirely in terms of the generators of SU_6^{CS} it must be diagonal in SU_6^{CS} irreps though not necessarily in those of SU_2^S or SU_3^C.

The two-particle colour–spin operator must transform under $SU_2^S \otimes SU_3^C$ as a ${}^1 0$ state. This suggests that we construct a two-particle tensor operator out of scalar coupled products of single-particle tensor operators

$$x_i^a \equiv (21^4\,{}^3 21)_i^a , \tag{12}$$

where a designates the species of quark upon which the single-particle operator x_i acts. We note that any two-particle operator may be rewritten as

$$\sum_{i<j} \boldsymbol{x}_i^a \,.\, \boldsymbol{x}_j^b = \tfrac{1}{2}\Big((2-\delta_{ab})\boldsymbol{X}^a \,.\, \boldsymbol{X}^b - \sum_{i=1}^{N} (\boldsymbol{x}_i^a)^2 \delta_{ab}\Big) , \tag{13}$$

where

$$\boldsymbol{X}^a = \sum_{i=1}^{N} \boldsymbol{x}_i^a . \tag{14}$$

The last operator in equation (13) is simply related to the number operator.

We now try to represent the colour–spin operator Δ_{g}^{ab} of equation (3) in terms of a coupled scalar product of $SU_6 \supset SU_2 \otimes SU_3$ symmetrized tensor operators. Consider the operator

$$\boldsymbol{X}^{ab} \equiv [21_a^4\,{}^3 21,\, 21_b^4\,{}^3 21]_0^{10} . \tag{15}$$

Use of the Wigner–Eckart theorem followed by explicit evaluation of the colour–spin matrix elements for the nucleon N or Δ isospin multiplets shows that the desired relationship is

$$\Delta_{\mathrm{g}}^{ab} = [140\sqrt{6}\,(2-\delta_{ab})\boldsymbol{X}^{ab} + 8N\delta_{ab}] , \tag{16}$$

where N is the number of q_a quarks. Use of charge conjugation leads to the equivalences

$$\Delta_{\mathrm{g}}^{ab} \equiv \Delta_{\mathrm{g}}^{\bar{a}\bar{b}} \equiv -\Delta_{\mathrm{g}}^{a\bar{b}} \equiv -\Delta_{\mathrm{g}}^{\bar{a}b} . \tag{17}$$

We must now work towards the evaluation of the matrix elements of the colour–spin operator Δ_{g}^{ab} as expressed in equation (16). To do this we firstly compute the reduced matrix elements of our tensor operators.

Calculation of Reduced Matrix Elements

Any implementation of the Wigner–Eckart theorem in calculating matrix elements requires a knowledge of the 6*j* and 3*jm* symbols associated with the group–subgroup chain of interest, in our case the $SU_6 \supset SU_2 \otimes SU_3$ chain. Methods of computing such symbols have been outlined elsewhere (Butler 1975, 1981; Butler and Wybourne 1976; Butler *et al.* 1979). In our particular approach the 3*jm* symbols for $SU_6 \supset SU_2 \otimes SU_3$ arise only in the calculation of reduced matrix elements; after that only SU_2 and SU_3 6*j* symbols are needed and then only a very few. As such, it is more profitable to list the reduced matrix elements alone.

Table 3. Reduced matrix elements for SU_6 and $SU_6 \supset SU_2 \otimes SU_3$

In (*b*) for brevity $T \equiv 21^{4}\,{}^{3}21$; also a superscript plus or minus indicates whether or not the matrix element changes sign under permutation

Matrix element	Value	Matrix element	Value
(*a*) SU_6			
$\langle 1 \,\|21^4\|\, 1 \rangle$	1	$\langle 21^2 \,\|21^4\|\, 21^2 \rangle_0$	5
$\langle 1^2 \,\|21^4\|\, 1^2 \rangle$	2	$\langle 21^2 \,\|21^4\|\, 21^2 \rangle_1$	$-3i\sqrt{3}$
$\langle 1^3 \,\|21^4\|\, 1^3 \rangle$	$-\sqrt{6}$	$\langle 21 \,\|21^4\|\, 21 \rangle_0$	$-\frac{13}{3}$
$\langle 1^4 \,\|21^4\|\, 1^4 \rangle$	2	$\langle 21 \,\|21^4\|\, 21 \rangle_1$	$-\frac{8}{3}i\sqrt{2}$
$\langle 2 \,\|21^4\|\, 2 \rangle$	$-2\sqrt{2}$	$\langle 2^2 \,\|21^4\|\, 2^2 \rangle$	-8
(*b*) $SU_6 \supset SU_2 \otimes SU_3$			
$\langle 1\ {}^{2}1 \,\|T\|\, 1\ {}^{2}1 \rangle^{+}$	$\frac{2}{35}\sqrt{210}$	$\langle 21\ {}^{2}21 \,\|T\|\, 21\ {}^{2}0 \rangle^{+}$	$-\frac{4}{35}\sqrt{35}$
$\langle 1^2\ {}^{3}1^2 \,\|T\|\, 1^2\ {}^{3}1^2 \rangle^{+}$	$-\frac{2}{35}\sqrt{210}$	$\langle 21\ {}^{2}21 \,\|T\|\, 21\ {}^{2}21 \rangle_{0}^{-}$	0
$\langle 1^2\ {}^{3}1^2 \,\|T\|\, 1^2\ {}^{1}2 \rangle^{+}$	$\frac{6}{35}\sqrt{35}$	$\langle 21\ {}^{2}21 \,\|T\|\, 21\ {}^{2}21 \rangle_{1}^{+}$	$\frac{4}{35}i\sqrt{35}$
$\langle 2\ {}^{3}2 \,\|T\|\, 2\ {}^{3}2 \rangle^{+}$	$\frac{2}{7}\sqrt{42}$	$\langle 2^2\ {}^{3}1 \,\|T\|\, 2^2\ {}^{3}1 \rangle^{+}$	$\frac{1}{14}\sqrt{210}$
$\langle 2\ {}^{3}2 \,\|T\|\, 2\ {}^{1}1^2 \rangle^{+}$	$-\frac{6}{35}\sqrt{35}$	$\langle 2^2\ {}^{3}1 \,\|T\|\, 2^2\ {}^{1}2^2 \rangle^{+}$	$\frac{3}{35}\sqrt{70}$
$\langle 1^3\,{}^{2}21 \,\|T\|\, 1^3\,{}^{2}21 \rangle_{0}^{+}$	$-\frac{4}{7}\sqrt{7}$	$\langle 2^2\ {}^{3}1 \,\|T\|\, 2^2\ {}^{5}2^2 \rangle^{+}$	$-\frac{3}{7}\sqrt{7}$
$\langle 1^3\,{}^{2}21 \,\|T\|\, 1^3\,{}^{2}21 \rangle_{1}^{-}$	0	$\langle 21^2\ {}^{5}1 \,\|T\|\, 21^2\ {}^{5}1 \rangle^{+}$	$\frac{1}{7}\sqrt{42}$
$\langle 1^3\ {}^{4}0 \,\|T\|\, 1^3\,{}^{2}21 \rangle^{-}$	$-\frac{4}{35}\sqrt{70}$	$\langle 21^2\ {}^{5}1 \,\|T\|\, 21^2\ {}^{3}1 \rangle^{-}$	$-\frac{3}{7}\sqrt{7}$
$\langle 21\ {}^{4}21 \,\|T\|\, 21\ {}^{4}21 \rangle_{0}^{-}$	0	$\langle 21^2\ {}^{3}1 \,\|T\|\, 21^2\ {}^{3}1 \rangle^{+}$	$-\frac{1}{70}\sqrt{210}$
$\langle 21\ {}^{4}21 \,\|T\|\, 21\ {}^{4}21 \rangle_{1}^{+}$	$\frac{4}{7}i\sqrt{14}$	$\langle 21^2\ {}^{3}1 \,\|T\|\, 21^2\ {}^{1}1 \rangle^{-}$	$-\frac{3}{35}\sqrt{35}$
$\langle 21\ {}^{4}21 \,\|T\|\, 21\ {}^{2}21 \rangle_{0}^{-}$	$-\frac{4}{7}\sqrt{7}$	$\langle 21^2\ {}^{5}1 \,\|T\|\, 21^2\ {}^{3}2^2 \rangle^{+}$	$-\frac{1}{7}\sqrt{42}$
$\langle 21\ {}^{4}21 \,\|T\|\, 21\ {}^{2}21 \rangle_{1}^{-}$	$-\frac{4}{35}i\sqrt{35}$	$\langle 21^2\ {}^{3}2^2 \,\|T\|\, 21^2\ {}^{1}1 \rangle^{+}$	$-\frac{2}{35}\sqrt{210}$
$\langle 21\ {}^{4}21 \,\|T\|\, 21\ {}^{2}0 \rangle^{-}$	$-\frac{4}{35}\sqrt{35}$	$\langle 21^2\ {}^{3}2^2 \,\|T\|\, 21^2\ {}^{3}1 \rangle^{-}$	$-\frac{3}{35}\sqrt{35}$

The SU_6 reduced matrix elements $\langle \lambda^{CS} \| 21^4 \| \lambda^{CS} \rangle$ may be calculated by noting that the operator S_z must transform as a $21^{4}\,{}^{3}00000$ tensor operator component and hence (cf. Butler *et al.* 1979)

$$\langle \lambda^{CS} S \mu^{C} S_z I^{C} Y^{C} I_z^{C} \,|\, 21^{4}\,{}^{3}00000 \,|\, \lambda^{CS} S \mu^{C} S_z I^{C} Y^{C} I_z^{C} \rangle = k S_z, \qquad (18)$$

where k is a proportionality constant associated with the normalization of our reduced matrix elements. Explicit use of the Wigner–Eckart theorem then leads to the result

$$\sum_r \begin{pmatrix} \lambda^{CS*} & 21^4 & \lambda^{CS} \\ S\mu^{C*} & {}^3 0 & S\mu^{C} \end{pmatrix}^r \langle \lambda^{CS} \| 21^4 \| \lambda^{CS} \rangle_r = k\surd |\mu^C| S(S+1)(2S+1), \tag{19}$$

where r is a product multiplicity index and $|\mu^C|$ is the dimension of the μ^C irrep of SU_3^C. Use of the known values of the $SU_6 \supset SU_2 \otimes SU_3$ $3jm$ symbols leads to equations in the reduced matrix elements. Choosing $\langle 1 \| 21^4 \| 1 \rangle = 1$ requires that we take $k = \surd(210)/105$, which leads to the results given in Table 3*a*.

The $SU_2 \otimes SU_3$ dependence of the matrix elements may be obtained by noting that

$$\langle \lambda_a^{CS} S_a \mu_a^C \| 21^4\,{}^3 21 \| \lambda_a^{CS} S_a' \mu_a^{C'} \rangle_s = \sum_r \begin{pmatrix} \lambda_a^{CS*} & 21^4 & \lambda_a^{CS} \\ S_a \mu_a^{C*} & {}^3 21 & S_a' \mu_a^{C'} \end{pmatrix}_s^r \langle \lambda_a^{CS} \| 21^4 \| \lambda_a^{CS} \rangle_r , \tag{20}$$

where s is an SU_3 product multiplicity index. The necessary reduced matrix elements are given in Table 3*b*. In this table we have written for brevity $T \equiv 21^4\,{}^3 21$. Also, a plus or minus sign is given as a superscript to indicate whether or not the matrix element changes sign under permutation. (Note that the reduced matrix elements in Table 3 depend on phase choices and multiplicity resolutions for the $SU_6 \supset SU_2 \otimes SU_3$ $3jm$ factors.)

Calculation of Colour–Spin Matrix Elements

We now direct our attention to the evaluation of the matrix elements of the colour–spin operator Δ_g^{ab}. Firstly we note that for a group of N quarks of the same flavour (q_a^N) we have the known result (Jaffe 1977*b*)

$$\langle q_a^N \lambda^{CS} S \mu^C i | \Delta_g^{aa} | q_a^N \lambda^{CS'} S' \mu^{C'} i' \rangle$$

$$= (8N - \tfrac{1}{2} C_6(\lambda^{CS}) + \tfrac{4}{3} S(S+1) + \tfrac{1}{2} C_3(\mu^C)) \delta(\lambda^{CS}, \lambda^{CS'}) \delta(S, S') \delta(\mu^C, \mu^{C'}) \delta(i, i') , \tag{21}$$

where for printing convenience we have put $\delta(x, x') \equiv \delta_{xx'}$. Here the eigenvalues of the Casimir invariants are given by

$$3C_6(\lambda^{CS}) = 12 \sum_{i=1}^{n} \lambda_i(\lambda_i + 7 - 2i) - 2m^2 , \tag{22}$$

where n is the number of parts of λ^{CS} and m is the weight of λ^{CS}, and

$$3C_3(\mu^C) = 4(\mu_1^2 + \mu_2^2 - \mu_1 \mu_2 + 3\mu_1) . \tag{23}$$

The results for other matrix elements of Δ_g^{ab} are rather more complicated and require full use of the Racah–Wigner calculus (see Butler 1981). Let us firstly recall some properties of scalar coupled products of tensor operators which are generalizations of the familiar SU_2 angular momentum tensor operator results (Judd 1963) to arbitrary compact groups. One property is

$$\langle x_1 \lambda_1 i_1 | [P^\kappa Q^{\kappa*}]_0^0 | x_2 \lambda_2 i_2 \rangle = \delta_{i_1 i_2} \delta_{\lambda_1 \lambda_2} |\lambda_1|^{-1} |\kappa|^{-\frac{1}{2}}$$

$$\times \sum_{s x_3 \lambda_3} \cdot \{\lambda_1\}\{\lambda_1^* \kappa \lambda_3 s\} \langle x_1 \lambda_1 \| P^\kappa \| x_3 \lambda_3 \rangle_s \langle x_3 \lambda_3 \| Q^{\kappa*} \| x_2 \lambda_2 \rangle_s , \tag{24}$$

where as usual we let $|\omega|$ stand for the dimension of the irrep ω, while s is a product multiplicity, $\{\lambda_1\}$ is a $2j$ phase and $\{\lambda_1^* \kappa \lambda_3 s\}$ is a $3j$ phase (Butler 1975, 1981; Butler and Wybourne 1976). (Use of equation (24) to evaluate the matrix elements of $\boldsymbol{X}^{ab}$ defined by equation (15) led directly to the result (16).)

A further useful result for computing with coupled scalar product operators acting on coupled kets is given by

$$\begin{aligned}\langle(\lambda_1 \lambda_2)r_1 \lambda i|\,[P^{\kappa} Q^{\kappa^*}]^0_0\,|(\mu_1 \mu_2)r_2 \mu j\rangle &\\ = \delta_{ij}\delta_{\lambda\mu}|\kappa|^{-\frac{1}{2}} \sum_{s_1 s_2} \{\lambda_2\}\{\lambda_2 \kappa \mu_2^* s_2\}\{\lambda_1 \lambda_2 \lambda^* r_1\} &\begin{Bmatrix} \mu_1 & \mu_2 & \lambda^* \\ \lambda_2^* & \lambda_1 & \kappa \end{Bmatrix}_{s_1 s_2 r_1 r_2} \\ \times\langle\lambda_1 \| P^{\kappa} \| \mu_1\rangle_{s_1} \langle\lambda_2 \| Q^{\kappa^*} \| \mu_2\rangle_{s_2}. &\qquad (25)\end{aligned}$$

The enumeration of essential results is completed by considering the reduced matrix elements for operators that act on only one part of a coupled ket. Thus if P^{κ_1} acts only on part 1 of a system then

$$\begin{aligned}\langle(\lambda_1 \lambda_2)r_1 \lambda \| P^{\kappa_1} \| (\mu_1 \mu_2)r_2 \mu\rangle_s &\\ = \delta_{\lambda_2\mu_2}|\lambda|^{\frac{1}{2}}|\mu|^{\frac{1}{2}} \sum_{s_1} \{\lambda_1\}\{\lambda_1 \lambda_2 \lambda^* r_1\}\{\lambda_1^* \kappa_1 \mu_1 s_1\} &\begin{Bmatrix} \lambda^* & \kappa_1 & \mu \\ \mu_1 & \lambda_2^* & \lambda_1 \end{Bmatrix}_{r_1 s_1 r_2 s} \\ \times\langle\lambda_1 \| P^{\kappa_1} \| \mu_1\rangle_{s_1}, &\qquad (26)\end{aligned}$$

while if Q^{κ_2} acts only on part 2 of a system then

$$\begin{aligned}\langle(\lambda_1 \lambda_2)r_1 \lambda \| Q^{\kappa_2} \| (\mu_1 \mu_2)r_2 \mu\rangle_s &\\ = \delta_{\lambda_1\mu_1}|\lambda|^{\frac{1}{2}}|\mu|^{\frac{1}{2}} \sum_{s_2} \{\mu_2\}\{\mu_1 \mu_2 \mu^* r_2\}\{\lambda_2^* \kappa_2 \mu_2 s_2\} &\begin{Bmatrix} \lambda^* & \kappa_2 & \mu \\ \mu_2 & \lambda_1^* & \lambda_2 \end{Bmatrix}_{r_1 s_2 r_2 s} \\ \times\langle\lambda_2 \| Q^{\kappa_2} \| \mu_2\rangle_{s_2}. &\qquad (27)\end{aligned}$$

(Note that equations (24)–(27) use Butler's (1981) canonical phases.)

Let us illustrate the use of the above results to derive a formula for the colour–spin operator $\Delta_{\mathrm{g}}^{a\bar{c}}$ acting on the colour singlet states of a configuration $q_a^x q_b^y \bar{q}_c^z$. A typical matrix element will be of the form

$$\begin{aligned}&\langle(\lambda_a^{CS} S_a \mu_a^C \lambda_b^{CS} S_b \mu_b^C)r_{ab} S_{ab} \mu_{ab}^C \lambda_c^{CS} S_c \mu_c^C;\ S\,0^C\ i|\,\Delta_{\mathrm{g}}^{a\bar{c}}\,| \\ &\qquad\times(\lambda_a^{CS} S'_a \mu_a^{C'} \lambda_b^{CS} S'_b \mu_b^{C'})r'_{ab} S'_{ab} \mu_{ab}^{C'} \lambda_c^{CS} S'_c \mu_c^{C'};\ S\,0^C\ i\rangle. \qquad (28)\end{aligned}$$

Noting the result (16) and then using equation (25) we find that the expression (28) becomes

$$\begin{aligned}&-280\sqrt{6}\sqrt{\tfrac{1}{24}} \sum_{s_1 s_2} (-1)^{2S_c}\{\mu_c^C\}(-1)^{S_c+1+S'_c}\{\mu_c^C\, 21^C\, \mu_c^{C'*}\, s_2\} \\ &\times(-1)^{S_{ab}+S_c+S}\{\mu_{ab}^C\, \mu_c^C\, 0^C\}\begin{Bmatrix} \mu_{ab}^{C'} & \mu_c^{C'} & 0^C \\ \mu_c^{C*} & \mu_{ab}^C & 21^C \end{Bmatrix}_{s_1 s_2 00} \begin{Bmatrix} S'_{ab} & S'_c & S \\ S_c & S_{ab} & 1 \end{Bmatrix} \\ &\times\langle(\lambda_a^{CS} S_a \mu_a^C \lambda_b^{CS} S_b \mu_b^C)r_{ab} S_{ab} \mu_{ab}^C \,\|\, 21_a^{4}\,{}^{3}21^C \,\|\, (\lambda_a^{CS} S'_a \mu_a^{C'} \lambda_b^{CS} S'_b \mu_b^{C'})r'_{ab} S'_{ab} \mu_{ab}^{C'}\rangle_{s_1} \\ &\times\langle\lambda_c^{CS} S_c \mu_c^C \,\|\, 21_c^{4}\,{}^{3}21^C \,\|\, \lambda_c^{CS} S'_c \mu_c^{C'}\rangle_{s_2}, \qquad (29)\end{aligned}$$

where we have made use of the fact that for SU_2^S we have for the $2j$ and $3j$ phases respectively

$$\{S\} = (-1)^{2S} \qquad \text{and} \qquad \{S_1\, S_2\, S_3\} = (-)^{S_1+S_2+S_3}. \tag{30}$$

The SU_2 $6j$ symbol may be evaluated from the tables of Rotenberg *et al.* (1959). The factor 24 in the expression (29) is just the dimension of the ${}^{3}21^{C}$ irrep of $SU_2^S \otimes SU_3^C$. The remaining $2j$ and $3j$ phases and the $6j$ symbols all pertain to SU_3. The SU_3 $2j$ phases may all be taken as unity while the $3j$ phases have been specified by Butler *et al.* (1979).

The appearance of the identity irrep in the $6j$ symbol leads to considerable simplification in the expression (29), to give

$$\begin{aligned}
&140\,\delta(\mu_c^C, \mu_{ab}^{C*})\,\delta(\mu_c^{C'}, \mu_{ab}^{C'*})\,(-1)^{S_{ab}+S_c'+S}\,|\mu_c^C|^{-\frac{1}{2}}\,|\mu_c^{C'}|^{-\frac{1}{2}}\begin{Bmatrix} S_{ab}' & S_c' & S \\ S_c & S_{ab} & 1 \end{Bmatrix} \\
&\times \sum_{s} \langle(\lambda_a^{CS}\, S_a\, \mu_a^C\, \lambda_b^{CS}\, S_b\, \mu_b^C) r_{ab}\, S_{ab}\, \mu_c^{C*} \,\|\, 21_a^{4}\, {}^{3}21^{C} \,\|\, (\lambda_a^{CS}\, S_a'\, \mu_a^{C'}\, \lambda_b^{CS}\, S_b'\, \mu_b^{C'}) r_{ab}'\, S_{ab}'\, \mu_c^{C'*}\rangle_s \\
&\qquad \times \langle \lambda_c^{CS}\, S_c\, \mu_c^C \,\|\, 21_c^{4}\, {}^{3}21^{C} \,\|\, \lambda_c^{CS}\, S_c'\, \mu_c^{C'}\rangle_s .
\end{aligned} \tag{31}$$

The second reduced matrix element may be lifted from the results given in Table 3*b*. The first reduced matrix element may be further evaluated by use of equation (26) to give

$$\begin{aligned}
&\langle(\lambda_a^{CS}\, S_a\, \mu_a^C\, \lambda_b^{CS}\, S_b\, \mu_b^C) r_{ab}\, S_{ab}\, \mu_c^{C*} \,\|\, 21_a^{4}\, {}^{3}21^{C} \,\|\, (\lambda_a^{CS}\, S_a'\, \mu_a^{C'}\, \lambda_b^{CS}\, S_b'\, \mu_b^{C'}) r_{ab}'\, S_{ab}'\, \mu_c^{C'*}\rangle_s \\
&= \delta(S_b, S_b')\,\delta(\mu_b^C, \mu_b^{C'})\,|\mu_c^C|^{\frac{1}{2}}\,|\mu_c^{C'}|^{\frac{1}{2}}\,[(2S_{ab}+1)(2S_{ab}'+1)]^{\frac{1}{2}} \\
&\times \sum_{s_1} (-1)^{S_b+S_{ab}+1+S_a'} \{\mu_a^C\, \mu_b^C\, \mu_c^C\, r_{ab}\}\{\mu_a^{C*}\, 21^C\, \mu_a^{C'}\, s_1\} \begin{Bmatrix} S_{ab} & 1 & S_{ab}' \\ S_a' & S_b & S_a \end{Bmatrix} \begin{Bmatrix} \mu_c^C & 21^C & \mu_c^{C'*} \\ \mu_a^{C'} & \mu_b^{C*} & \mu_a^C \end{Bmatrix}_{r_{ab} s_1 r_{ab}' s} \\
&\times \langle \lambda_a^{CS}\, S_a\, \mu_a^C \,\|\, 21^{4}\, {}^{3}21 \,\|\, \lambda_a^{CS}\, S_a'\, \mu_a^{C'}\rangle_{s_1} .
\end{aligned} \tag{32}$$

This completes the desired formula. Proceeding in this manner it is a simple matter to derive all the relevant formulae and then to compute all the colour–spin matrix elements for the generic configurations associated with $q^4\bar{q}$ and q^4q^2. The results are given in Tables 4*a* and 4*b* respectively. It is important to note that all these matrix elements have been multiplied by a common factor of 3 to eliminate denominators.

Checking of Colour–Spin Matrices

It is essential to be able to verify that the colour–spin matrices are correct and to this end a checking procedure must be devised. To make such a check we try to choose relationships between the various radial integrals that force the colour–spin matrices to have simple predictable eigenvalues.

In the case of $q_a^4 q_b^2$, making all the integrals *aa*, *bb* and *ab* equal to unity must yield a set of states that can be mapped onto those of q_a^6. Consider for example the states of $q_a^4 q_b^2$ with $\lambda_a^{CS} = 21^2$ and $\lambda_b^{CS} = 1^2$ that couple together to produce ${}^{3}0$ states under $SU_2^S \otimes SU_3^C$. There are just four such states (a_1, a_2, a_3, a_4). We note that

$$21^2 \times 1^2 \rightarrow 321 + 31^3 + 2^3 + 2^21^2 + 21^4 .$$

Table 4. **Colour–spin matrix elements for $q^4\bar{q}$ and $q_a^4 q_b^2$ configurations**

Note that all matrix elements here (but not their checking eigenvalues) have been multiplied by 3 to eliminate denominators

(a) Generic configurations of $q^4\bar{q}$

State	Matrix elements	Checking eigenvalues
Configuration $q_a^4\bar{q}_b$		
$^{2}0$	$\begin{array}{cc} & \begin{array}{cc}(a_1) & (a_2)\end{array} \\ \begin{array}{c}a_1\\a_2\end{array} & \begin{bmatrix}8(aa+a\bar{b}) & -24a\bar{b}\\ -24a\bar{b} & 0\end{bmatrix}\end{array}$	$(8, -8)$
$^{4}0$	$\begin{array}{cc} & \begin{array}{cc}(b_1) & (b_2)\end{array} \\ \begin{array}{c}b_1\\b_2\end{array} & \begin{bmatrix}4(2aa-a\bar{b}) & -12\sqrt{10}\,a\bar{b}\\ -12\sqrt{10}\,a\bar{b} & 24(aa-a\bar{b})\end{bmatrix}\end{array}$	$(24, -4)$
$^{6}0$	$\overset{(c)}{c\ [8(3aa+2a\bar{b})\ ]}$	$(\frac{8}{3})$
$^{2}0$	$\overset{(d)}{d\ [-8(2aa+5a\bar{b})]}$	(8)
$^{4}0$	$\overset{(e)}{e\ [-4(4aa-5a\bar{b})]}$	(-12)
$^{2}0$	$\overset{(f)}{f\ [8(7aa+4a\bar{b})\ ]}$	(8)
$^{4}0$	$\overset{(g)}{g\ [8(7aa-2a\bar{b})\ ]}$	(24)
Configuration $q_a^3 q_b \bar{q}_c$		
$^{2}0$	$\begin{array}{cc} & \begin{array}{ccccc}(a_1) & (a_2) & (a_3) & (a_4) & (a_5)\end{array} \\ \begin{array}{c}a_1\\ \\a_2\\ \\a_3\\a_4\\a_5\end{array} & \begin{bmatrix}\begin{array}{l}6aa-30ab\\-2b\bar{c}-30a\bar{c}\end{array} & 8(ab+2a\bar{c}) & -16\sqrt{3}\,a\bar{c} & 8(2ab+a\bar{c}) & -8\sqrt{3}\,a\bar{c}\\ 8(ab+2a\bar{c}) & \begin{array}{l}-6aa+6ab\\+4b\bar{c}-12a\bar{c}\end{array} & -2\sqrt{3}\,(b\bar{c}+3a\bar{c}) & 8(ab+2a\bar{c}) & 8\sqrt{3}\,a\bar{c}\\ -16\sqrt{3}\,a\bar{c} & -2\sqrt{3}\,(b\bar{c}+3a\bar{c}) & -6(aa+3ab) & 8\sqrt{3}\,a\bar{c} & -24ab\\ 8(2ab+a\bar{c}) & 8(ab+2a\bar{c}) & 8\sqrt{3}\,a\bar{c} & -24aa-32b\bar{c} & 16\sqrt{3}\,b\bar{c}\\ -8\sqrt{3}\,a\bar{c} & 8\sqrt{3}\,a\bar{c} & -24ab & 16\sqrt{3}\,b\bar{c} & -24aa\end{bmatrix}\end{array}$	$(-24, -8,$ $8, 8, 8)$
$^{4}0$	$\begin{array}{cc} & \begin{array}{cccc}(b_1) & (b_2) & (b_3) & (b_4)\end{array} \\ \begin{array}{c}b_1\\ \\b_2\\ \\b_3\\ \\b_4\end{array} & \begin{bmatrix}\begin{array}{l}6aa-30ab\\+b\bar{c}+15a\bar{c}\end{array} & -\sqrt{15}\,(b\bar{c}+3a\bar{c}) & 8(ab-a\bar{c}) & 4(4ab-a\bar{c})\\ -\sqrt{15}\,(b\bar{c}+3a\bar{c}) & \begin{array}{l}6aa+18ab\\+3b\bar{c}-27a\bar{c}\end{array} & -8\sqrt{15}\,a\bar{c} & -4\sqrt{15}\,a\bar{c}\\ 8(ab-a\bar{c}) & -8\sqrt{15}\,a\bar{c} & \begin{array}{l}-6(aa-ab)\\-2b\bar{c}+6a\bar{c}\end{array} & 8(ab-a\bar{c})\\ 4(4ab-a\bar{c}) & -4\sqrt{15}\,a\bar{c} & 8(ab-a\bar{c}) & -8(3aa-2b\bar{c})\end{bmatrix}\end{array}$	$(24, -20,$ $-12, -4)$
$^{6}0$	$\overset{(c)}{c\ [6aa+18ab-2b\bar{c}+18a\bar{c}]}$	$(\frac{8}{3})$

Table 4 (*Continued*)

State	Matrix elements	Checking eigenvalues
$^2 0$	$\begin{array}{c\|ccc} & (d_1) & (d_2) & (d_3) \\ \hline d_1 & 8(3aa+2b\bar{c}) & 8\sqrt{2}\,(2ab+a\bar{c}) & -8\sqrt{6}\,a\bar{c} \\ d_2 & 8\sqrt{2}\,(2ab+a\bar{c}) & 30aa+10ab+4b\bar{c}+20a\bar{c} & -2\sqrt{3}\,(b\bar{c}-5a\bar{c}) \\ d_3 & -8\sqrt{6}\,a\bar{c} & -2\sqrt{3}\,(b\bar{c}-5a\bar{c}) & 30(aa-ab) \end{array}$	(8, 8, −8)
$^4 0$	$\begin{array}{c\|ccc} & (e_1) & (e_2) & (e_3) \\ \hline e_1 & 24(aa-b\bar{c}) & 8\sqrt{15}\,b\bar{c} & -4\sqrt{30}\,a\bar{c} \\ e_2 & 8\sqrt{15}\,b\bar{c} & 8(3aa-b\bar{c}) & 4\sqrt{2}\,(4ab-a\bar{c}) \\ e_3 & -4\sqrt{30}\,a\bar{c} & 4\sqrt{2}\,(4ab-a\bar{c}) & 30aa+10ab-2b\bar{c}-10a\bar{c} \end{array}$	(24, 24, −4)
$^6 0$	$\begin{array}{c\|c} & (f) \\ \hline f & [8(3aa+2b\bar{c})] \end{array}$	($\frac{8}{3}$)
Configuration $q_a^2 q_b^2 \bar{q}_c$		
$^2 0$	$\begin{array}{c\|cccc} & (g_1) & (g_2) & (g_3) & (g_4) \\ \hline g_1 & -4aa+8bb-20ab-20a\bar{c}+4b\bar{c} & -4\sqrt{2}\,(5a\bar{c}+b\bar{c}) & -24(ab+a\bar{c}) & 0 \\ g_2 & -4\sqrt{2}\,(5a\bar{c}+b\bar{c}) & -4aa+8bb-40ab & -12\sqrt{2}\,a\bar{c} & -12\sqrt{3}\,ab \\ g_3 & -24(ab+a\bar{c}) & -12\sqrt{2}\,a\bar{c} & -24aa+8bb-16b\bar{c} & -12\sqrt{6}\,b\bar{c} \\ g_4 & 0 & -12\sqrt{3}\,ab & -12\sqrt{6}\,b\bar{c} & -24aa+12bb \end{array}$	(8, 8, −8, −24)
$^4 0$	$\begin{array}{c\|ccc} & (h_1) & (h_2) & (h_3) \\ \hline h_1 & -4aa+8bb-20ab+10a\bar{c}-2b\bar{c} & -2\sqrt{5}\,(b\bar{c}+5a\bar{c}) & -24ab+12a\bar{c} \\ h_2 & -2\sqrt{5}\,(b\bar{c}+5a\bar{c}) & -4aa+8bb+20ab+6b\bar{c}-30a\bar{c} & 12\sqrt{5}\,a\bar{c} \\ h_3 & -24ab+12a\bar{c} & 12\sqrt{5}\,a\bar{c} & -24aa+8bb+8b\bar{c} \end{array}$	(−4, −20, 24)
$^6 0$	$\begin{array}{c\|c} & (i) \\ \hline i & [-4aa+8bb+20ab+20a\bar{c}-4b\bar{c}] \end{array}$	($\frac{8}{3}$)
$^2 0$	$\begin{array}{c\|cccc} & (k_1) & (k_2) & (k_3) & (k_4) \\ \hline k_1 & 8(aa+bb-ab)-8(b\bar{c}+a\bar{c}) & -8\sqrt{2}\,(a\bar{c}-b\bar{c}) & -24(ab+b\bar{c}) & -24(ab+a\bar{c}) \\ k_2 & -8\sqrt{2}\,(a\bar{c}-b\bar{c}) & 8(aa+bb-2ab) & 12\sqrt{2}b\,\bar{c} & -12\sqrt{2}a\,\bar{c} \\ k_3 & -24(ab+b\bar{c}) & 12\sqrt{2}\,b\bar{c} & 8aa+12bb+8a\bar{c} & 12ab \\ k_4 & -24(ab+a\bar{c}) & -12\sqrt{2}\,a\bar{c} & 12ab & 12aa+8bb+8b\bar{c} \end{array}$	(8, 8, −8, 8)
$^4 0$	$\begin{array}{c\|cccc} & (l_1) & (l_2) & (l_3) & (l_4) \\ \hline l_1 & 8(aa+bb-ab)+4(b\bar{c}+a\bar{c}) & -4\sqrt{5}\,(a\bar{c}-b\bar{c}) & -12(2ab-b\bar{c}) & -12(2ab-a\bar{c}) \\ l_2 & -4\sqrt{5}\,(a\bar{c}-b\bar{c}) & 8(aa+bb+ab)-12(a\bar{c}+b\bar{c}) & -12\sqrt{5}\,b\bar{c} & 12\sqrt{5}\,a\bar{c} \\ l_3 & -12(2ab-b\bar{c}) & -12\sqrt{5}\,b\bar{c} & 8aa+12bb-4a\bar{c} & 12ab \\ l_4 & -12(2ab-a\bar{c}) & 12\sqrt{5}\,a\bar{c} & 12ab & 12aa+8bb-4b\bar{c} \end{array}$	(24, 24, −12, −4)
$^6 0$	$\begin{array}{c\|c} & (m) \\ \hline m & [8(aa+bb+ab+a\bar{c}+b\bar{c})] \end{array}$	($\frac{8}{3}$)

Table 4 (*Continued*)

(*b*) *Generic configurations of* $q_a^4 q_b^2$

State	Matrix elements	Checking eigenvalues
$^{3}0$	$\begin{array}{c\|cccc} & (a_1) & (a_2) & (a_3) & (a_4) \\ \hline a_1 & 24aa+8bb-24ab & -4\sqrt{30}\,ab & 0 & -8\sqrt{15}\,ab \\ a_2 & -4\sqrt{30}\,ab & 8aa+8bb+4ab & -8\sqrt{6}\,ab & 12\sqrt{2}\,ab \\ a_3 & 0 & -8\sqrt{6}\,ab & 8bb & -16\sqrt{3}\,ab \\ a_4 & -8\sqrt{15}\,ab & 12\sqrt{2}\,ab & -16\sqrt{3}\,ab & 20aa+12bb \end{array}$	$(\frac{8}{3}, \frac{8}{3}, \frac{80}{3}, -\frac{28}{3})$
$^{1}0$	$\begin{array}{c\|c} & (b) \\ \hline b & 8aa+8bb+8ab \end{array}$	(8)
$^{5}0$	$\begin{array}{c\|cc} & (c_1) & (c_2) \\ \hline c_1 & 24aa+8bb-8ab & -12\sqrt{6}\,ab \\ c_2 & -12\sqrt{6}\,ab & 8aa+8bb-4ab \end{array}$	(16, —4)
$^{7}0$	$\begin{array}{c\|c} & (d) \\ \hline d & 24aa+8bb+16ab \end{array}$	(16)
$^{1}0$	$\begin{array}{c\|cc} & (e_1) & (e_2) \\ \hline e_1 & -16aa+8bb-40ab & 24\sqrt{3}\,ab \\ e_2 & 24\sqrt{3}\,ab & -12(aa-bb) \end{array}$	(8, −24)
$^{3}0$	$\begin{array}{c\|c} & (f) \\ \hline f & -16aa+8bb-20ab \end{array}$	$(-\frac{28}{3})$
$^{5}0$	$\begin{array}{c\|cc} & (g_1) & (g_2) \\ \hline g_1 & -16aa+8bb+20ab & -12\sqrt{6}\,ab \\ g_2 & -12\sqrt{6}\,ab & 12(aa+bb) \end{array}$	(16, −4)
$^{1}0$	$\begin{array}{c\|cc} & (h_1) & (h_2) \\ \hline h_1 & 56aa+8bb+32ab & 24\sqrt{6}\,ab \\ h_2 & 24\sqrt{6}\,ab & 60aa+12bb \end{array}$	(8, 48)
$^{3}0$	$\begin{array}{c\|c} & (i) \\ \hline i & 56aa+8bb+16ab \end{array}$	$(\frac{80}{3})$
$^{5}0$	$\begin{array}{c\|c} & (j) \\ \hline j & 56aa+8bb-16ab \end{array}$	(16)
$^{3}0$	$\begin{array}{c\|cc} & (k_1) & (k_2) \\ \hline k_1 & 56aa-24bb & -24\sqrt{2}\,ab \\ k_2 & -24\sqrt{2}\,ab & 60aa-4bb \end{array}$	$(\frac{80}{3}, \frac{8}{3})$

However, only 321, 31^3, 2^3 and 21^4 will yield a $^{3}0$ state under $SU_6^{CS} \rightarrow SU_2^S \otimes SU_3^C$ and hence the four states a_i of $q_a^4 q_b^2$ must go into the four states of q_a^6 associated with the 321, 31^3, 2^3 and 21^4 irreps of SU_6^{CS}. It follows from equation (21) (Wybourne 1978) that the four states of q_a^6 will have colour–spin matrix elements of

$$321(-28/3), \qquad 31^3(8/3), \qquad 2^3(8/3), \qquad 21^4(80/3). \tag{33}$$

Thus if the colour spin matrix for $q_a^4 q_b^2$ is correctly calculated then placing all the radial integrals equal to unity and diagonalizing the 4×4 matrix must yield the four eigenvalues (33), as is indeed the case. The checking eigenvalues for the other matrices of $q_a^4 q_b^2$ are given in the last column of Table 4*b*.

A similar check can be devised for the matrices associated with the generic configurations of $q^4 \bar{q}$. In this case we make all the radial integrals involving a pair of quarks equal to $+1$ and those involving a quark with an antiquark equal to -1.

Table 5. Colour–spin matrix elements of q^3 configurations

Quantum numbers	State	Matrix element	Checking eigenvalue
Configuration q_a^3			
$\alpha \vert 21\,{}^20\rangle$	${}^20 \quad \alpha$	(α) $[-8aa]$	(-8)
$\beta \vert 1^3\,{}^40\rangle$	${}^40 \quad \beta$	(β) $[8aa]$	(8)
Configuration $q_a^2\, q_b$			
$\gamma \vert 1^2\,{}^31^2\;1\,{}^21;\,{}^20\rangle$	${}^20 \quad \gamma$	(γ) $[\frac{8}{3}(aa-4ab)]$	(-8)
$\delta \vert 2\,{}^11^2\;1\,{}^21;\;{}^20\rangle$	${}^20 \quad \delta$	(δ) $[-8aa]$	(-8)
$\varepsilon \vert 1^2\,{}^31^2\;1\,{}^21;\,{}^40\rangle$	${}^40 \quad \varepsilon$	(ε) $[\frac{8}{3}(aa+2ab)]$	(8)

Table 6. Colour–spin matrices in baryon octet and decuplet

Here *o* denotes the nonstrange u and d quarks and *s* denotes the strange s quark

State	Matrices			
^{20}	(N) $\mathrm{N}[-8oo]$	(Σ) $\Sigma\,[\frac{8}{3}(oo-4os)]$	(Λ) $\Lambda\,[-8oo]$	(Ξ) $\Xi\,[\frac{8}{3}(ss-4os)]$
^{40}	(Δ) $\Delta\,[8oo]$	(Σ^*) $\Sigma^*\,[\frac{8}{3}(oo+2os)]$	(Ξ^*) $\Xi^*\,[\frac{8}{3}(ss+2os)]$	(Ω) $\Omega\,[8ss]$

In this case the states of $q^4 \bar{q}$ map onto those of q^9, since the transformation properties of an antiquark under SU_6^{CS} are those of a five-quark system. We can then use equation (21) (*but* with $N = 5$ and *not* $N = 9$) to predict the eigenvalues that should arise in diagonalizing the colour–spin matrices. Similar checks can be devised for other quark–antiquark configurations such as for example $q^3 \bar{q}^3$.

Example of q^3

As an illustration of the construction of colour–spin matrices from those for generic configurations, let us consider the familiar case of the baryon octet and decuplet states in q^3. The colour–spin matrix elements for q_a^3 and $q_a^2 q_b$ are given in Table 5. Neglecting the mass difference between the u and d quarks, which amounts to ignoring mass splittings within an isospin multiplet, we may make the associations

$$q_o^3 \sim \mathrm{N}, \Delta, \qquad q_o^2 q_s \sim \Sigma, \Lambda, \Sigma^*, \qquad q_o q_s^2 \sim \Xi, \Xi^*, \qquad q_s^3 \sim \Omega.$$

We may now make the appropriate identification of the radial integrals and extract the relevant matrix elements from Table 5 to yield the results of Table 6. Note that, had we used the Jaffe (1977*a*) approximation, states of the same strangeness and spin would be degenerate, e.g. Λ and Σ.

Conclusions

The colour–spin matrices for $q^4 \bar{q}$ and $q_a^4 q_b^2$ have been constructed and verified by making use of the full Racah–Wigner calculus. These results now make it possible to perform bag model calculations giving an adequate treatment of the colour–spin interaction. The methods used are quite general and can be easily extended to other interesting quark configurations such as $q^3 \bar{q}^3$. Explicit calculations using the colour–spin matrices presented here are currently being completed.

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