Analogies between finite-dimensional irreducible representations of SO(2n) and infinite-dimensional irreducible representations of Sp(2n,$\mathbb{R}$). II. Plethysms

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The basic spin difference character Δ'' of SO$(2n)$ is a useful device in dealing with characters of irreducible spinor representations of SO$(2n)$. It is shown here that its kth-fold symmetrized powers, or plethysms, associated with partitions κ of k factorize in such a way that $\Delta''\otimes\{\kappa\}=(\Delta'')^{r(\kappa)}\Pi_\kappa$, where $r(\kappa)$ is the Frobenius rank of κ. The analogy between SO$(2n)$ and Sp$(2n,\mathbb{R})$ is shown to be such that the plethysms of the basic harmonic or metaplectic character $\tilde{\Delta}$ of Sp$(2n,\mathbb{R})$ factorize in the same way to give $\tilde{\Delta}\otimes\{\kappa\}=(\tilde{\Delta})^{r(\kappa)}\tilde{\Pi}_\kappa$. Moreover, the analogy is shown to extend to the explicit decompositions into characters of irreducible representations of SO$(2n)$ and Sp$(2n,\mathbb{R})$ not only for the plethysms themselves, but also for their factors Π_κ and $\tilde{\Pi}_\kappa$. Explicit formulas are derived for each of these decompositions, expressed in terms of various group–subgroup branching rule multiplicities, particularly those defined by the restriction from O(k) to the symmetric group S_k. Illustrative examples are included, as well as an extension to the symmetrized powers of certain basic tensor difference characters of both SO$(2n)$ and Sp$(2n,\mathbb{R})$.
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I. INTRODUCTION

In a preceding paper[1] (hereafter referred to as KWI) the analogy between finite-dimensional representations of SO$(2N)$ and infinite-dimensional representations of Sp$(2n,\mathbb{R})$ was made highly explicit at the level of the characters of these representations and the decompositions of their various tensor products and powers. However, as pointed out in KWI a central problem in making applications of Sp$(2n,\mathbb{R})$ to various models of physical systems such as nuclei[2,3] and quantum dots[4,5] is the resolution of tensor powers of the fundamental metaplectic representation which has character[6,7] $\tilde{\Delta}$. Considerable progress[8–11] has been made on this problem, which amounts to the evaluation of symmetrized powers, or plethysms, of $\tilde{\Delta}$. Here we tackle this problem by emphasizing the remarkable analogies discussed in KWI that exist between SO$(2n)$ and Sp$(2n,\mathbb{R})$. In this context the precise analog of the basic metaplectic character, $\tilde{\Delta}$, of Sp$(2n,\mathbb{R})$ is the basic spin-difference character,[12,13] Δ'', of SO$(2n)$. While progress[14–16] has also been made on the problem of evaluating plethysms of such characters of SO$(2n)$, the aim here is to draw on the analogy that exists between the two problems so as to solve both problems in a unified manner.

Our notation follows that developed in KWI and in references contained therein. In the case of the orthogonal group O$(2n)$ the spin representation of dimension 2^n with character Δ decomposes

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on restriction to the proper orthogonal group SO$(2n)$ into a direct sum of two irreducible representations each of dimension 2^{n-1} with characters Δ_+ and Δ_-.

The relevant character formulas for SO$(2n)$ take the form:

$$\Delta=\Delta_+ + \Delta_- = \prod_{i=1}^{n} (x_i^{1/2}+x_i^{-1/2}), \tag{1.1a}$$

$$\Delta''=\Delta_+ - \Delta_- = \prod_{i=1}^{n} (x_i^{1/2}-x_i^{-1/2}), \tag{1.1b}$$

where x_i and x_i^{-1} for $i=1,2,\ldots,n$ are the eigenvalues of an arbitrary group element of SO$(2n)$. At the identity element I we have $x_i=1$ for $i=1,2,\ldots,n$ so that $\dim\Delta=2^n$ while $\dim\Delta''=0$.

The sum $\tilde{\Delta}$ and difference $\tilde{\Delta}''$ characters of the infinite-dimensional irreducible representations of Sp$(2n,\mathbb{R})$ are given by

$$\tilde{\Delta}=\tilde{\Delta}_+ + \tilde{\Delta}_- = \prod_{i=1}^{n} (x_i^{-1/2}-x_i^{1/2})^{-1}, \tag{1.2a}$$

$$\tilde{\Delta}''=\tilde{\Delta}_+ - \tilde{\Delta}_- = \prod_{i=1}^{n} (x_i^{-1/2}+x_i^{1/2})^{-1}, \tag{1.2b}$$

where now x_i and x_i^{-1} for $i=1,2,\ldots,n$ are the eigenvalues of an arbitrary group element of Sp$(2n,\mathbb{R})$.

The symmetric and antisymmetric squares of Δ and Δ'' are given by[14–16]

$$\Delta\otimes\{2\}=[1^n]_+ + [1^n]_- + \sum_{x=0}^{\infty} ([1^{n-1-4x}]+[1^{n-3-4x}]+2[1^{n-4-4x}]), \tag{1.3a}$$

$$\Delta\otimes\{1^2\}=\sum_{x=0}^{\infty} ([1^{n-1-4x}]+2[1^{n-2-4x}]+[1^{n-3-4x}]), \tag{1.3b}$$

$$\Delta''\otimes\{2\}=[1^n]_+ + \sum_{x=0}^{\infty} (-1)^{1+x}[1^{n-1-x}], \tag{1.3c}$$

$$\Delta''\otimes\{1^2\}=[1^n]_- + \sum_{x=0}^{\infty} (-1)^{1+x}[1^{n-1-x}], \tag{1.3d}$$

where $[1^k]$ is the character of the kth fold antisymmetrized power of the defining irreducible representation $[1]$ of SO$(2n)$. These representations are irreducible for $k=1,2\ldots,n-1$, while for $k=n$ we have $[1^n]=[1^n]_+ + [1^n]_-$.

Similarly, the symmetric squares of $\tilde{\Delta}$ and $\tilde{\Delta}''$ are given by[9–11]

$$\tilde{\Delta}\otimes\{2\}=\langle 1(0)\rangle + \sum_{x=0}^{\infty} \langle 1(x)\rangle, \tag{1.4a}$$

$$\tilde{\Delta}\otimes\{1^2\}=\langle 1(0)\rangle^* + \sum_{x=0}^{\infty} \langle 1(x)\rangle, \tag{1.4b}$$

$$\tilde{\Delta}''\otimes\{2\}=\langle 1(0)\rangle+\langle 1(0)\rangle^*+\sum_{x=0}^{\infty}(-\langle 1(1+4x)\rangle-\langle 1(3+4x)\rangle+2\langle 1(4+4x)\rangle), \quad (1.4c)$$

$$\tilde{\Delta}''\otimes\{1^2\}=\sum_{x=0}^{\infty}(-\langle 1(1+4x)\rangle+2\langle 1(2+4x)\rangle-\langle 1(3+4x)\rangle), \quad (1.4d)$$

where $\langle 1(m)\rangle$ is the character of a certain harmonic series infinite-dimensional irreducible representations of $Sp(2n,\mathbb{R})$ and the asterisk signifies the associate[11] of an irreducible representation of $Sp(2n,\mathbb{R})$.

Comparison of (1.1) and (1.2) gives a formal connection between the characters Δ and Δ'' of $SO(2n)$ and the characters $\tilde{\Delta}$ and $\tilde{\Delta}''$ of $Sp(2n,\mathbb{R})$. The formal connection is brought home rather forcibly in (1.3) and (1.4) through the analogy between the symmetrized squares of Δ and $\tilde{\Delta}''$, and between those of Δ'' and $\tilde{\Delta}$. It is the latter analogy which is explored further here through some observations on the somewhat unexpected factorization of the plethysms $\Delta''\otimes\{\kappa\}$ of $SO(2n)$ and $\tilde{\Delta}\otimes\{\kappa\}$ of $Sp(2n,\mathbb{R})$.

For $SO(2n)$, since[16]

$$\Delta_{\pm}\Delta_{\pm}=[1^n]_{\pm}+\sum_{x=0}^{\infty}[1^{n-2-2x}], \quad \Delta_{\pm}\Delta_{\mp}=\sum_{x=0}^{\infty}[1^{n-1-2x}], \quad (1.5)$$

it follows that (1.3c) and (1.3d) can be written in the form:

$$\Delta''\otimes\{2\}=\Delta''\Delta_{+}, \quad \Delta''\otimes\{1^2\}=-\Delta''\Delta_{-}, \quad (1.6)$$

with

$$\dim(\Delta_{+})=2^{n-1}, \quad \dim(-\Delta_{-})=-2^{n-1}. \quad (1.7)$$

These factorizations and the accompanying dimensionality formulas may appear somewhat unremarkable, however, it is also the case that[16]

$$\Delta''\otimes\{21\}=\Delta''\sum_{x=0}^{\infty}(-1)^x(-[1^{n-1-3x}]+[1^{n-2-3x}]), \quad (1.8)$$

with

$$\dim\left(\sum_{x=0}^{\infty}(-1)^x(-[1^{n-1-3x}]+[1^{n-2-3x}])\right)=-3^{n-1}. \quad (1.9)$$

This factorization and the accompanying dimensionality formula is far from trivial to derive, but taken in conjunction with (1.6) and (1.7) it is tempting to explore to what extent one might have

$$\Delta''\otimes\{\kappa\}=\Delta''\Pi_{\kappa}, \quad (1.10)$$

with Π_{κ} both belonging to the ring over $\mathbb{Z}$ of characters of irreducible representations of $SO(2n)$ and having dimension given by

$$\dim\Pi_{\kappa}=\pm k^{n-1}, \quad (1.11)$$

where κ is a partition of k.

Similarly for $Sp(2n,\mathbb{R})$ it is known that[9]

$$\tilde{\Delta}_{\pm}\tilde{\Delta}_{\pm}=\langle 1(0)\rangle_{\pm}+\sum_{x=0}^{\infty}\langle 1(2+2x)\rangle, \quad \tilde{\Delta}_{\pm}\tilde{\Delta}_{\mp}=\sum_{x=0}^{\infty}\langle 1(1+2x)\rangle, \tag{1.12}$$

where it has been convenient to denote $\langle 1(0)\rangle$ and $\langle 1(0)\rangle^*$ by $\langle 1(0)\rangle_+$ and $\langle 1(0)\rangle_-$, respectively. It then follows that (1.4a) and (1.4b) can be written in the following form:

$$\tilde{\Delta}\otimes\{2\}=\tilde{\Delta}\tilde{\Delta}_+, \quad \tilde{\Delta}\otimes\{1^2\}=\tilde{\Delta}\tilde{\Delta}_-. \tag{1.13}$$

Once again we have a rather trivial looking factorization leading us to seek an Sp$(2n,\mathbb{R})$ analog of (1.10) of the form

$$\tilde{\Delta}\otimes\{\kappa\}=\tilde{\Delta}\tilde{\Pi}_{\kappa}, \tag{1.14}$$

where $\tilde{\Pi}_{\kappa}$ belongs to the ring over $\mathbb{Z}$ of characters of irreducible representations of Sp$(2n,\mathbb{R})$, but now we would expect

$$\dim\tilde{\Pi}_{\kappa}=\infty. \tag{1.15}$$

Before embarking on the evaluation of the plethysms of interest here, namely $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$, some general formulas are given in Sec. II for the evaluation of arbitrary plethysms of the form $S\otimes\{\kappa\}$, emphasizing the advantages that follow from expressing the partition κ in Frobenius notation and from distinguishing between even and odd weight contributions to series of S functions. In conjunction with a crucial proposition due to Scharf and Thibon,[17] rederived here in Sec. II, some of these formulas are then used in Sec. III to evaluate quite explicitly the plethysms $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$. The results are expressed in terms of the branching rule coefficients appropriate to the restriction from the orthogonal group O(k) to its finite subgroup, the symmetric group S_k. In the case of the plethysms $\tilde{\Delta}\otimes\{\kappa\}$ of Sp$(2n,\mathbb{R})$ this connection with such branching rule coefficients was first pointed out by Carvalho.[8] The coefficients themselves may be evaluated in a variety of ways.[18–23]

The remaining formulas of Sec. II are then used in Sec. IV to derive factorizations of these same plethysms in the form

$$\Delta''\otimes\{\kappa\}=(\Delta'')^{r(\kappa)}\Pi_{\kappa} \tag{1.16}$$

and

$$\tilde{\Delta}\otimes\{\kappa\}=(\tilde{\Delta})^{r(\kappa)}\tilde{\Pi}_{\kappa}, \tag{1.17}$$

where $r(\kappa)$ is the Frobenius rank of the partition κ. Explicit formulas are given for Π_{κ} and $\tilde{\Pi}_{\kappa}$ in terms of characters of the symmetric group and certain symmetric functions. Furthermore, certain determinantal expansions are derived for both Π_{κ} and $\tilde{\Pi}_{\kappa}$, leading to a very simple dimension formula for Π_{κ}, but not of course for $\tilde{\Pi}_{\kappa}$, which is infinite dimensional.

However, these formulas do not reveal whether Π_{κ} and $\tilde{\Pi}_{\kappa}$ can be expressed as linear combinations of characters of irreducible representations of SO$(2n)$ and Sp$(2n,\mathbb{R})$, as appropriate, with integer coefficients. This is accomplished in Sec. V, where formulas interpolating between $\Delta''\otimes\{\kappa\}$ and Π_{κ}, and between $\tilde{\Delta}\otimes\{\kappa\}$ and $\tilde{\Pi}_{\kappa}$ are established. The coefficients in these expansions are all integers, determined once again by group–subgroup branching rules and their inverses. Numerous examples of the explicit calculation of Π_{κ}, $\tilde{\Pi}_{\kappa}$, $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$ are provided in Sec. VI.

Finally, in Sec. VII, the procedures are extended to the case of the plethysms of the basic tensor difference characters of SO$(2n)$ and Sp$(2n,\mathbb{R})$. Once again factorization occurs, and the remarkable analogy between SO$(2n)$ and Sp$(2n,\mathbb{R})$ is shown to hold true yet again.

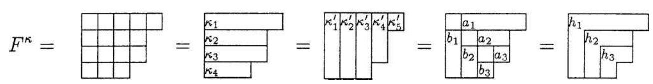
II. SOME TECHNIQUES FOR EVALUATING PLETHYSMS

Before embarking on the evaluation of pleythysms of Δ'' and $\tilde{\Delta}$, it is worth recalling from KWI some of the Schur-function and character-theoretic background to these problems. This relies heavily on the exploitation of partitions and Young diagrams.

Each partition $\kappa=(\kappa_1,\kappa_2,\ldots,\kappa_p)$ of k specifies a Young diagram F^κ consisting of $k=|\kappa|$ boxes arranged in $p=\ell(\kappa)$ left-adjusted rows of lengths κ_i for $i=1,2,\ldots,p$. The lengths κ_j' for $j=1,2,\ldots,q$ of the $q=b(\kappa)$ top-adjusted columns of F^κ serve to define the conjugate partition $\kappa'=(\kappa_1',\kappa_2',\ldots,\kappa_q')$. The number of boxes $r=r(\kappa)$ on the principal diagonal of F^κ is known as the Frobenius rank of the partition κ. In Frobenius notation

$$\kappa=\begin{pmatrix} a_1 & a_2\cdots a_r \\ b_1 & b_2\cdots b_r \end{pmatrix},$$

where for $k=1,2,\ldots,r$ the parameters $a_k=\kappa_k-k$ and $b_k=\kappa_k'-k$ are the arm and leg lengths, respectively, of F^κ with respect to its main diagonal of length r. With this notation the Young diagram can also be viewed as the union of a set of nested hooks of length $h_k=a_k+b_k+1$ with $k=1,2,\ldots,r$. All this is illustrated schematically by



(2.1)

With this notation there exist a number of distinct determinantal expansions of the Schur function $\{\kappa\}$. These include the following:[13,24,25]

$$\{\kappa\}=|\{\kappa_i-i+j\}|_{p\times p}=|\{1^{\kappa_j'-j+i}\}|_{q\times q}=|\{\kappa_i-i+1,1^{\kappa_j'-j}\}|_{r\times r}=\left|\begin{Bmatrix} a_i \\ b_j \end{Bmatrix}\right|_{r\times r}. \tag{2.2}$$

In the present context the significance of the last of these expansions is that for any linear combination, S, of Schur functions the evaluation of its plethysm $S\otimes\{\kappa\}$ can be effected by means of the determinantal expansion:

$$S\otimes\{\kappa\}=\left|S\otimes\begin{Bmatrix} a_s \\ b_t \end{Bmatrix}\right|_{r\times r}. \tag{2.3}$$

An alternative expansion of $\{\kappa\}$, entirely different to those of (2.2), takes the form:[13,23,24]

$$\{\kappa\}=\sum_{\rho\vdash k}\frac{1}{z_\rho}\chi_\rho^\kappa p_\rho, \tag{2.4}$$

where the sum is taken over all partitions ρ of k, and p_ρ is the power sum symmetric function specified by ρ. The coefficient χ_ρ^κ is the character in the irreducible representation (κ) of the symmetric group S_k of the conjugacy class of elements having cycle structure specified by the partition ρ. If the length of ρ is $\ell(\rho)$ then

$$\rho=(\rho_1,\rho_2,\ldots,\rho_{\ell(\rho)})=(k^{m_k},\ldots,2^{m_2},1^{m_1}) \tag{2.5}$$

with

$$\sum_{j=1}^{k} m_j=\ell(\rho),\quad \sum_{j=1}^{k} jm_j=k. \tag{2.6}$$

With this notation

$$z_\rho=\prod_{j=1}^{k} m_j!\,j^{m_j}, \quad p_\rho=\prod_{j=1}^{k} p_j^{m_j}, \tag{2.7}$$

where p_j is just the elementary power sum function defined for all $j\geqslant 1$ by

$$p_j(x_1,x_2,\ldots,x_n)=\sum_s x_s^j, \tag{2.8}$$

for whatever is the appropriate set of indeterminates $\{x_1,x_2,\ldots\}$ that is denumerable but not necessarily finite.

For any given κ the summation over ρ in (2.4) may be restricted to those partitions for which the characters χ_ρ^κ of S_k are nonvanishing. The Murnaghan–Nakayama recurrence relation for characters of the symmetric group takes the form:[12,13,24,25]

$$\chi_\rho^\kappa=\sum_\xi (-1)^{\ell\ell(\xi)}\chi_\sigma^\lambda, \tag{2.9}$$

where if $\rho=(\rho_1,\rho_2,\ldots,\rho_{\ell(\rho)})$ then $\sigma=(\rho_2,\rho_3,\ldots,\rho_{\ell(\rho)})$. The summation is over all continuous boundary strips ξ of length ρ_1 such that their removal from the Young diagram F^κ leaves F^λ for some partition λ. The parameter $\ell\ell(\xi)$ is the leg length of ξ, which is one less than the number of rows containing boxes within the boundary strip ξ.

For χ_ρ^κ to be nonvanishing it is necessary under the iteration of (2.9) to remove all boxes of F^κ through various sequences of precisely $\ell(\rho)$ continuous boundary strip removals leading to $\chi_0^0=1$ multiplied by some combination of leg length factors $(-1)^{\ell\ell(\xi)}$. Since these continuous boundary strips each have at most one box on any diagonal and the longest diagonal of F^κ is the principal diagonal whose length is the Frobenius rank $r=r(\kappa)$, it follows that[25]

$$\chi_\rho^\kappa=0 \quad \text{if} \quad \ell(\rho)<r(\kappa). \tag{2.10}$$

Just as (2.2) could be used by way of (2.3) to simplify the evaluation of plethysms, so a further time-honored method of evaluating $S\otimes\{\kappa\}$ makes use of the expansion (2.4). This approach, supplemented by the multiplicative expansion of p_ρ in (2.7) and the simple observation (2.10), yields the following formula:

$$S\otimes\{\kappa\}=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)} \frac{1}{z_\rho}\chi_\rho^\kappa\prod_{j=1}^{k} (S\otimes p_j)^{m_j}. \tag{2.11}$$

It might be stressed that the bound $\ell(\rho)=r(\kappa)$ can always be saturated in such a way that $\chi_\rho^\kappa\neq 0$. This is done most simply by setting $\rho=(h_1,h_2,\ldots,h_r)$. In fact $\chi_{h_1h_2\cdots h_r}^\kappa=(-1)^{b_1+b_2+\cdots+b_r}$. Moreover, for any $S=S(x_1,x_2,\ldots,x_n)$ and $j\geqslant 1$ we not only have $S\otimes p_j=p_j\otimes S$ but also

$$S(x_1,x_2,\ldots,x_n)\otimes p_j=S(x_1^j,x_2^j,\ldots,x_n^j). \tag{2.12}$$

In what follows a rather general lemma on plethysms is of use, namely:

Lemma 2.1: Let X and Y be two series of S-functions all of whose terms are of even and of odd weight, respectively, and let κ be an arbitrary partition. If

$$(X+Y)\otimes\{\kappa\}=\sum_\mu p_\kappa^\mu\{\mu\}, \tag{2.13}$$

then

$$(X'-Y')\otimes\{\kappa\}=\sum_{\mu}(-1)^{|\mu|}p_\kappa^\mu\{\mu'\}. \tag{2.14}$$

Proof: It should be noted from Lemma 5.3 of an earlier paper[11] that Littlewood's conjugacy formula[26] can be generalized to give in the present context

$$(X\otimes\{\rho\})'=X'\otimes\{\rho\},\quad (Y\otimes\{\rho\})'=Y'\otimes\{\rho'\}, \tag{2.15}$$

for any partition ρ. Moreover, Littlewood's algebra of plethysm[13,26] is such that

$$(X+Y)\otimes\{\kappa\}=\sum_{\sigma,\tau}c_{\sigma,\tau}^{\kappa}(X\otimes\{\sigma\})(Y\otimes\{\tau\})=\sum_{\mu:|\mu|\mathrm{even}}p_\kappa^\mu\{\mu\}+\sum_{\mu:|\mu|\mathrm{odd}}p_\kappa^\mu\{\mu\}=\sum_{\mu}p_\kappa^\mu\{\mu\}, \tag{2.16}$$

where the coefficients $c_{\sigma,\tau}^{\kappa}$ are the famous Littlewood–Richardson coefficients[24,25] defining products of S-functions, and the second step involves evaluating further products of the various S-functions appearing in $X\otimes\{\sigma\}$ and $Y\otimes\{\tau\}$, distinguishing between those of even and odd weight. Furthermore, thanks again in the first step to Littlewood's algebra of plethysm and in the second to (2.15), we have

$$\begin{aligned}(X'-Y')\otimes\{\kappa\}&=\sum_{\sigma,\tau}(-1)^{|\tau|}c_{\sigma,\tau}^{\kappa}(X'\otimes\{\sigma\})(Y'\otimes\{\tau'\})=\sum_{\sigma,\tau}(-1)^{|\tau|}c_{\sigma,\tau}^{\kappa}(X\otimes\{\sigma\})'(Y\otimes\{\tau\})'\\&=\sum_{\sigma,\tau}(-1)^{|\tau|}c_{\sigma,\tau}^{\kappa}((X\otimes\{\sigma\})(Y\otimes\{\tau\}))'=\sum_{\mu:|\mu|\mathrm{even}}p_\kappa^\mu\{\mu'\}-\sum_{\mu:|\mu|\mathrm{odd}}p_\kappa^\mu\{\mu'\}\\&=\sum_{\mu}(-1)^{|\mu|}p_\kappa^\mu\{\mu'\},\end{aligned} \tag{2.17}$$

as required. The penultimate step depends on the fact that the only terms $\{\mu\}$ appearing in (2.16) of odd weight are those that arise from products of the necessarily even weight terms of $X\otimes\{\sigma\}$ with some odd weight term of $Y\otimes\{\tau\}$. Such terms arise precisely when τ has odd weight.□

As an application of Lemma 2.1 we may apply it directly to the four S-function series denoted in KWI by M, L, Q, and P, thereby obtaining:

Corollary 2.2: Let

$$M\otimes\{\kappa\}=\left(\sum_{m=0}^{\infty}\{m\}\right)\otimes\{\kappa\}=\sum_{\mu}m_\kappa^\mu\{\mu\}, \tag{2.18a}$$

$$L\otimes\{\kappa\}=\left(\sum_{m=0}^{\infty}(-1)^m\{1^m\}\right)\otimes\{\kappa\}=\sum_{\mu}\ell_\kappa^\mu\{\mu\}, \tag{2.18b}$$

$$Q\otimes\{\kappa\}=\left(\sum_{m=0}^{\infty}\{1^m\}\right)\otimes\{\kappa\}=\sum_{\mu}q_\kappa^\mu\{\mu\}, \tag{2.18c}$$

$$P\otimes\{\kappa\}=\left(\sum_{m=0}^{\infty}(-1)^m\{m\}\right)\otimes\{\kappa\}=\sum_{\mu}p_\kappa^\mu\{\mu\}, \tag{2.18d}$$

then

$$\ell_\kappa^{\mu'}=(-1)^{|\mu|}m_\kappa^\mu\quad \textit{and}\quad p_\kappa^{\mu'}=(-1)^{|\mu|}q_\kappa^\mu. \tag{2.19}$$

Proof: If we write $M=X+Y$ with $X=\{0\}+\{2\}+\cdots$ and $Y=\{1\}+\{3\}+\cdots$, then $X'=\{0\}+\{1^2\}+\cdots$ and $Y'=\{1\}+\{1^3\}+\cdots$, so that $L=X'-Y'$. The application of Lemma 2.1 then leads immediately to the first part of (2.19). Likewise, if we write $Q=X+Y$ with $X=\{0\}+\{1^2\}+\cdots$ and $Y=\{1\}+\{1^3\}+\cdots$, then $X'=\{0\}+\{2\}+\cdots$ and $Y'=\{1\}+\{3\}+\cdots$, so that $P=X'-Y'$. The application of Lemma 2.1 then leads immediately to the second part of (2.19). □

At least for reasonably small values of k, the weight of κ, it is not difficult, although it is certainly tedious, to evaluate the various coefficients m_κ^μ, ℓ_κ^μ, q_κ^μ, and p_κ^μ appearing in the plethysms (2.18) up to any preassigned weight $|\mu|$ through the use, for example, of the software package SCHUR.[27] However, it is well worth noting that the following proposition has been derived by Scharf and Thibon[17] as part of a Hopf algebra approach to inner plethysms:

Proposition 2.3: Let μ be a partition which is $U(k)$-standard in the sense that $\ell(\mu)=\mu_1'\leqslant k$ and let the coefficients g_κ^μ be defined by the $U(k)$ to S_k branching rule:

$$U(k)\rightarrow S_k:\quad \{\mu\}\rightarrow\sum_{\kappa:\kappa\vdash k} g_\kappa^\mu(\kappa), \tag{2.20}$$

where the summation is over all partitions κ of k, then

$$M\otimes\{\kappa\}=\sum_{\mu:\ell(\mu)\leqslant k} g_\kappa^\mu\{\mu\}\quad \textit{and}\quad L\otimes\{\kappa\}=\sum_{\mu:\ell(\mu)\leqslant k}(-1)^{|\mu|}g_\kappa^\mu\{\mu'\}. \tag{2.21}$$

The validity of the crucial first part of (2.21) was established[17] as a reciprocity theorem linking characters of U(n) and S_k. The second part of (2.21) is then a trivial consequence of the conjugacy relation (2.19). However, we can also offer an alternative proof of (2.21) using one of Littlewood's results[28] on inner plethysms.

First it should be noted that the irreducible representation (κ) of the symmetric groups S_k specified by the partition κ of k, may also be specified in reduced notation by $\langle\nu\rangle$ where $(\kappa)=(k-|\nu|,\nu)$. With this notation we have:

Lemma 2.4: Let λ be a partition of $|\lambda|$ with $|\lambda|\leqslant k$, *and let $p=k-|\lambda|$. Then*

$$\langle\lambda/M\rangle=(p\cdot\lambda), \tag{2.22}$$

where / and $\cdot$ signify S-function quotients and products, respectively.

Proof: The reduced notation used on the left-hand side of (2.22) is such that in more conventional standard notation we have

$$\langle\lambda/M\rangle=\sum_m\langle\lambda/m\rangle=\sum_m (k-|\lambda|+m,\lambda/m). \tag{2.23}$$

However on the right-hand side of (2.22) the application of the special case of the Littlewood–Richardson rule known as the Pieri rule gives

$$(p\cdot\lambda)=\sum_m (p+m,\lambda/m). \tag{2.24}$$

Since $p=k-|\lambda|$, comparison of (2.23) and (2.24) yields (2.22), as required. □

Now we can return to the proof of Proposition 2.3.

Proof: From the definition of M and the algebra of plethysms[13] it follows that

$$M\otimes\{\kappa\}=(\{0\}+\{1\}+\{2\}+\cdots)\otimes\{\kappa\}=\sum_{\pi,\rho,\sigma,\ldots} c^{\kappa}_{\pi\rho\sigma\cdots}((\{0\}\otimes\{\pi\})\cdot(\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdots)$$

$$=\sum_{p,\rho,\sigma,\ldots} c^{\kappa}_{p\rho\sigma\cdots}((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdots)=\sum_{p,\rho,\sigma,\cdots} c^{\kappa}_{p\rho\sigma\cdots}c^{\mu}_{(\{1\}\otimes\{\rho\})(\{2\}\otimes\{\sigma\})\cdots}\{\mu\}, \tag{2.25}$$

where the coefficients $c^{\kappa}_{\pi\rho\sigma\ldots}$ are defined by the S-function product

$$\{\pi\}\cdot\{\rho\}\cdot\{\sigma\}\cdots=\sum_{\kappa} c^{\kappa}_{\pi\rho\sigma\cdots}\{\kappa\} \tag{2.26}$$

and use of the Littlewood–Richardson rule as many times as appropriate. Similarly, the coefficients $c^{\mu}_{(\{1\}\otimes\{\rho\})(\{2\}\otimes\{\sigma\})\cdots}$ are defined by

$$(\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdots=\sum_{\mu} c^{\mu}_{(\{1\}\otimes\{\rho\})(\{2\}\otimes\{\sigma\})\cdots}\{\mu\}. \tag{2.27}$$

The second step of (2.25) makes use of the fact that $\{0\}\otimes\{\pi\}=0$ if π is not a one-part partition, while in the case of a one-part partition p we have $\{0\}\otimes\{p\}=\{0\}=1$.

Turning to (2.20), the branching rule for the restriction from $U(k)$ to $U(k-1)$ may be expressed in the form

$$\mathrm{U}(k)\rightarrow\mathrm{U}(k-1):\{\mu\}\rightarrow\{\mu/M\}=\sum_{a}\{\mu\}/\{a\}=\sum_{a}\{\mu\}/(\{1\}\otimes\{a\}), \tag{2.28}$$

where, largely for aesthetic reasons in what follows, use has been made of the fact that $\{1\}\otimes\{a\}=\{a\}$. Littlewood[28] has provided the branching rule for the restriction from $\mathrm{U}(k-1)$ to S_k in his Theorem XI. This takes the following form:

$$U(k-1)\rightarrow S_k:\quad \{\nu\}\rightarrow\sum_{b,c,\ldots,\eta,\zeta,\ldots}\langle((\{\eta\}\cdot\{\zeta\}\cdots)\cdot(\{\nu\}/((\{2\}\otimes\{\eta\})\cdot(\{3\}\otimes\{\zeta\})\cdots$$
$$\cdot(\{2\}\otimes\{b\})\cdot(\{3\}\otimes\{c\})\cdots))\rangle, \tag{2.29}$$

where the angular brackets $\langle\cdots\rangle$ have been used again to signify characters of S_k expressed in reduced notation.

Combining (2.28) and (2.29), and using the fact that $\{\mu\}/X=\Sigma_{\xi}c^{\mu}_{\xi X}\{\xi\}$ for all X, with $\{\xi\}=\{1\}\otimes\{\xi\}$, we obtain

$$\mathrm{U}(k)\to S_k:$$

$$\begin{aligned}\{\mu\}\to &\sum_{a,b,c,\dots,\xi,\eta,\zeta,\dots} c^{\mu}_{(\{1\}\otimes\{\xi\})\cdot(\{2\}\otimes\{\eta\})\cdot(\{3\}\otimes\{\zeta\})\cdots(\{1\}\otimes\{a\})\cdot(\{2\}\otimes\{b\})\cdot(\{3\}\otimes\{c\})\cdots}\langle \xi\cdot\eta\cdot\zeta\cdots\rangle\\
&=\sum_{a,b,c,\dots,\xi,\eta,\zeta,\dots} c^{\mu}_{(\{1\}\otimes(\{a\cdot\xi\}))\cdot(\{2\}\otimes(\{b\cdot\eta\}))\cdot(\{3\}\otimes(\{c\cdot\zeta\}))\cdots}\langle \xi\cdot\eta\cdot\zeta\cdots\rangle\\
&=\sum_{a,b,c,\dots,\rho,\sigma,\tau,\dots} c^{\mu}_{((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdot(\{3\}\otimes\{\tau\})\cdots)}\langle(\rho/a)\cdot(\eta/b)\cdot(\zeta/c)\cdots\rangle\\
&=\sum_{m,\rho,\sigma,\tau,\dots} c^{\mu}_{((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdot(\{3\}\otimes\{\tau\})\cdots)}\langle(\rho\cdot\sigma\cdot\tau\cdots)/m\rangle\\
&=\sum_{\rho,\sigma,\tau,\dots} c^{\mu}_{((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdot(\{3\}\otimes\{\tau\})\cdots)}\langle(\rho\cdot\sigma\cdot\tau\cdots)/M\rangle\\
&=\sum_{\rho,\sigma,\tau,\dots} c^{\mu}_{((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdot(\{3\}\otimes\{\tau\})\cdots)}(p\cdot\rho\cdot\sigma\cdot\tau\cdots)\\
&=\sum_{\rho,\sigma,\tau,\dots} c^{\mu}_{((\{1\}\otimes\{\rho\})\cdot(\{2\}\otimes\{\sigma\})\cdot(\{3\}\otimes\{\tau\})\cdots)}c^{\kappa}_{p\rho\sigma\tau\dots}(\kappa),\end{aligned}\tag{2.30}$$

where in the penultimate step use has been made of Lemma 2.4, extended by virtue of its linearity in λ to the case in which λ is replaced by $\rho\cdot\sigma\cdot\tau\cdots$ and $p=k-|\rho|-|\sigma|-|\tau|-\cdots$.

Comparison of (2.30) with (2.25) then completes the proof of the first part of (2.21) and hence of Proposition 2.3, since the second part follows, as we have seen, from (2.19). □

III. EVALUATION OF THE PLETHYSMS $\Delta''\otimes\{\kappa\}$ AND $\tilde{\Delta}\otimes\{\kappa\}$

It follows from KWI Sec. II that

$$\Delta=\epsilon^{1/2}\overline{Q}=\epsilon^{1/2}\prod_{i=1}^{n}(1+x_i^{-1}),\quad \Delta''=\epsilon^{1/2}\overline{L}=\epsilon^{1/2}\prod_{i=1}^{n}(1-x_i^{-1}),\tag{3.1a}$$

$$\tilde{\Delta}=\epsilon^{1/2}M=\epsilon^{1/2}\prod_{i=1}^{n}(1-x_i)^{-1},\quad \tilde{\Delta}''=\epsilon^{1/2}P=\epsilon^{1/2}\prod_{i=1}^{n}(1+x_i)^{-1},\tag{3.1b}$$

where it has been convenient to introduce the S-function series $\overline{Q}$ and $\overline{L}$, which are contragredient to Q and L, respectively. It follows that in order to evaluate the required SO$(2n)$ and Sp$(2n,\mathbb{R})$ plethysms of Δ, Δ'', $\tilde{\Delta}$, and $\tilde{\Delta}''$ at the level of U(n) it is only necessary to evaluate plethysms of Q, L, M, and P, and take the contragredient where appropriate. However, general expressions for these plethysms are only available through Proposition 2.3 for $M\otimes\{\kappa\}$ and $L\otimes\{\kappa\}$. These are related through (3.1) to the plethysms of $\tilde{\Delta}$ and Δ''.

Taking the case $\tilde{\Delta}$ first, we arrive at a result first enunciated without proof by Carvalho.[8]

Proposition 3.1: Let λ be a partition which is O(k)*-standard in the sense that* $\lambda_1'+\lambda_2'\leqslant k$ *and let the coefficients* b_κ^λ *be defined by the* O(k) *to* S_k *branching rule:*

$$\mathrm{O}(k)\to S_k:\quad [\lambda]\to\sum_{\kappa:\kappa\vdash k} b_\kappa^\lambda(\kappa),\tag{3.2}$$

where the summation is over all partitions κ *of* k*. Then for any partition* κ *of* k *the corresponding plethysm of the representation* $\tilde{\Delta}$ *of* Sp$(2n,\mathbb{R})$ *decomposes in accordance with the rule*

$$\tilde{\Delta}\otimes\{\kappa\}=\sum_{\lambda:\lambda_1'+\lambda_2'\leq k} b_\kappa^\lambda\langle\tfrac{1}{2}k(\lambda)\rangle, \tag{3.3}$$

or, equivalently,

$$\tilde{\Delta}\otimes\{\kappa\}=\begin{cases}\displaystyle\sum_{\lambda:\ell(\lambda)<m}(b_\kappa^\lambda\langle m(\lambda)\rangle+b_{\kappa'}^\lambda\langle m(\lambda)\rangle^*)+\sum_{\lambda:\ell(\lambda)=m} b_\kappa^\lambda\langle m(\lambda)\rangle & \text{if } k=2m\\ \displaystyle\sum_{\lambda:\ell(\lambda)\leq m}(b_\kappa^\lambda\langle\tilde{\Delta};m(\lambda)\rangle+b_{\kappa'}^\lambda\langle\tilde{\Delta};m(\lambda)\rangle^*) & \text{if } k=2m+1,\end{cases} \tag{3.4}$$

where the asterisk (*) *signifies an associate*[11] *irreducible representation of* $\mathrm{Sp}(2n,\mathbb{R})$, *and it has been convenient to denote* $\langle\frac{1}{2}+m(\lambda)\rangle$ *by* $\langle\tilde{\Delta};m(\lambda)\rangle$.

Proof: From (3.1c) and Proposition 2.3 we have

$$\tilde{\Delta}\otimes\{\kappa\}=(\epsilon^{1/2}M)\otimes\{\kappa\}=\epsilon^{k/2}(M\otimes\{\kappa\})=\epsilon^{k/2}\sum_{\mu:\ell(\mu)\leq k} g_\kappa^\mu\{\mu\}, \tag{3.5}$$

where the coefficients g_κ^μ are defined by the $\mathrm{U}(k)\to S_k$ branching rule (2.20). We can refine this branching rule by noting that $\mathrm{O}(k)$ is a subgroup of $\mathrm{U}(k)$ which itself contains S_k as a subgroup. For μ such that $\ell(\mu)\leq k$ let the coefficients R_λ^μ be defined by the $\mathrm{U}(k)$ to $\mathrm{O}(k)$ branching rule:

$$\mathrm{U}(k)\to\mathrm{O}(k):\quad \{\mu\}\to\sum_{\lambda:\lambda_1'+\lambda_2'\leq k} R_\lambda^\mu[\lambda]. \tag{3.6}$$

Combining this with (3.2) gives

$$\mathrm{U}(k)\to\mathrm{O}(k)\to S_k:\quad \{\mu\}\to\sum_{\lambda:\lambda_1'+\lambda_2'\leq k} R_\lambda^\mu[\lambda]\to\sum_{\substack{\lambda:\lambda_1'+\lambda_2'\leq k\\ \kappa:\kappa\vdash k}} R_\lambda^\mu b_\kappa^\lambda(\kappa). \tag{3.7}$$

Comparison with (2.20) reveals that

$$g_\kappa^\mu=\sum_{\lambda:\lambda_1'+\lambda_2'\leq k} R_\lambda^\mu b_\kappa^\lambda \quad \text{for } \ell(\mu)\leq k \text{ and } \kappa\vdash k. \tag{3.8}$$

However, it is also known that[7]

$$\mathrm{Sp}(2n,\mathbb{R})\to\mathrm{U}(n):\quad \langle\tfrac{1}{2}k(\lambda)\rangle\to\epsilon^{k/2}\sum_{\mu:\ell(\mu)\leq k} R_\lambda^\mu\{\mu\} \quad \text{for } \lambda_1'+\lambda_2'\leq k, \tag{3.9}$$

where this expression serves to define the character $\langle\frac{1}{2}k(\lambda)\rangle$ of $\mathrm{Sp}(2n,\mathbb{R})$ completely since $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{U}(n)$ are of the same rank. It follows that the successive use of (3.8) and (3.9) in (3.5) leads directly to (3.3) as follows:

$$\tilde{\Delta}\otimes\{\kappa\}=\epsilon^{k/2}\sum_{\substack{\mu:\ell(\mu)\leq k\\ \lambda:\lambda_1'+\lambda_2'\leq k}} R_\lambda^\mu b_\kappa^\lambda\{\mu\}=\sum_{\lambda:\lambda_1'+\lambda_2'\leq k} b_\kappa^\lambda\langle\tfrac{1}{2}k(\lambda)\rangle \quad \text{for } \kappa\vdash k. \tag{3.10}$$

The passage from (3.3) to (3.4) is effected by noting that the summation over λ in (3.3) yields mutually associated pairs of irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$ together with self-associate irreducible representations in the case $k=2m$. Hence (3.3) can be rewritten in the form

$$\tilde{\Delta}\otimes\{\kappa\}=\begin{cases}\sum_{\lambda:\ell(\lambda)<m}(b_\kappa^\lambda\langle m(\lambda)\rangle+b_\kappa^{\lambda^*}\langle m(\lambda)\rangle^*)+\sum_{\lambda:\ell(\lambda)=m}b_\kappa^\lambda\langle m(\lambda)\rangle & \text{if } k=2m\\ \sum_{\lambda:\ell(\lambda)\leqslant m}(b_\kappa^\lambda\langle\tilde{\Delta};m(\lambda)\rangle+b_\kappa^{\lambda^*}\langle\tilde{\Delta};m(\lambda)\rangle^*) & \text{if } k=2m+1.\end{cases} \tag{3.11}$$

The notation[11] is such that $\langle m(\lambda)\rangle^*=\langle m(\lambda^*)\rangle$ and $\langle\tilde{\Delta};m(\lambda)\rangle^*=\langle\tilde{\Delta};m(\lambda^*)\rangle$, where λ^* is the k associate of the partition λ for $k=2m$ and $k=2m+1$, respectively. However, in $\mathrm{O}(k)$ we have $[\lambda^*]=[\lambda]^*=[0]^*[\lambda]$ and on restriction from $\mathrm{O}(k)$ to S_k we have $[0]^*\to(1^k)$. Moreover in S_k we have $(1^k)\cdot(\kappa)=(\kappa')$ for all κ. It follows that $b_\kappa^{\lambda^*}=b_{\kappa'}^\lambda$. Using this in (3.11) gives (3.4), as required to complete the proof of Proposition 3.1. □

Turning to the case of Δ'', the analog of Proposition 3.1 takes the form:

Proposition 3.2: Let λ *be such that* $\lambda_1'+\lambda_2'\leqslant k$ *and let the coefficients* b_κ^λ *be defined by the* $\mathrm{O}(k)$ *to* S_k *branching rule (3.2). Then for any partition* κ *of* k *the corresponding plethysm of the difference character* Δ'' *of* $\mathrm{SO}(2n)$ *decomposes in accordance with the rule*

$$\Delta''\otimes\{\kappa\}$$

$$=\begin{cases}\sum_{\lambda:l(\lambda)<m}(-1)^{|\lambda|}(b_\kappa^\lambda[m^n/\lambda']_++b_{\kappa'}^\lambda[m^n/\lambda']_-)+\sum_{\lambda:l(\lambda)=m}(-1)^{|\lambda|}b_\kappa^\lambda[m^n/\lambda'] & \text{if } k=2m\\ \sum_{\lambda:\ell(\lambda)\leqslant m}(-1)^{|\lambda|}(b_\kappa^\lambda[\Delta;m^n/\lambda']_+-b_{\kappa'}^\lambda[\Delta;m^n/\lambda']_-) & \text{if } k=2m+1.\end{cases} \tag{3.12}$$

Proof: From (3.1), Proposition 2.3, and (3.8) we have

$$\begin{aligned}\Delta''\otimes\{\kappa\}&=(\epsilon^{1/2}\overline{L})\otimes\{\kappa\}=\epsilon^{k/2}(\overline{L}\otimes\{\kappa\})=\sum_{\mu:\ell(\mu)\leqslant k}(-1)^{|\mu|}\epsilon^{k/2}g_\kappa^\mu\{\overline{\mu'}\}\\ &=\sum_{\substack{\mu:\ell(\mu)\leqslant k\\ \lambda:\lambda_1'+\lambda_2'\leqslant k}}(-1)^{|\mu|}\epsilon^{k/2}R_\lambda^\mu b_\kappa^\lambda\{\overline{\mu'}\}\\ &=\begin{cases}\sum_{\substack{\mu:\ell(\mu)\leqslant 2m\\ \lambda:\ell(\lambda)<m}}(-1)^{|\mu|}\epsilon^m(R_\lambda^\mu b_\kappa^\lambda\{\overline{\mu'}\}+R_{\lambda^*}^\mu b_\kappa^{\lambda^*}\{\overline{\mu'}\})\\ \qquad+\sum_{\substack{\mu:\ell(\mu)\leqslant 2m\\ \lambda:\ell(\lambda)=m}}(-1)^{|\mu|}\epsilon^m R_\lambda^\mu b_\kappa^\lambda\{\overline{\mu'}\} \quad \text{if } k=2m\\ \sum_{\substack{\mu:\ell(\mu)\leqslant 2m+1\\ \lambda:\ell(\lambda)\leqslant m}}(-1)^{|\mu|}\epsilon^{m+1/2}(R_\lambda^\mu b_\kappa^\lambda\{\overline{\mu'}\}+R_{\lambda^*}^\mu b_\kappa^{\lambda^*}\{\overline{\mu'}\})\\ \quad \text{if } k=2m+1.\end{cases}\end{aligned} \tag{3.13}$$

This time it is necessary to convert the combination of characters of $\mathrm{U}(n)$ appearing on the right-hand side of (3.13) into linear combinations of characters of $\mathrm{SO}(2n)$. To this end we require an analog of (3.9).

It will be recalled that (3.9) arises from a comparison of the branching rule[6,7]

$$\mathrm{Sp}(2nk,\mathbb{R})\to\mathrm{Sp}(2n,\mathbb{R})\times\mathrm{O}(k):\quad \tilde{\Delta}\to\sum_{\lambda:\lambda_1'+\lambda_2'\leqslant k}\langle\tfrac{1}{2}k(\lambda)\rangle\times[\lambda], \tag{3.14}$$

with the sequence of branching rules associated with the group–subgroup chain

$$\mathrm{Sp}(2nk,\mathbb{R})\rightarrow \mathrm{U}(nk)\rightarrow \mathrm{U}(n)\times \mathrm{U}(k)\rightarrow \mathrm{U}(n)\times \mathrm{O}(k). \tag{3.15}$$

This chain is such that

$$\tilde{\Delta}\rightarrow \epsilon^{1/2}M\rightarrow \sum_{\mu:\ell(\mu)\leqslant k} \epsilon^{k/2}\{\mu\}\times \epsilon^{n/2}\{\mu\}\rightarrow \sum_{\substack{\mu:\ell(\mu)\leqslant k\\ \lambda:\lambda_1'+\lambda_2'\leqslant k}} \epsilon^{k/2}R^{\mu}_{\lambda}\{\mu\}\times[\lambda]. \tag{3.16}$$

In the last step of (3.16) use has been made of the U(k) to O(k) branching rule (3.6). In addition the k-independent factor $\epsilon^{n/2}$ with $\epsilon=\pm 1$ has been dropped for convenience since its retention would only involve various, but essentially equivalent, embeddings of O(k) in U(k). Now comparison of (3.14) and (3.16) leads directly to the required Sp($2n,\mathbb{R}$) to U(n) branching rule (3.9).

Mimicking this procedure in the case of Δ'' it is necessary to distinguish between the cases of k even and odd. The branching rule of KWI for the restriction of Δ'' from SO($2nk$) to SO($2n$) $\times$O(k) takes the form:

$$\Delta''\rightarrow\begin{cases} \displaystyle\sum_{\lambda:\ell(\lambda)<m}(-1)^{|\lambda|}([m^n/\lambda']_+\times[\lambda]+[m^n/\lambda']_-\times[\lambda]^*)+\sum_{\lambda:\ell(\lambda)=m}(-1)^{|\lambda|}[m^n/\lambda']\times[\lambda] & \text{if } k=2m\\ \displaystyle\sum_{\lambda:\ell(\lambda)\leqslant m}(-1)^{|\lambda|}([\Delta;m^n/\lambda']_+\times[\lambda]-[\Delta;m^n/\lambda']_-\times[\lambda]^*) & \text{if } k=2m+1. \end{cases} \tag{3.17}$$

The analog of (3.15) is the group–subgroup chain

$$\mathrm{SO}(2nk)\rightarrow \mathrm{U}(nk)\rightarrow \mathrm{U}(n)\times \mathrm{U}(k)\rightarrow \mathrm{U}(n)\times \mathrm{O}(k), \tag{3.18}$$

for which we have

$$\Delta''\rightarrow \epsilon^{1/2}\overline{L}\rightarrow \sum_{\mu:\ell(\mu)\leqslant k}(-1)^{|\mu|}\epsilon^{k/2}\overline{\{\mu'\}}\times \epsilon^{n/2}\{\overline{\mu}\}\rightarrow \sum_{\substack{\mu:\ell(\mu)\leqslant k\\ \lambda:\lambda_1'+\lambda_2'\leqslant k}}(-1)^{|\mu|}\epsilon^{k/2}R^{\mu}_{\lambda}\overline{\{\mu'\}}\times[\lambda]. \tag{3.19}$$

In the last step use has been made of the U(k) to O($\underline{k}$) branching rule (3.6) together with the fact that the restriction from U(k) to O(k) is such that $R^{\overline{\mu}}_{\lambda}=R^{\mu}_{\lambda}$. As before, the k-independent factor $\epsilon^{n/2}$ with $\epsilon=\pm 1$ has been dropped. Distinguishing in the usual way between even and odd values of k and between irreducible representations of O(k) and their associates, this gives for the branching from SO($2nk$) to U(n)$\times$O(k):

$$\Delta''\to\begin{cases}\displaystyle\sum_{\substack{\mu:\ell(\mu)\le 2m\\ \lambda:\ell(\lambda)<m}}(-1)^{|\mu|}\epsilon^m(R^\mu_\lambda\{\overline{\mu'}\}\times[\lambda]+R^\mu_{\lambda*}\{\overline{\mu'}\}\times[\lambda]^*)\\ \displaystyle\qquad+\sum_{\substack{\mu:\ell(\mu)\le 2m\\ \lambda:\ell(\lambda)=m}}(-1)^{|\mu|}\epsilon^m R^\mu_\lambda\{\overline{\mu'}\}\times[\lambda] & \text{if } k=2m\\ \displaystyle\sum_{\substack{\mu:\ell(\mu)\le 2m+1\\ \lambda:\ell(\lambda)\le m}}(-1)^{|\mu|}\epsilon^{m+1/2}(R^\mu_\lambda\{\overline{\mu'}\}\times[\lambda]+R^\mu_{\lambda*}\{\overline{\mu'}\}\times[\lambda]^*) & \text{if } k=2m+1.\end{cases}\tag{3.20}$$

Comparison of (3.17) and (3.20) then yields the required branching rules for the restriction from SO$(2n)$ to U(n):

$$[m^n/\lambda']_+\to\sum_{\mu:\ell(\mu)\le 2m}(-1)^{|\mu|-|\lambda|}\epsilon^m R^\mu_\lambda\{\overline{\mu'}\}\quad\text{for } \ell(\lambda)<m,\tag{3.21a}$$

$$[m^n/\lambda']_-\to\sum_{\mu:\ell(\mu)\le 2m}(-1)^{|\mu|-|\lambda|}\epsilon^m R^\mu_{\lambda*}\{\overline{\mu'}\}\quad\text{for } \ell(\lambda)<m,\tag{3.21b}$$

$$[m^n/\lambda']\to\sum_{\mu:\ell(\mu)\le 2m}(-1)^{|\mu|-|\lambda|}\epsilon^m R^\mu_\lambda\{\overline{\mu'}\}\quad\text{for } \ell(\lambda)=m,\tag{3.21c}$$

$$[\Delta;m^n/\lambda']_+\to\sum_{\mu:\ell(\mu)\le 2m+1}(-1)^{|\mu|-|\lambda|}\epsilon^{m+1/2} R^\mu_\lambda\{\overline{\mu'}\}\quad\text{for } \ell(\lambda)\le m,\tag{3.21d}$$

$$[\Delta;m^n/\lambda']_-\to\sum_{\mu:\ell(\mu)\le 2m+1}(-1)^{|\mu|-|\lambda|+1}\epsilon^{m+1/2} R^\mu_{\lambda*}\{\overline{\mu'}\}\quad\text{for } \ell(\lambda)\le m.\tag{3.21e}$$

As in the case of (3.9), these branching rules furnish identities expressing characters of irreducible representations, in this case of SO$(2n)$, in terms of those of U(n), a subgroup of the same rank. Using these identities in (3.13) and recalling that $b^{\lambda*}_\kappa=b^\lambda_{\kappa'}$ gives (3.12), and thereby completes the proof of Proposition 3.2. □

IV. THE FACTORIZATION OF PLETHYSMS OF Δ'' AND $\tilde{\Delta}$

We are now in a position to derive the following:

Proposition 4.1: Let κ be a partition of k of Frobenius rank $r=r(\kappa)$. Then

$$\Delta''\otimes\{\kappa\}=(\Delta'')^{r(\kappa)}\Pi_\kappa,\tag{4.1}$$

with

$$\Pi_\kappa=\sum_{\rho\vdash k,\ell(\rho)\ge r(\kappa)}\frac{1}{z_\rho}\chi^\kappa_\rho(\Delta'')^{\ell(\rho)-r(\kappa)}\prod_{j=1}^{k}P_j^{m_j},\tag{4.2}$$

where

$$P_j=\prod_{i=1}^{n}\left(x_i^{(j-1)/2}+x_i^{(j-3)/2}+\cdots+x_i^{-(j-1)/2}\right).\tag{4.3}$$

Proof: Setting $S=\Delta''$ in (2.12), with Δ'' given by the character formula (1.1b), immediately gives

$$\Delta''\otimes p_j=\prod_{i=1}^{n}(x_i^{j/2}-x_i^{-j/2})=\prod_{i=1}^{n}(x_i^{1/2}-x_i^{-1/2})(x_i^{(j-1)/2}+x_i^{(j-3)/2}+\cdots+x_i^{-(j-1)/2})=\Delta''P_j, \tag{4.4}$$

with P_j as defined in (4.3). It then follows from (2.11) that

$$\begin{aligned}\Delta''\otimes\{\kappa\}&=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)}\frac{1}{z_\rho}\chi_\rho^\kappa\prod_{j=1}^{k}(\Delta''\otimes p_j)^{m_j}\\&=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)}\frac{1}{z_\rho}\chi_\rho^\kappa\prod_{j=1}^{k}(\Delta''P_j)^{m_j}\\&=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)}\frac{1}{z_\rho}\chi_\rho^\kappa(\Delta'')^{\ell(\rho)}\prod_{j=1}^{k}P_j^{m_j},\end{aligned} \tag{4.5}$$

where use has been made of (2.7). As required, this gives (4.1) with Π_κ as defined in (4.2). □

The factorization of $\Delta''\otimes\{\kappa\}$ spelt out in (4.1) and (4.2) serves to both confirm and refine (1.10). In seeking to do the same for (1.11) the following result may be derived:

Corollary 4.2: Let κ be a partition of k which in Frobenius notation takes the form

$$\kappa=\begin{pmatrix}a_1 & a_2\cdots a_r\\ b_1 & b_2\cdots b_r\end{pmatrix}$$

with $r=r(\kappa)$. Then $\Delta''\otimes\{\kappa\}=(\Delta'')^{r(\kappa)}\Pi_\kappa$ with

$$\Pi_\kappa=|\Pi_{\binom{a_s}{b_t}}|_{r\times r} \tag{4.6}$$

and

$$\dim\,\Pi_\kappa=(-1)^{b_1+b_2+\cdots+b_r}|(a_s+b_t+1)^{n-1}|_{r\times r}, \tag{4.7}$$

where $\dim\Pi_\kappa$ *is the value of* Π_κ *at the identity, that is the value at* $x_i=1$ *for all* $i=1,2,\ldots,n$.

Proof: Setting $S=\Delta''$ in (2.3) gives

$$\Delta''\otimes\{\kappa\}=\left|\Delta''\otimes\begin{Bmatrix}a_s\\ b_t\end{Bmatrix}\right|_{r\times r}=|\Delta''\Pi_{\binom{a_s}{b_t}}|_{r\times r}=(\Delta'')^{r(\kappa)}|\Pi_{\binom{a_s}{b_t}}|_{r\times r}. \tag{4.8}$$

Comparison with the definition of Π_κ in (4.1) gives (4.6).

To derive (4.7) we first consider the special case where $\kappa=\binom{a}{b}=(1+a,1^b)$ with $k=|\kappa|=a+b+1$ and $r=r(\kappa)=1$. Then, since $\dim\Delta''=0$, it follows that in (4.2) the only terms contributing to $\dim\Pi_\kappa$ are those for which $\ell(\rho)=r(\kappa)=1$. But there is only one such term and that corresponds to the one part partition $\rho=(k)$ for which $z_\rho=z_k=k$. Moreover, $\chi_k^{1+a,1^b}=(-1)^b$. Hence

$$\dim\Pi_\kappa=\frac{1}{z_k}\chi_k^{1+a,1^b}\dim P_k=\frac{1}{k}(-1)^b k^n=(-1)^b(a+b+1)^{n-1}. \tag{4.9}$$

This confirms the validity of (4.7) in the case $k=1$. Thanks to (4.6) we then have in the general case

$$\dim\Pi_\kappa=|\dim\Pi_{\binom{a_s}{b_t}}|_{r\times r}=|(-1)^{b_t}(a_s+b_t+1)^{n-1}|_{r\times r}, \tag{4.10}$$

giving (4.7), as required. □

Plethysms of $\tilde{\Delta}$ can be dealt with in exactly the same way and one arrives at the general result:

Proposition 4.3: Let κ be a partition of k of Frobenius rank $r=r(\kappa)$. Then

$$\tilde{\Delta}\otimes\{\kappa\}=(\tilde{\Delta})^{r(\kappa)}\tilde{\Pi}_{\kappa}, \tag{4.11}$$

with

$$\tilde{\Pi}_{\kappa}=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)}\frac{1}{z_{\rho}}\chi_{\rho}^{\kappa}(\tilde{\Delta})^{\ell(\rho)-r(\kappa)}\prod_{j=1}^{k}P_{j}^{-m_j}, \tag{4.12}$$

where

$$\tilde{P}_{j}=\prod_{i=1}^{n}(x_i^{(j-1)/2}+x_i^{(j-3)/2}+\cdots+x_i^{-(j-1)/2})^{-1}. \tag{4.13}$$

Moreover,

$$\tilde{\Pi}_{\kappa}=|\tilde{\Pi}_{\binom{a_s}{b_t}}|_{r\times r}. \tag{4.14}$$

Proof: Proceeding as before, we set $S=\tilde{\Delta}$ in (2.12) and use the character formula (1.2a) for $\tilde{\Delta}$ to obtain instead of (4.4) the analogous formula

$$\begin{aligned}\tilde{\Delta}\otimes p_j&=\prod_{i=1}^{n}(x_i^{-j/2}-x_i^{j/2})^{-1}=\prod_{i=1}^{n}(x_i^{-1/2}-x_i^{1/2})^{-1}(x_i^{(j-1)/2}+x_i^{(j-3)/2}+\cdots+x_i^{-(j-1)/2})^{-1}\\&=\tilde{\Delta}P_j^{-1}.\end{aligned} \tag{4.15}$$

Using this in (2.11) with $S=\tilde{\Delta}$ then leads to the analog of (4.5), namely:

$$\tilde{\Delta}\otimes\{\kappa\}=\sum_{\rho\vdash k,\ell(\rho)\geqslant r(\kappa)}\frac{1}{z_{\rho}}\chi_{\rho}^{\kappa}(\tilde{\Delta})^{\ell(\rho)}\prod_{j=1}^{k}P_{j}^{-m_j}. \tag{4.16}$$

Extracting the appropriate factors of $\tilde{\Delta}$ then gives (4.11) with $\tilde{\Pi}_{\kappa}$ as in (4.12).

This time because of the infinite-dimensional nature of both $\tilde{\Delta}$ and P_j^{-1} there exists no complete analog of Corollary 4.2. However, the use of $S=\tilde{\Delta}$ in (2.3) leads, as in the derivation of (4.6), to the identity (4.14). □

To close this section we provide some conjugacy rules for both Π_{κ} and $\tilde{\Pi}_{\kappa}$. The outer automorphism, *, of SO$(2n)$ is such that[16]

$$(\Delta''\otimes\{\kappa\})^{*}=(\Delta'')^{*}\otimes\{\kappa\}=(-\Delta'')\otimes\{\kappa\}=(-1)^{k}\Delta''\otimes\{\kappa'\}, \tag{4.17}$$

where, as usual, $k=|\kappa|$. From the factorization formula (4.1) it follows that

$$(\Delta''^{*})^{r(\kappa)}(\Pi_{\kappa})^{*}=(-1)^{r(\kappa)}(\Delta'')^{r(\kappa)}(\Pi_{\kappa})^{*}=(-1)^{k}(\Delta'')^{r(\kappa)}\Pi_{\kappa'}, \tag{4.18}$$

since $r(\kappa')=r(\kappa)$. Hence

$$\Pi_{\kappa'}=(-1)^{|\kappa|+r(\kappa)}(\Pi_{\kappa})^{*}. \tag{4.19}$$

Similarly, the properties of associate irreducible representations of Sp$(2n,\mathbb{R})$ are such that[12]

$$(\tilde{\Delta}\otimes\{\kappa\})^{*}=\tilde{\Delta}\otimes\{\kappa'\}. \tag{4.20}$$

It thus follows from the factorization formula (4.11) that

$$(\tilde{\Delta}^*)^{r(\kappa)}(\tilde{\Pi}_\kappa)^* = (\tilde{\Delta})^{r(\kappa)}(\tilde{\Pi}_\kappa)^* = (\tilde{\Delta})^{r(\kappa)}\tilde{\Pi}_{\kappa'}, \tag{4.21}$$

and hence

$$\tilde{\Pi}_{\kappa'} = (\tilde{\Pi}_\kappa)^*. \tag{4.22}$$

There remain several problems with Propositions 3.1 and 3.3. First, it is not at all clear whether or not Π_κ (respectively, $\tilde{\Pi}_\kappa$) can be expressed as linear combinations of characters of irreducible representations of SO$(2n)$ [respectively, Sp$(2n,\mathbb{R})$]. Second, if this is indeed true then the explicit formulas we have given for Π_κ and $\tilde{\Pi}_\kappa$ are not amenable to re-writing them in terms of such characters. Third, even if this can be done it is by no means obvious that the resulting coefficients of these characters are integers so that Π_κ (respectively, $\tilde{\Pi}_\kappa$) belongs to the rings of the characters of SO$(2n)$ [respectively, Sp$(2n,\mathbb{R})$] over the integers $\mathbb{Z}$. These problems are addressed in the following sections where Π_κ and $\tilde{\Pi}_\kappa$ are evaluated.

V. THE EVALUATION OF Π_κ AND $\tilde{\Pi}_\kappa$

Having established the factorization of $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$ as in (4.1) and (4.11), respectively, the evaluation of Π_κ and $\tilde{\Pi}_\kappa$ can be accomplished in a number of different ways. In principle one could proceed by exploiting (3.1) to express the required plethysms in the form $\epsilon^{k/2}(\overline{L}\otimes\{\kappa\})$ and $\epsilon^{k/2}(M\otimes\{\kappa\})$, then evaluating $(\overline{L}\otimes\{\kappa\})$ and $(M\otimes\{\kappa\})$, factoring out the $r(\kappa)$th power of $\overline{L}$ and M, and finally re-expressing the resulting characters of U(n) as characters of SO$(2n)$ or Sp$(2n,\mathbb{R})$, as appropriate.

In the case of SO$(2n)$ this may indeed be accomplished[16] at least for small k and n since the relevant series are finite. The extension to arbitrary n may be carried out inductively and checked dimensionally using (4.7). Some short cuts may be found using the algebra of plethysms. In this way one arrives at the following results:

$$\Pi_1 = 1, \tag{5.1a}$$

$$\Pi_2 = \Delta_+, \tag{5.1b}$$

$$\Pi_{1^2} = -\Delta_-, \tag{5.1c}$$

$$\Pi_3 = [1^n]_+ - \sum_{x=0}^{\infty} (-1)^x [1^{n-3-3x}], \tag{5.1d}$$

$$\Pi_{21} = \sum_{x=0}^{\infty} (-1)^x (-[1^{n-1-3x}] + [1^{n-2-3x}]), \tag{5.1e}$$

$$\Pi_{1^3} = [1^n]_- - \sum_{x=0}^{\infty} (-1)^x [1^{n-3-3x}], \tag{5.1f}$$

$$\Pi_4 = \sum_{x,y=0}^{\infty} (-1)^x ([\Delta;1^{n-3x-4y}]_+ - [\Delta;1^{n-6-3x-4y}]_-), \tag{5.1g}$$

$$\Pi_{31} = \sum_{x,y=0}^{\infty} (-1)^x (-[\Delta;1^{n-1-x-4y}]_+ + [\Delta;1^{n-3-x-4y}]_-), \tag{5.1h}$$

$$\Pi_{21^2}=\sum_{x,y=0}^{\infty}(-1)^x(-[\Delta;1^{n-3-x-4y}]_+ + [\Delta;1^{n-1-x-4y}]_-), \tag{5.1i}$$

$$\Pi_{1^4}=\sum_{x,y=0}^{\infty}(-1)^x([\Delta;1^{n-6-3x-4y}]_+ - [\Delta;1^{n-3x-4y}]_-). \tag{5.1j}$$

The above partitions are all of Frobenius rank 1. The partition $\kappa=(2^2)$ has Frobenius rank 2 and may be calculated in terms of the above rank 1 results by use of (4.6) of Corollary 4.2 as follows:

$$\Delta''\otimes\{2^2\}=\Delta''\otimes\begin{Bmatrix}1 & 0\\ 1 & 0\end{Bmatrix}=\begin{vmatrix}\Delta''\otimes\begin{Bmatrix}1\\1\end{Bmatrix} & \Delta''\otimes\begin{Bmatrix}1\\0\end{Bmatrix}\\ \Delta''\otimes\begin{Bmatrix}0\\1\end{Bmatrix} & \Delta''\otimes\begin{Bmatrix}0\\0\end{Bmatrix}\end{vmatrix}=(\Delta'')^2\begin{vmatrix}\Pi_{21} & \Pi_2\\ \Pi_{1^2} & \Pi_1\end{vmatrix}. \tag{5.2}$$

Hence

$$\Pi_{2^2}=\Pi_{21}\Pi_1-\Pi_2\Pi_{1^2}=\sum_{x=0}^{\infty}(-1)^x(-[1^{n-1-3x}]+[1^{n-2-3x}])+\sum_{x=0}^{\infty}[1^{n-1-2x}], \tag{5.3}$$

where use has been made of (5.1a)–(5.1c), (5.1e) and (1.5). The result (5.3) can be recast in the simpler form:

$$\Pi_{2^2}=\sum_{x=0}^{\infty}([1^{n-2-6x}]+[1^{n-3-6x}]+[1^{n-4-6x}]). \tag{5.4}$$

Proceeding in exactly the same way for other Frobenius rank 2 partitions one obtains, for example,

$$\Pi_{32}=\sum_{x,y=0}^{\infty}(-1)^x([\Delta;1^{n-2-3x-4y}]_+ - [\Delta;1^{n-4-3x-4y}]_-), \tag{5.5a}$$

$$\Pi_{2^21}=\sum_{x,y=0}^{\infty}(-1)^x([\Delta;1^{n-4-3x-4y}]_+ - [\Delta;1^{n-2-3x-4y}]_-). \tag{5.5b}$$

It might be noted that these two results are in conformity, as they must be, with the conjugacy formula (4.19). Before turning to alternative ways of identifying the multiplicities of the various characters that appear in Π_κ, it is worth pointing out that in all the examples of (5.1), (5.4), and (5.5) the multiplicities are integers. At first sight it would appear that the multiplicities we have obtained are all ± 1 but this is not the case. In (5.1g), for example, the multiplicity of $[\Delta;1^{n-12}]$ is 2, corresponding to the terms in the summation for which $x=4$, $y=0$ and $x=0$, $y=3$. However it is true that the multiplicities are indeed always integers.

In order to establish the general result it is helpful to consider the group–subgroup chain

$$\mathrm{O}(k-r(\kappa))\rightarrow\cdots\rightarrow\mathrm{O}(k-s)\rightarrow\cdots\rightarrow\mathrm{O}(k-1)\rightarrow\mathrm{O}(k)\rightarrow S_k \tag{5.6}$$

and the corresponding branching and inverse branching rules which are such that

$$[\lambda/M^{r(\kappa)}]\rightarrow\cdots\rightarrow[\lambda/M^s]\rightarrow\cdots\rightarrow[\lambda/M]\rightarrow[\lambda]\rightarrow\sum_{\kappa:\kappa\vdash k}b_\kappa^\lambda(\kappa). \tag{5.7}$$

For $s=0,1,\ldots,r(\kappa)$ this chain may be used to define coefficients $b_{\kappa,s}^{\mu}$ associated with the subchain extending from $\mathrm{O}(k-s)$ to S_k through the rule

$$[\mu]\rightarrow\cdots\rightarrow[\mu/L^{s-1}]\rightarrow[\mu/L^s]\rightarrow\sum_{\kappa:\kappa\vdash k} b^{\mu}_{\kappa,s}(\kappa), \tag{5.8}$$

where L is the inverse of M. It follows from (5.7) and (5.8) that

$$b^{\lambda/M^s}_{\kappa,s}=b^{\lambda}_{\kappa}, \quad b^{\mu}_{\kappa,s}=b^{\mu/L^s}_{\kappa}. \tag{5.9}$$

Consequently,

$$b^{\nu/M}_{\kappa,s}=b^{\nu/ML^s}_{\kappa}=b^{\nu/L^{s-1}}_{\kappa}=b^{\nu}_{\kappa,s-1}. \tag{5.10}$$

In addition, for all $\mathrm{O}(k-s)$ we have $[\mu^*]=[\mu]^*=[0]^*[\mu]$ and under the restriction from $\mathrm{O}(k-s)$ to S_k we have $[0]^*\rightarrow(1^k)$ with $(1^k)\cdot(\kappa)=(\kappa')$ for all partitions κ of k. It follows that quite generally $b^{\mu^*}_{\kappa,s}=b^{\mu}_{\kappa',s}$, while

$$b^{\mu}_{\kappa,s}=b^{\mu^*}_{\kappa,s}=b^{\mu}_{\kappa',s} \quad \text{if } k-s=2x \text{ and } \ell(\mu)=x. \tag{5.11}$$

With the use of these coefficients $b^{\mu}_{\kappa,s}$ we may interpolate between $\Delta''\otimes\{\kappa\}$ and Π_{κ} by means of the following:

Definition 5.1: Let κ be any partition of k with Frobenius rank $r(\kappa)$, and let the coefficients $b^{\mu}_{\kappa,s}$ be defined by (5.8) for $s=0,1,\ldots,r(\kappa)$. Then, let

$$X^{(s)}_{\kappa}=\begin{cases}\displaystyle\sum_{\mu:\ell(\mu)<x}(-1)^{|\mu|}(b^{\mu}_{\kappa,s}[x^n/\mu')]_{+}+b^{\mu}_{\kappa',s}[x^n/\mu']_{-})+\sum_{\mu:\ell(\mu)=x}(-1)^{|\mu|}b^{\mu}_{\kappa,s}[x^n/\mu'] \\ \quad\text{if } k-s=2x \\ \displaystyle\sum_{\mu:\ell(\mu)\leqslant x}(-1)^{|\mu|}(b^{\mu}_{\kappa,s}[\Delta;x^n/\mu']_{+}-b^{\mu}_{\kappa',s}[\Delta;x^n/\mu']_{-}) \quad \text{if } k-s=2x+1.\end{cases} \tag{5.12}$$

With this notation we have

Lemma 5.2: Let κ be any partition of k with Frobenius rank $r(\kappa)$, then

$$\Delta''X^{(s)}_{\kappa}=X^{(s-1)}_{\kappa} \quad \text{for } s=1,2,\ldots,r(\kappa). \tag{5.13}$$

Proof: In the case $k-s=2x$ with $s\geqslant1$, the product of Δ'' with $X^{(s)}_{\kappa}$ may be evaluated as follows:

$$\begin{aligned}\Delta''X^{(s)}_{\kappa}&=\sum_{\mu:\ell(\mu)<x}(-1)^{|\mu|}(b^{\mu}_{\kappa,s}[\Delta;x^n/\mu'L)]_{+}-b^{\mu}_{\kappa',s}[\Delta;x^n/\mu'L]_{-})\\&\quad+\sum_{\mu:\ell(\mu)=x}(-1)^{|\mu|}b^{\mu}_{\kappa,s}([\Delta;x^n/\mu'L)]_{+}-[\Delta;x^n/\mu'L]_{-})\\&=\sum_{\mu:\ell(\mu)\leqslant x}(-1)^{|\mu|}(b^{\mu}_{\kappa,s}[\Delta;x^n/\mu'L]_{+}-b^{\mu}_{\kappa',s}[\Delta;x^n/\mu'L]_{-})\\&=\sum_{\mu:\ell(\mu)\leqslant x}\sum_{p=0}^{\infty}(-1)^{|\mu|+p}(b^{\mu}_{\kappa,s}[\Delta;x^n/(\mu\cdot p)']_{+}-b^{\mu}_{\kappa',s}[\Delta;x^n/(\mu\cdot p)']_{-})\\&=\sum_{\mu:\ell(\mu)\leqslant x}\sum_{p=0}^{\infty}\sum_{\nu:\ell(\nu)\leqslant x}(-1)^{|\nu|}c^{\nu}_{\mu p}(b^{\mu}_{\kappa,s}[\Delta;x^n/\nu']_{+}-b^{\mu}_{\kappa',s}[\Delta;x^n/(\nu)']_{-})\end{aligned}$$

$$= \sum_{\nu:\ell(\nu)\leq x} \sum_{p=0}^{\infty} (-1)^{|\nu|}(b^{\nu/p}_{\kappa,s}[\Delta;x^n/\nu']_+ - b^{\nu/p}_{\kappa',s}[\Delta;x^n/\nu']_-)$$

$$= \sum_{\nu:\ell(\nu)\leq x} (-1)^{|\nu|}(b^{\nu/M}_{\kappa,s}[\Delta;x^n/\nu']_+ - b^{\nu/M}_{\kappa',s}[\Delta;x^n/\nu']_-) = X^{(s-1)}_{\kappa}. \tag{5.14}$$

The first step involves the use of the product rules enunciated in KWI. Thanks to (5.11) it is possible to regroup all the terms into a single sum as in the second step. The next four steps depend on the fact that $L=\Sigma_{p=0}^{\infty}(-1)^p\{1^p\}$ and $M=\Sigma_{p=0}^{\infty}\{p\}$, while

$$\{\mu\cdot p\} = \sum_{\nu} c^{\nu}_{\mu p}\{\nu\}, \quad \{\nu/p\} = \sum_{\mu} c^{\nu}_{\mu p}\{\mu\}, \tag{5.15}$$

where $c^{\nu}_{\mu p}$ are the usual Littlewood–Richardson coefficients. These are nonvanishing only if $|\nu| = |\mu| + p$ and $\ell(\mu)\leq\ell(\nu)\leq\ell(\mu)+1$. In fact, potential terms for which $\ell(\nu)=x+1$ all vanish since they all involve x^n/ν'. The final step is then accomplished by the use of (5.10) and a comparison of the resulting expression with the second case of (5.12) in which s is replaced by $s-1$ so that $k-(s-1)=2x+1$, as required.

Proceeding in the same way in the case $k-s=2x+1$, the product of Δ'' with $X^{(s)}_{\kappa}$ gives

$$\Delta'' X^{(s)}_{\kappa} = \sum_{\mu:\ell(\mu)\leq x} (-1)^{|\mu|}(b^{\mu}_{\kappa,s}[(x+1)^n/\mu' L)]_{(+)} + b^{\mu}_{\kappa',s}[(x+1)^n/\mu' L]_{(-)})$$

$$= \sum_{\mu:\ell(\mu)\leq x} \sum_{p=0}^{\infty} \sum_{\nu:\ell(\nu)\leq x+1} (-1)^{|\mu|+p} c^{\nu}_{\mu p}(b^{\mu}_{\kappa,s}[(x+1)^n/\nu']_{(+)} + b^{\mu}_{\kappa',s}[(x+1)^n/\nu']_{(-)})$$

$$= \sum_{\nu:\ell(\nu)\leq x} (-1)^{|\nu|}(b^{\nu/M}_{\kappa,s}[(x+1)^n/\nu']_+ + b^{\nu/M}_{\kappa',s}[(x+1)^n/\nu']_-)$$

$$+ \sum_{\nu:\ell(\nu)=x+1} \sum_{p=0}^{\infty} \sum_{\mu:\ell(\mu)=x} (-1)^{|\nu|} c^{\nu}_{\mu p}(b^{\mu}_{\kappa,s} + b^{\mu}_{\kappa',s})[(x+1)^n/\nu']. \tag{5.16}$$

Thus far, the only new features are the use in the first step of the notation introduced in KWI whereby, for characters of SO($2n$), we have

$$[\lambda]_{(\pm)} = \begin{cases} [\lambda] & \text{if } \lambda_1' < n \\ [\lambda]_{\pm} & \text{if } \lambda_1' = n, \end{cases} \tag{5.17}$$

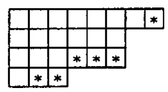
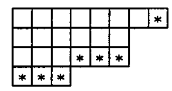
and the occurrence in (5.16) of the final set of terms for which $\ell(\nu)=x+1$. These cannot be discarded in this case as they now involve $[(x+1)^n/\nu']$ rather than $[x^n/\nu']$. However, for all such terms with $\ell(\nu)=x+1$ we have

$$b^{\nu/M}_{\kappa,s} = \sum_{p=0}^{\infty} \sum_{\lambda:x\leq\ell(\lambda)\leq x+1} c^{\nu}_{\lambda p} b^{\lambda}_{\kappa,s} = \sum_{p=0}^{\infty} \sum_{\mu:\ell(\mu)=x} (c^{\nu}_{\mu p} b^{\mu}_{\kappa,s} + c^{\nu}_{\mu^* p} b^{\mu^*}_{\kappa,s}), \tag{5.18}$$

where it has been recognized that the only partitions λ with $\ell(\lambda)=x+1$ for which $b^{\lambda}_{\kappa,s}$ is nonvanishing are those of the form $\lambda=\mu^*$ with $\ell(\mu)=x$, where $\mu^*=(\mu,1)$ is the O($k-s$) $=$O($2x+1$)-associate of μ. For such terms we have $b^{\mu^*}_{\kappa,s}=b^{\mu}_{\kappa',s}$ as usual. Moreover, for $\ell(\nu) = x+1$ and $\ell(\mu)=x$ we have

$$\sum_{p=0}^{\infty} c^{\nu}_{\mu^* p} = c^{\nu}_{\mu^*,|\nu|-|\mu|-1} = c^{\nu}_{(\mu,1),|\nu|-|\mu|-1} = c^{\nu}_{\mu,|\nu|-|\mu|} = \sum_{p=0}^{\infty} c^{\nu}_{\mu p}, \tag{5.19}$$

where the crucial third equality relies on the fact that the relevant coefficients are 1 or 0 according to whether the skew Young diagrams $F^{\nu/(\mu,1)}$ and $F^{\nu/\mu}$ either are or are not, respectively, horizontal strips in the terminology of Macdonald.[24] Such horizontal strips are indicated by the boxes with an asterisk in the following illustrative diagrams appropriate to the case $x=3$, $\nu=(8663)$, $\mu=(763)$, and $\mu^*=(7631)$:

$F^{\nu/(\mu,1)} =$  $F^{\nu/\mu} =$  (5.20)

It follows from the use of (5.19) in (5.18) that

$$b_{\kappa,s}^{\nu/M}=\sum_{p=0}^{\infty}\sum_{\mu:\ell(\mu)=x} c_{\mu p}^{\nu}(b_{\kappa,s}^{\mu}+b_{\kappa',s}^{\mu}). \tag{5.21}$$

Substituting this into (5.16) gives

$$\begin{aligned}\Delta'' X_{\kappa}^{(s)}=&\sum_{\nu:\ell(\nu)<x+1}(-1)^{|\nu|}(b_{\kappa,s}^{\nu/M}[(x+1)^n/\nu']_{+}+b_{\kappa',s}^{\nu/M}[(x+1)^n/\nu']_{-})\\ &+\sum_{\nu:\ell(\nu)=x+1}(-1)^{|\nu|}b_{\kappa,s}^{\nu/M}[(x+1)^n/\nu']\\ =&X_{\kappa}^{(s-1)}.\end{aligned} \tag{5.22}$$

The final step involves the use of (5.10) and the observation that in this case $k-(s-1)=2(x+1)$. This is necessary to make the connection with the first case of (5.12) with s replaced by $s-1$, and x by $x+1$.

Taken together, (5.14) and (5.22) imply the validity of (5.13) for $k-s$ both even and odd, thereby completing the proof of Lemma 5.2. □

This leads directly to

Proposition 5.3: Let κ be any partition, and let $r(\kappa)$ be its Frobenius rank. Then

$$\Delta''\otimes\{\kappa\}=(\Delta'')^s X_{\kappa}^{(s)} \quad \text{for } s=0,1,\ldots,r(\kappa). \tag{5.23}$$

Proof: Comparing (3.12) with the $s=0$ case of the definition (5.12), and noting from (5.9) that $b_{\kappa,0}^{\mu}=b_{\kappa}^{\mu}$, shows that

$$\Delta''\otimes\{\kappa\}=X_{\kappa}^{(0)}. \tag{5.24}$$

Then starting from this expression, factors of Δ'' may be extracted one-by-one through the application of Lemma 5.2 in the cases $s=1,2,\ldots,r(\kappa)$ to give

$$\Delta''\otimes\{\kappa\}=X_{\kappa}^{(0)}=\Delta'' X_{\kappa}^{(1)}=(\Delta'')^2X_{\kappa}^{(2)}=\cdots=(\Delta'')^{r(\kappa)}X_{\kappa}^{(r(\kappa))}. \tag{5.25}$$

This completes the proof of Proposition 5.3 □

Recalling the definition of Π_κ given in (4.1), it follows immediately from (5.25) that $\Pi_\kappa=X_{\kappa}^{(r(\kappa))}$. Thanks to Definition 5.1 this then implies:

Corollary 5.4: For any partition κ of k with Frobenius rank $r(\kappa)$,

$$\Pi_\kappa=\begin{cases}\displaystyle\sum_{\mu:\ell(\mu)<x}(-1)^{|\mu|}(b^{\mu}_{\kappa,r(\kappa)}[x^n/\mu'])]_{+}+b^{\mu}_{\kappa',r(\kappa)}[x^n/\mu']_{-})\\ \displaystyle\quad+\sum_{\mu:\ell(\mu)=x}(-1)^{|\mu|}b^{\mu}_{\kappa,r(\kappa)}[x^n/\mu'] \quad \text{if } k-r(\kappa)=2x\\ \displaystyle\sum_{\mu:\ell(\mu)\leqslant x}(-1)^{|\mu|}(b^{\mu}_{\kappa,r(\kappa)}[\Delta;x^n/\mu']_{+}-b^{\mu}_{\kappa',r(\kappa)}[\Delta;x^n/\mu']_{-}) \quad \text{if } k-r(\kappa)=2x+1.\end{cases} \tag{5.26}$$

The procedure used to evaluate Π_κ may now be used to evaluate $\tilde{\Pi}_\kappa$. In fact, for technical reasons that are largely a matter of notation and the absence of factors of -1, the case of $Sp(2n,\mathbb{R})$ is slightly easier to deal with than that of $SO(2k)$. This shows itself in the statement of the analog of Definition 5.1, namely:

Definition 5.5: Let κ be any partition of k with Frobenius rank $r(\kappa)$, and let the coefficients $b^{\lambda}_{\kappa,s}$ be defined by (5.8) for $s=0,1,\ldots,r(\kappa)$. Then let

$$\tilde{X}^{(s)}_\kappa=\sum_{\mu:\mu_1'+\mu_2'\leqslant k-s} b^{\mu}_{\kappa,s}\langle \tfrac{1}{2}(k-s)(\mu)\rangle. \tag{5.27}$$

With this notation we have:

Lemma 5.6: Let κ be any partition of k with Frobenius rank $r(\kappa)$, then

$$\tilde{\Delta}\tilde{X}^{(s)}_\kappa=\tilde{X}^{(s-1)}_\kappa \quad \text{for } s=1,2,\ldots,r(\kappa). \tag{5.28}$$

Proof: For $s\geqslant 1$ we have

$$\begin{aligned}\tilde{\Delta}\tilde{X}^{(s)}_\kappa&=\sum_{\mu:\mu_1'+\mu_2'\leqslant k-s} b^{\mu}_{\kappa,s}\tilde{\Delta}\langle \tfrac{1}{2}(k-s)(\mu)\rangle\\ &=\sum_{\mu:\mu_1'+\mu_2'\leqslant k-s} b^{\mu}_{\kappa,s}\langle \tfrac{1}{2}(k-s+1)(\mu\cdot M)\rangle\\ &=\sum_{\nu:\nu_1'+\nu_2'\leqslant k-s+2} b^{\nu/M}_{\kappa,s}\langle \tfrac{1}{2}(k-s+1)(\nu)\rangle\\ &=\sum_{\nu:\nu_1'+\nu_2'\leqslant k-s+1} b^{\nu}_{\kappa,s-1}\langle \tfrac{1}{2}(k-s+1)(\nu)\rangle=\tilde{X}^{(s-1)}_\kappa,\end{aligned} \tag{5.29}$$

as required. The first step involves the use of the $Sp(2n,\mathbb{R})$ product rule (5.8) of KWI. Then it should be noted that multiplication of μ by M may give rise to terms ν in which a box has been added to each of the first two columns of F^μ to form F^ν. This is the origin of the condition on ν that $\nu_1'+\nu_2'=k-s+2$. However, the fact that $\langle \tfrac{1}{2}(k-s+1)(\nu)\rangle=0$ if $\nu_1'+\nu_2'=k-s+2$ allows this condition to be relaxed to $\nu_1'+\nu_2'=k-s+1=k-(s-1)$, thereby leading to the identification made in the final step. □

An immediate consequence of this Lemma is:

Proposition 5.7: Let κ be any partition, and let $r(\kappa)$ be its Frobenius rank. Then

$$\tilde{\Delta}\otimes\{\kappa\}=(\tilde{\Delta})^s\tilde{X}^{(s)}_\kappa \quad \text{for } s=0,1,\ldots,r(\kappa). \tag{5.30}$$

Proof: Comparing (3.10) with the $s=0$ case of the definition (5.27), and noting once again that $b^{\mu}_{\kappa,0}=b^{\mu}_{\kappa}$, shows that

$$\tilde{\Delta}\otimes\{\kappa\}=\tilde{X}_\kappa^{(0)}. \tag{5.31}$$

Then starting from this expression (5.31) factors of $\tilde{\Delta}$ may be extracted one-by-one through the application of Lemma 5.6 in the cases $s=1,2,\ldots,r(\kappa)$ to give

$$\tilde{\Delta}\otimes\{\kappa\}=\tilde{X}_\kappa^{(0)}=\tilde{\Delta}\tilde{X}_\kappa^{(1)}=(\tilde{\Delta})^2\tilde{X}_\kappa^{(2)}=\cdots=(\tilde{\Delta})^{r(\kappa)}\tilde{X}_\kappa^{(r(\kappa))}. \tag{5.32}$$

This completes the proof of Proposition 5.7. □

This, in turn, implies the following:

Corollary 5.8: For any partition κ of k with Frobenius rank $r(\kappa)$,

$$\tilde{\Pi}_\kappa=\sum_{\mu:\mu_1'+\mu_2'\leqslant k-r(\kappa)} b^\mu_{\kappa,r(\kappa)}\langle\tfrac{1}{2}(k-r(\kappa))(\mu)\rangle. \tag{5.33}$$

or equivalently,

$$\tilde{\Pi}_\kappa=\begin{cases}\displaystyle\sum_{\mu:\ell(\mu)<x}(b^\mu_{\kappa,r(\kappa)}\langle x(\mu)\rangle+b^\mu_{\kappa',r(\kappa)}\langle x(\mu)\rangle^*)+\sum_{\mu:\ell(\mu)=x}b^\mu_{\kappa,r(\kappa)}\langle x(\mu)\rangle & \text{if } k-r(\kappa)=2x\\ \displaystyle\sum_{\mu:\ell(\mu)\leqslant x}(b^\mu_{\kappa,r(\kappa)}\langle\tilde{\Delta};x(\mu)\rangle+b^\mu_{\kappa',r(\kappa)}\langle\tilde{\Delta};x(\mu)\rangle^*) & \text{if } k-r(\kappa)=2x+1.\end{cases} \tag{5.34}$$

Proof: The result (5.33) follows immediately from Proposition 5.7 and the definition of $\tilde{\Pi}_\kappa$ given in (4.11) together imply that $\tilde{\Pi}_\kappa=\tilde{X}_\kappa^{(r(\kappa))}$. Finally, the passage from (5.33) to (5.34) is a straightforward consequence of the definition[12] of associate irreducible representations of $\mathrm{Sp}(2n,\mathbb{R})$. The form (5.34) is included merely to stress once again the analogy between $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SO}(2n)$, exemplified this time by the direct correspondence between (5.34) and (5.26). □

VI. EXAMPLES

Although the formulas (5.26) and (5.34) may look formidable, they depend only on the coefficients $b^\mu_{\kappa,r(\kappa)}$. These coefficients are themselves defined by (5.8). Fortunately, the relevant branching rules for restrictions from $\mathrm{O}(k-1)$ and $\mathrm{O}(k)$ to S_k, as well as the inverse branching rules from $\mathrm{O}(k-s)$ to $\mathrm{O}(k-s+1)$, that are needed to exploit (5.8) to the full, are well understood. The relevant coefficients may be found in a variety of ways[18–23] for both $\mathrm{O}(k-1)$ and $\mathrm{O}(k)$ to S_k, and for the inverse restriction[29] from $\mathrm{O}(m)$ to $\mathrm{O}(m+1)$. They are implemented, for example, in the software package SCHUR.[27]

For low values of k even this level of sophistication is not really required. For example, in the case $k=2$ the only irreducible representations (κ) of S_2 are (2) and (1^2). Each of these is such that $r(\kappa)=1$ and $k-r(\kappa)=2x+1$ with $x=0$. With these values of k and $r(\kappa)$ the relevant coefficients $b^\mu_{\kappa,r(\kappa)}$ are the branching rule coefficients associated with restriction from $\mathrm{O}(1)$ to S_2. The complete set of such branchings consists merely of $[0]\to(2)$ and $[0]^*\to(1^2)$. Using this information in (5.34) immediately yields

$$\tilde{\Pi}_2=\langle\tfrac{1}{2}(0)\rangle=\tilde{\Delta}_+,\quad \tilde{\Pi}_{1^2}=\langle\tfrac{1}{2}(0)\rangle^*=\tilde{\Delta}_-. \tag{6.1}$$

Similarly, for $k=3$ all irreducible representations (κ) of S_3 are such that $r(\kappa)=1$, and the relevant $\mathrm{O}(2)\to S_3$ branching rules are $[0]\to(3)$, $[0]^*\to(1^3)$ and

$$[m]\to\begin{cases}(3)+(1^3) & \text{if } m=0(\mathrm{mod}\,3)\\ (21) & \text{if } m=1,2(\mathrm{mod}\,3).\end{cases} \tag{6.2}$$

This enables us to conclude that

$$\tilde{\Pi}_3 = \langle 1(0)\rangle + \sum_{x=0}^{\infty} \langle 1(3+3x)\rangle, \tag{6.3a}$$

$$\tilde{\Pi}_{21} = \sum_{x=0}^{\infty} (\langle 1(1+3x)\rangle + \langle 1(2+3x)\rangle), \tag{6.3b}$$

$$\tilde{\Pi}_{1^3} = \langle 1(0)\rangle^* + \sum_{x=0}^{\infty} \langle 1(3+3x)\rangle. \tag{6.3c}$$

The case $k=4$ is more difficult. In those cases for which $r(\kappa)=1$ the relevant branching rules are those for $\mathrm{O}(3)\to S_4$. These are such that $[0]\to(4)$, $[1^3]=[0]^*\to(1^4)$ and if $[r]\to\Sigma_\kappa\, b^r_{\kappa,1}(\kappa)$, then $[r,1]=[r]^*\to\Sigma_\kappa\, b^r_{\kappa',1}(\kappa)$. The coefficients $b^r_{\kappa,1}$ are tabulated below:

$[r]\backslash(\kappa)$	(4)	(31)	(2^2)	(21^2)	(1^4)	
[0]	1					
[1]		1				
[2]		1	1			
[3]	1	1		1		
[4]	1	1	1	1		
[5]		2	1	1		
[6]	1	2	1	1	1	(6.4)
[7]	1	2	1	2		
[8]	1	2	2	2		
[9]	1	3	1	2	1	
[10]	1	3	2	2	1	
[11]	1	3	2	3		
[12]	2	3	2	3	1	

Finally, it should be noted that

$$b_{\kappa^{s+12t},1} = b_{\kappa^s,1} + t f^\kappa, \quad \text{for } s=0,1,2,\ldots,11 \text{ and } t=0,1,2,\ldots, \tag{6.5}$$

where f^κ is the dimension of the irreducible representation (κ) of S_4.

Using these results in (5.34), with $b^\mu_{\kappa,r(\kappa)} = b^r_{\kappa,1}$ and $b^\mu_{\kappa',r(\kappa)} = b^r_{\kappa',1}$ for $r=0,1,\ldots,$ one obtains the following results:

$$\tilde{\Pi}_4 = \sum_{x,y=0}^{\infty} (\langle \tfrac{3}{2}(3x+4y)\rangle + \langle \tfrac{3}{2}(6+3x+4y)\rangle^*), \tag{6.6a}$$

$$\tilde{\Pi}_{31} = \sum_{x,y=0}^{\infty} (\langle \tfrac{3}{2}(1+x+4y)\rangle + \langle \tfrac{3}{2}(3+x+4y)\rangle^*), \tag{6.6b}$$

$$\tilde{\Pi}_{21^2} = \sum_{x,y=0}^{\infty} (\langle \tfrac{3}{2}(3+x+4y)\rangle + \langle \tfrac{3}{2}(1+x+4y)\rangle^*), \tag{6.6c}$$

$$\tilde{\Pi}_{1^4} = \sum_{x,y=0}^{\infty} (\langle \tfrac{3}{2}(6+3x+4y)\rangle + \langle \tfrac{3}{2}(3x+4y)\rangle^*). \tag{6.6d}$$

The analogy between (6.1), (6.3), and (6.6) and the corresponding results for Π_κ in (5.1) could not be more striking. Of course, given the validity of (5.26) all the results (5.1) now follow from the information obtained here on branching rule coefficients.

Furthermore in direct analogy to (5.2) we have

$$\tilde{\Delta}\otimes\{2^2\}=\tilde{\Delta}\otimes\begin{Bmatrix}1 & 0\\ 1 & 0\end{Bmatrix}=\begin{vmatrix}\tilde{\Delta}\otimes\begin{Bmatrix}1\\1\end{Bmatrix} & \tilde{\Delta}\otimes\begin{Bmatrix}1\\0\end{Bmatrix}\\ \tilde{\Delta}\otimes\begin{Bmatrix}0\\1\end{Bmatrix} & \tilde{\Delta}\otimes\begin{Bmatrix}0\\0\end{Bmatrix}\end{vmatrix}=(\tilde{\Delta})^2\begin{vmatrix}\tilde{\Pi}_{21} & \tilde{\Pi}_2\\ \tilde{\Pi}_{1^2} & \tilde{\Pi}_1\end{vmatrix}. \tag{6.7}$$

Hence,

$$\begin{aligned}\tilde{\Pi}_{2^2}=\tilde{\Pi}_{21}\tilde{\Pi}_1-\tilde{\Pi}_2\tilde{\Pi}_{1^2}&=\sum_{x=0}^{\infty}\left(\langle 1(1+3x)\rangle+\langle 1(2+3x)\rangle\right)-\sum_{x=0}^{\infty}\langle 1(1+2x)\rangle\\ &=\sum_{x=0}^{\infty}\left(\langle 1(2+6x)\rangle-\langle 1(3+6x)\rangle+\langle 1(4+6x)\rangle\right).\end{aligned} \tag{6.8}$$

Again the analogy to (5.4) is clear. It should be noted that negative coefficients may and indeed do appear in $\tilde{\Pi}_\kappa$ for some κ, as in this example (6.8).

In this case for which $k=4$, $(\kappa)=(2^2)$, and $r(\kappa)=2$, the same result (6.8) may also be obtained directly from (5.34) and a consideration of the chain $\mathrm{O}(2)\to\mathrm{O}(3)\to S_4$. Under the restriction $\mathrm{O}(2)\to\mathrm{O}(3)$ we have $[m]\to[m/L]=[m]-[m-1]$. Combining this with the tabulation (6.4) of the branching rule multiplicities for $\mathrm{O}(3)\to S_4$ it follows that we have $[m]\to\Sigma_\kappa\, b^r_{\kappa,1}(\kappa)$ with the coefficients $b^r_{\kappa,1}$ now given by

$[r]\backslash(\kappa)$	(4)	(31)	(2^2)	(21^2)	(1^4)
[0]	1				
[1]	-1	1			
[2]			1		
[3]	1		-1	1	
[4]			1		
[5]	-1	1			
[6]	1				1
[7]				1	-1
[8]			1		
[9]		1	-1		1
[10]			1		
[11]				1	-1
[12]	1				1

(6.9)

In addition $[1^2]\to(1^4)$ and

$$b^{s+12t}_{\kappa,1}=b^s_{\kappa,1}+\delta_{s0}m^{1^2}_\kappa \quad \text{for } s=0,1,2,\ldots,11 \text{ and } t=1,2,\ldots\,. \tag{6.10}$$

It is then easy to see that (6.8) follows directly from (6.9) and (6.10).

A more testing example of the use of Corollary 5.8 is provided by the calculation of $\tilde{\Pi}_{3^3}$. In this case we have $k=9$, $\kappa=(3^3)$, and $r(\kappa)=3$ so that the relevant chain of groups is $\mathrm{O}(6)\to\mathrm{O}(7)\to\mathrm{O}(8)\to S_9$ and the relevant coefficients are $b^\mu_{\kappa,3}$. Consideration of this chain for the branching of all irreducible representations $[\mu]$ of O(6) of weight $|\mu|\leqslant 10$ leads to the following:

$$\begin{aligned}\tilde{\Pi}_{3^3}=&\langle 3(32^2)\rangle+\langle 3(3^2)\rangle+\langle 3(3^2)\rangle^*-\langle 3(3^21)\rangle-\langle 3(3^22)\rangle\\ &+\langle 3(3^3)\rangle+\langle 3(421)\rangle-2\langle 3(42^2)\rangle-2\langle 3(43)\rangle-2\langle 3(43)\rangle^*+2\langle 3(431)\rangle\\ &+3\langle 3(432)\rangle-3\langle 3(43^2)\rangle+3\langle 3(4^2)\rangle+3\langle 3(4^2)\rangle^*-3\langle 3(4^21)\rangle-2\langle 3(4^22)\rangle\end{aligned}$$

$$
\begin{aligned}
&+\langle 3(52)\rangle+\langle 3(52)\rangle^*-\langle 3(521)\rangle+2\langle 3(52^2)\rangle+\langle 3(53)\rangle\\
&+\langle 3(53)\rangle^*-2\langle 3(532)\rangle-3\langle 3(54)\rangle-3\langle 3(54)\rangle^*+4\langle 3(541)\rangle+4\langle 3(5^2)\rangle\\
&+4\langle 3(5^2)\rangle^*+\langle 3(61^2)\rangle-\langle 3(62)\rangle-\langle 3(62)\rangle^*+\langle 3(621)\rangle+2\langle 3(62^2)\rangle+3\langle 3(63)\rangle\\
&+3\langle 3(63)\rangle^*-\langle 3(631)\rangle-\langle 3(64)\rangle-\langle 3(64)\rangle^*+\langle 3(71)\rangle+\langle 3(71)\rangle^*-2\langle 3(71^2)\rangle\\
&+\langle 3(72)\rangle+\langle 3(72)\rangle^*+3\langle 3(721)\rangle-3\langle 3(73)\rangle-3\langle 3(73)\rangle^*-\langle 3(81)\rangle-\langle 3(81)\rangle^*\\
&+3\langle 3(81^2)\rangle+2\langle 3(82)\rangle+2\langle 3(82)\rangle^*+\langle 3(9)\rangle+\langle 3(9)\rangle^*-2\langle 3(10)\rangle-2\langle 3(10)\rangle^*+\cdots
\end{aligned}
\tag{6.11}
$$

for all $n\geqslant 6$. In accordance with (5.34) the coefficients are just the multiplicities of (3^3) in the branching from O(6) to S_9. In this case we have $k-r(\kappa)=2x$ with $x=3$, so that in (5.34) the summation is carried out only over those partitions μ for which $\ell(\mu)\leqslant 3$. In addition, the fact that in this case κ is self-conjugate, that is $\kappa=\kappa'$, implies that the multiplicity of an irreducible representation $\langle 3(\mu)\rangle$ is the same as that of its associate[11] $\langle 3(\mu)\rangle^*$. Thus the coefficients of $\langle 3(pq)\rangle$ and $\langle 3(pq)\rangle^*=\langle 3(pq1^2)\rangle$ are necessarily the same for all $p\geqslant q\geqslant 1$, as are those of $\langle 3(p)\rangle$ and $\langle 3(p)\rangle^*=\langle 3(p1^4)\rangle$ for all $p\geqslant 1$. The irreducible representations $\langle 3(pqr)\rangle$ for which $p\geqslant q\geqslant r\geqslant 1$ are self-associate.

This expression, when multiplied by $\tilde{\Delta}^3$ in the case Sp$(2n,\mathbb{R})$ with $n\geqslant 8$, yields up to weight 10 the following result:

$$
\begin{aligned}
\tilde{\Delta}\otimes\{3^3\}=&\langle\tilde{\Delta};4(32^2)\rangle+\langle\tilde{\Delta};4(32^2)\rangle^*+3\langle\tilde{\Delta};4(32^21)\rangle+3\langle\tilde{\Delta};4(32^21)\rangle^*+5\langle\tilde{\Delta};4(32^3)\rangle\\
&+5\langle\tilde{\Delta};4(32^3)\rangle^*+\langle\tilde{\Delta};4(3^2)\rangle+\langle\tilde{\Delta};4(3^2)\rangle^*+2\langle\tilde{\Delta};4(3^21)\rangle+2\langle\tilde{\Delta};4(3^21)\rangle^*\\
&+\langle\tilde{\Delta};4(3^21^2)\rangle+\langle\tilde{\Delta};4(3^21^2)\rangle^*+5\langle\tilde{\Delta};4(3^22)\rangle+5\langle\tilde{\Delta};4(3^22)\rangle^*+8\langle\tilde{\Delta};4(3^221)\rangle\\
&+8\langle\tilde{\Delta};4(3^221)\rangle^*+11\langle\tilde{\Delta};4(3^22^2)\rangle+11\langle\tilde{\Delta};4(3^22^2)\rangle^*+5\langle\tilde{\Delta};4(3^3)\rangle+5\langle\tilde{\Delta};4(3^3)\rangle^*\\
&+6\langle\tilde{\Delta};4(3^31)\rangle+6\langle\tilde{\Delta};4(3^31)\rangle^*+\langle\tilde{\Delta};4(421)\rangle+\langle\tilde{\Delta};4(421)\rangle^*+3\langle\tilde{\Delta};4(421^2)\rangle\\
&+3\langle\tilde{\Delta};4(421^2)\rangle^*+4\langle\tilde{\Delta};4(42^2)\rangle+4\langle\tilde{\Delta};4(42^2)\rangle^*+12\langle\tilde{\Delta};4(42^21)\rangle\\
&+12\langle\tilde{\Delta};4(42^21)\rangle^*+13\langle\tilde{\Delta};4(42^3)\rangle+13\langle\tilde{\Delta};4(42^3)\rangle^*+\langle\tilde{\Delta};4(43)\rangle+\langle\tilde{\Delta};4(43)\rangle^*\\
&+5\langle\tilde{\Delta};4(431)\rangle+5\langle\tilde{\Delta};4(431)\rangle^*+10\langle\tilde{\Delta};4(431^2)\rangle+10\langle\tilde{\Delta};4(431^2)\rangle^*+15\langle\tilde{\Delta};4(432)\rangle\\
&+15\langle\tilde{\Delta};4(432)\rangle^*+38\langle\tilde{\Delta};4(4321)\rangle+38\langle\tilde{\Delta};4(4321)\rangle^*+15\langle\tilde{\Delta};4(43^2)\rangle\\
&+15\langle\tilde{\Delta};4(43^2)\rangle^*+6\langle\tilde{\Delta};4(4^21)\rangle+6\langle\tilde{\Delta};4(4^21)\rangle^*+18\langle\tilde{\Delta};4(4^21^2)\rangle\\
&+18\langle\tilde{\Delta};4(4^21^2)\rangle^*+18\langle\tilde{\Delta};4(4^22)\rangle+18\langle\tilde{\Delta};4(4^22)\rangle^*+\langle\tilde{\Delta};4(52)\rangle+\langle\tilde{\Delta};4(52)\rangle^*\\
&+5\langle\tilde{\Delta};4(521)\rangle+5\langle\tilde{\Delta};4(521)\rangle^*+10\langle\tilde{\Delta};4(521^2)\rangle+10\langle\tilde{\Delta};4(521^2)\rangle^*+14\langle\tilde{\Delta};4(52^2)\rangle\\
&+14\langle\tilde{\Delta};4(52^2)\rangle^*+35\langle\tilde{\Delta};4(52^21)\rangle+35\langle\tilde{\Delta};4(52^21)\rangle^*+4\langle\tilde{\Delta};4(53)\rangle+4\langle\tilde{\Delta};4(53)\rangle^*\\
&+18\langle\tilde{\Delta};4(531)\rangle+18\langle\tilde{\Delta};4(531)\rangle^*+34\langle\tilde{\Delta};4(531^2)\rangle+34\langle\tilde{\Delta};4(531^2)\rangle^*\\
&+49\langle\tilde{\Delta};4(532)\rangle+49\langle\tilde{\Delta};4(532)\rangle^*+5\langle\tilde{\Delta};4(54)\rangle+5\langle\tilde{\Delta};4(54)\rangle^*+32\langle\tilde{\Delta};4(541)\rangle\\
&+32\langle\tilde{\Delta};4(541)\rangle^*+10\langle\tilde{\Delta};4(5^2)\rangle+10\langle\tilde{\Delta};4(5^2)\rangle^*+\langle\tilde{\Delta};4(61^2)\rangle+\langle\tilde{\Delta};4(61^2)\rangle^*\\
&+3\langle\tilde{\Delta};4(61^3)\rangle+3\langle\tilde{\Delta};4(61^3)\rangle^*+2\langle\tilde{\Delta};4(62)\rangle+2\langle\tilde{\Delta};4(62)\rangle^*+13\langle\tilde{\Delta};4(621)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+13\langle\tilde{\Delta};4(621)\rangle^*+29\langle\tilde{\Delta};4(621^2)\rangle+29\langle\tilde{\Delta};4(621^2)\rangle^*+33\langle\tilde{\Delta};4(62^2)\rangle\\
&+33\langle\tilde{\Delta};4(62^2)\rangle^*+10\langle\tilde{\Delta};4(63)\rangle+10\langle\tilde{\Delta};4(63)\rangle^*+49\langle\tilde{\Delta};4(631)\rangle+49\langle\tilde{\Delta};4(631)\rangle^*\\
&+17\langle\tilde{\Delta};4(64)\rangle+17\langle\tilde{\Delta};4(64)\rangle^*+\langle\tilde{\Delta};4(71)\rangle+\langle\tilde{\Delta};4(71)\rangle^*+4\langle\tilde{\Delta};4(71^2)\rangle\\
&+4\langle\tilde{\Delta};4(71^2)\rangle^*+7\langle\tilde{\Delta};4(71^3)\rangle+7\langle\tilde{\Delta};4(71^3)\rangle^*+7\langle\tilde{\Delta};4(72)\rangle+7\langle\tilde{\Delta};4(72)\rangle^*\\
&+34\langle\tilde{\Delta};4(721)\rangle+34\langle\tilde{\Delta};4(721)\rangle^*+25\langle\tilde{\Delta};4(73)\rangle+25\langle\tilde{\Delta};4(73)\rangle^*+2\langle\tilde{\Delta};4(81)\rangle\\
&+2\langle\tilde{\Delta};4(81)\rangle^*+9\langle\tilde{\Delta};4(81^2)\rangle+9\langle\tilde{\Delta};4(81^2)\rangle^*+15\langle\tilde{\Delta};4(82)\rangle+15\langle\tilde{\Delta};4(82)\rangle^*\\
&+\langle\tilde{\Delta};4(9)\rangle+\langle\tilde{\Delta};4(9)\rangle^*+6\langle\tilde{\Delta};4(91)\rangle+6\langle\tilde{\Delta};4(91)\rangle^*+\langle\tilde{\Delta};4(10)\rangle+\langle\tilde{\Delta};4(10)\rangle^*\\
&+\cdots. \qquad (6.12)
\end{aligned}
$$

Despite the fact that negative terms appear in (6.11), the coefficients in (6.12) are all positive, as required. The same result (6.12) can also be obtained directly from Proposition 3.1 using the branching rules for O(9)$\to S_9$ to determine the relevant coefficients b_κ^λ.

As can be seen from (3.4) in any case for which k is odd and $\kappa=\kappa'$, all the terms must appear in mutually associate pairs that share the same multiplicity. This is indeed the case in (6.12) for all Sp$(2n,\mathbb{R})$ with $n\geqslant 8$. However, more generally in the case Sp$(2n,\mathbb{R})$ it is necessary to delete all those terms of the form $\langle\tilde{\Delta};4(\lambda)\rangle$ for which $\ell(\lambda)>n$. Thanks to the modification rules of O(9), $\langle\tilde{\Delta};4(p)\rangle^*=\langle\tilde{\Delta};4(p1^7)\rangle$, $\langle\tilde{\Delta};4(pq)\rangle^*=\langle\tilde{\Delta};4(pq1^5)\rangle$, $\langle\tilde{\Delta};4(pqr)\rangle^*=\langle\tilde{\Delta};4(pqr1^3)\rangle$ and $\langle\tilde{\Delta};4(pqrs)\rangle^*=\langle\tilde{\Delta};4(pqrs1)\rangle$. It follows that in applying (6.12) to Sp$(12,\mathbb{R})$, for example, it is necessary to drop all the terms of the form $\langle\tilde{\Delta};4(pq)\rangle^*$ and $\langle\tilde{\Delta};4(p)\rangle^*$, but no others.

To give just one example of the calculation of $\Delta''\otimes\{\kappa\}$ for SO$(2n)$ by means of the determinantal expansion of Corollary 4.2, we consider the case $\kappa=(3^32)$. In Frobenius notation

$$
\{3^32\}=\begin{pmatrix}2&1&0\\3&2&0\end{pmatrix}, \qquad (6.13)
$$

so that

$$
\Pi_{(3^32)}=\begin{vmatrix}\Pi_{(31^3)}&\Pi_{(31^2)}&\Pi_{(3)}\\ \Pi_{(21^3)}&\Pi_{(21^2)}&\Pi_{(2)}\\ \Pi_{(1^4)}&\Pi_{(1^3)}&\Pi_{(1)}\end{vmatrix}, \qquad (6.14)
$$

where it has been typographically convenient to write $\binom{a}{b}$ in standard partition notation $(a+1,1^b)$. In the case of SO(8), for example, evaluating the individual $\Pi_{(a+1,1^b)}$ by means of Corollary 5.4 with $\kappa=(a+1,1^b)$ and $r(\kappa)=1$ and the use of the branching rules for O$(k-1)\to S_k$ for $k=1,2,\ldots,6$ gives

$$
\begin{aligned}
\Pi_{(31^3)}=&-[\Delta;2^3]_-+[\Delta;2^21]_-+[\Delta;21^3]_-+[\Delta;21^2]_+-2[\Delta;21^2]_--[\Delta;21]_+\\
&+[\Delta;21]_-+2[\Delta;2]_+-2[\Delta;2]_--[\Delta;1^4]_--2[\Delta;1^3]_++3[\Delta;1^3]_-\\
&+2[\Delta;1^2]_+-3[\Delta;1^2]_--3[\Delta;1]_++4[\Delta;1]_-+2[\Delta;0]_+-2[\Delta;0]_-,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{(31^2)}=&+[2^3]-[2^21]-[21^3]_+-[21^3]_-+2[21^2]-[21]+[2]+[1^4]_++[1^4]_-\\
&-2[1^3]+2[1^2]-2[1],
\end{aligned}
$$

$$
\Pi_{(3)}=+[1^4]_+-[1],
$$

$$\Pi_{(21^3)} = -[2^31]_- + [2^21^2]_- - [21^3]_- + [21^2] - [21] + [2] + 2[1^4]_- - 2[1^3] + [1^2] - [1] + [0],$$

$$\Pi_{(21^2)} = +[\Delta;1^3]_- - [\Delta;1^2]_- - [\Delta;1]_+ + [\Delta;1]_- + [\Delta;0]_+ - [\Delta;0]_- , \tag{6.15}$$

$$\Pi_{(2)} = +[\Delta;0]_+ ,$$

$$\Pi_{(1^4)} = -[\Delta;1^4]_- + [\Delta;1]_- - [\Delta;0]_- ,$$

$$\Pi_{(1^3)} = +[1^4]_- - [1],$$

$$\Pi_{(1)} = +[0].$$

Each of the above expansions of $\Pi_{(a+1,1^b)}$ may be set in SCHUR as an rvar and the determinant

$$\begin{vmatrix} rv1 & rv2 & rv3 \\ rv4 & rv5 & rv6 \\ rv7 & rv8 & rv9 \end{vmatrix} \tag{6.16}$$

evaluated in SCHUR to yield the result:

$$\Pi_{3^32} = -[43] + [421^2]_+ + [421^2]_- - [42] + [41^3]_+ - [41] + [3^21^2]_- - [3^21] - [3^2] - [32^3]_- - [32^2] + [321^2]_+ + [321^2]_- + [321] - 3[32] + 2[31^3]_+ + [31^3]_- - [31^2] - 2[31] - [2^4]_- - [2^31]_- + 2[2^21^2]_+ + 3[2^21^2]_- - [2^2] + [21^3]_+ + [21^3]_- + [21^2] - 3[21] + [2] + [1^4]_- - 2[1^3] - [1^2] + [1] + 2[0]. \tag{6.17}$$

The dimension is checked by noting that

$$\begin{vmatrix} -216 & 125 & 27 \\ -125 & -64 & 8 \\ -64 & 27 & 1 \end{vmatrix} = 3924. \tag{6.18}$$

Finally, since $\kappa = (3^32)$ has Frobenius rank $r(\kappa) = 3$, multiplication of (6.17) by $(\Delta'')^3$ gives

$$\Delta'' \otimes \{3^32\} = -[\Delta;541^2]_+ + [\Delta;541^2]_- + 3[\Delta;541]_+ - 3[\Delta;541]_- - 5[\Delta;54]_+ + 5[\Delta;54]_- + [\Delta;532^2]_+ - [\Delta;532^2]_- - 3[\Delta;5321]_+ + 3[\Delta;5321]_- + 4[\Delta;532]_+ - 4[\Delta;532]_- + 8[\Delta;531^2]_+ - 8[\Delta;531^2]_- - 17[\Delta;531]_+ + 17[\Delta;531]_- + 19[\Delta;53]_+ - 19[\Delta;53]_- - 2[\Delta;52^3]_+ + 3[\Delta;52^3]_- + 6[\Delta;52^21]_+ - 9[\Delta;52^21]_- - 9[\Delta;52^2]_+ + 11[\Delta;52^2]_- - 13[\Delta;521^2]_+ + 19[\Delta;521^2]_- + 28[\Delta;521]_+ - 33[\Delta;521]_- - 26[\Delta;52]_+ + 29[\Delta;52]_- + 6[\Delta;51^3]_+ - 16[\Delta;51^3]_- - 17[\Delta;51^2]_+ + 26[\Delta;51^2]_- + 17[\Delta;51]_+ - 24[\Delta;51]_- - 6[\Delta;5]_+ + 10[\Delta;5]_- - [\Delta;4^22^2]_- - [\Delta;4^221]_+ + 4[\Delta;4^221]_- + 4[\Delta;4^22]_+ - 6[\Delta;4^22]_- + 5[\Delta;4^21^2]_+ - 11[\Delta;4^21^2]_- - 18[\Delta;4^21]_+ + 23[\Delta;4^21]_- + 21[\Delta;4^2]_+ - 24[\Delta;4^2]_- + [\Delta;43^3]_- - 3[\Delta;43^22]_- - [\Delta;43^21]_+ + 4[\Delta;43^21]_- + 3[\Delta;43^2]_+ - 4[\Delta;43^2]_- - 2[\Delta;432^2]_+ + 11[\Delta;432^2]_- + 13[\Delta;4321]_+ - 30[\Delta;4321]_- - 32[\Delta;432]_+ + 41[\Delta;432]_-$$

$$
\begin{aligned}
&-33[\Delta;431^2]_+ +57[\Delta;431^2]_- +94[\Delta;431]_+ -112[\Delta;431]_- -92[\Delta;43]_+\\
&+102[\Delta;43]_- +5[\Delta;42^3]_+ -20[\Delta;42^3]_- -28[\Delta;42^21]_+ +58[\Delta;42^21]_-\\
&+64[\Delta;42^2]_+ -80[\Delta;42^2]_- +66[\Delta;421^2]_+ -111[\Delta;421^2]_- -178[\Delta;421]_+\\
&+211[\Delta;421]_- +160[\Delta;42]_+ -178[\Delta;42]_- -48[\Delta;41^3]_+ +84[\Delta;41^3]_-\\
&+136[\Delta;41^2]_+ -165[\Delta;41^2]_- -137[\Delta;41]_+ +157[\Delta;41]_- +52[\Delta;4]_+\\
&-61[\Delta;4]_- -2[\Delta;3^4]_- +7[\Delta;3^32]_- +3[\Delta;3^31]_+ -12[\Delta;3^31]_- -10[\Delta;3^3]_+\\
&+14[\Delta;3^3]_- +2[\Delta;3^22^2]_+ -24[\Delta;3^22^2]_- -24[\Delta;3^221]_+ +70[\Delta;3^221]_-\\
&+72[\Delta;3^22]_+ -98[\Delta;3^22]_- +55[\Delta;3^21^2]_+ -117[\Delta;3^21^2]_- -175[\Delta;3^21]_+\\
&+222[\Delta;3^21]_- +156[\Delta;3^2]_+ -181[\Delta;3^2]_- -8[\Delta;32^3]_+ +45[\Delta;32^3]_-\\
&+63[\Delta;32^21]_+ -144[\Delta;32^21]_- -166[\Delta;32^2]_+ +212[\Delta;32^2]_- -151[\Delta;321^2]_+\\
&+268[\Delta;321^2]_- +438[\Delta;321]_+ -525[\Delta;321]_- -383[\Delta;32]_+ +428[\Delta;32]_-\\
&+141[\Delta;31^3]_+ -209[\Delta;31^3]_- -396[\Delta;31^2]_+ +448[\Delta;31^2]_- +408[\Delta;31]_+\\
&-438[\Delta;31]_- -179[\Delta;3]_+ +186[\Delta;3]_- +9[\Delta;2^4]_+ -39[\Delta;2^4]_- -64[\Delta;2^31]_+\\
&+140[\Delta;2^31]_- +168[\Delta;2^3]_+ -215[\Delta;2^3]_- +156[\Delta;2^21^2]_+ -273[\Delta;2^21^2]_-\\
&-453[\Delta;2^21]_+ +544[\Delta;2^21]_- +395[\Delta;2^2]_+ -442[\Delta;2^2]_- -192[\Delta;21^3]_+\\
&+282[\Delta;21^3]_- +547[\Delta;21^2]_+ -620[\Delta;21^2]_- -585[\Delta;21]_+ +628[\Delta;21]_-\\
&+302[\Delta;2]_+ -313[\Delta;2]_- +109[\Delta;1^4]_+ -166[\Delta;1^4]_- -323[\Delta;1^3]_+\\
&+373[\Delta;1^3]_- +376[\Delta;1^2]_+ -411[\Delta;1^2]_- -262[\Delta;1]_+ +280[\Delta;1]_- +98[\Delta;0]_+\\
&-107[\Delta;0]_- . \qquad (6.19)
\end{aligned}
$$

This may also be checked, or arrived at very tediously, through the use of Proposition 3.2 and the branching rules for $O(11)\to S_{11}$.

VII. BASIC TENSOR DIFFERENCE CHARACTERS OF SO(2*n*) AND SP(2*N*,$\mathbb{R}$)

Basic tensor sum, $\Box$, and difference, $\Box''$, characters of $SO(2n)$ are specified most conveniently by writing[16]

$$\Box = \Box_+ + \Box_- = [1^n]_+ + [1^n]_- , \qquad (7.1a)$$

$$\Box'' = \Box_+ + \Box_- = [1^n]_+ - [1^n]_- . \qquad (7.1b)$$

Likewise, we may specify analogous basic tensor sum $\tilde{\Box}$ and difference $\tilde{\Box}''$ characters of $Sp(2n,\mathbb{R})$ by writing

$$\tilde{\Box} = \tilde{\Box}_+ + \tilde{\Box}_- = \langle 1(0)\rangle + \langle 1(0)\rangle^*, \qquad (7.2a)$$

$$\tilde{\Box}'' = \tilde{\Box}_+ + \tilde{\Box}_- = \langle 1(0)\rangle - \langle 1(0)\rangle^*. \qquad (7.2b)$$

It follows from (1.1) and (1.5) that

$$\Box'' = [1^n]_+ - [1^n]_- = (\Delta_+)^2 - (\Delta_-)^2 = \Delta\Delta''. \qquad (7.3)$$

Similarly, it follows from (1.2) and (1.12) that

$$\tilde{\Box}'' = \langle 1(0) \rangle - \langle 1(0) \rangle^* = (\tilde{\Delta}_+)^2 - (\tilde{\Delta}_-)^2 = \tilde{\Delta}\tilde{\Delta}''. \tag{7.4}$$

Hence, from (1.1) and (1.2) we have

$$\Box'' = \prod_{i=1}^{n} (x_i - x_i^{-1}) \tag{7.5}$$

and

$$\tilde{\Box}'' = \prod_{i=1}^{n} (x_i^{-1} - x_i)^{-1}, \tag{7.6}$$

respectively.

Turning to plethysms of the basic tensor sum and difference characters, the results (8.10a) and (8.10b) of Ref. 16 can be rewritten in the following form:

$$\Box_+ \otimes \{2\} = \sum_{x=0}^{\infty} [2^n/1^{4x}]_+ + \sum_{x,y=0}^{\infty} [2^n/(2^{2y+2}1^{4x})], \tag{7.7a}$$

$$\Box_+ \otimes \{1^2\} = \sum_{x=0}^{\infty} [2^n/1^{4x+2}]_+ + \sum_{x,y=0}^{\infty} [2^n/(2^{2y+2}1^{4x+2})], \tag{7.7b}$$

$$\Box_- \otimes \{2\} = \sum_{x=0}^{\infty} [2^n/1^{4x}]_- + \sum_{x,y=0}^{\infty} [2^n/(2^{2y+2}1^{4x})], \tag{7.7c}$$

$$\Box_- \otimes \{1^2\} = \sum_{x=0}^{\infty} [2^n/1^{4x+2}]_- + \sum_{x,y=0}^{\infty} [2^n/(2^{2y+2}1^{4x+2})]. \tag{7.7d}$$

By the adaptation of a procedure[30] developed for the evaluation of symmetrized products of $SO^*(2n)$ to the case of $Sp(2n,\mathbb{R})$ one obtains the following minor modification of the results (6.2a) and (6.2b) of Ref. 29:

$$\langle \tfrac{1}{2}k(0) \rangle \otimes \{2\} = \langle k(D_+)_N \rangle, \tag{7.8a}$$

$$\langle \tfrac{1}{2}k(0) \rangle \otimes \{1^2\} = \langle k(D_-)_N \rangle, \tag{7.8b}$$

where $N = \min(k,n)$ and $(D_+)_N$ is the infinite series of partitions with parts all even, of weight $0 (\mathrm{mod}\, 4)$ and number of parts $\leqslant N$ while $(D_-)_N$ is similar except that now the partitions are of weight $2 (\mathrm{mod}\, 4)$. Specializing (7.8a) and (7.8b) to the case $k=2$ then leads to

$$\langle 1(0) \rangle \otimes \{2\} = \tilde{\Box}_+ \otimes \{2\} = \langle 2(D_+)_2 \rangle, \tag{7.9a}$$

$$\langle 1(0) \rangle \otimes \{1^2\} = \tilde{\Box}_+ \otimes \{1^2\} = \langle 2(D_-)_2 \rangle. \tag{7.9b}$$

Since $\tilde{\Box}_- = \langle 1(0) \rangle^* = \tilde{\Box}_+^*$ we may use the conjugacy theorem (5.25) of Ref. 11 to write down the corresponding symmetrized powers of $\tilde{\Box}_-$. This allows us to summarize the results in a form analogous to (7.7), namely:

$$\tilde{\Box}_+ \otimes \{2\} = \sum_{x=0}^{\infty} \langle 2(4x) \rangle + \sum_{x,y=0}^{\infty} \langle 2(4x+2y+2,4x) \rangle, \tag{7.10a}$$

$$\tilde{\Box}_+\otimes\{1^2\}=\sum_{x=0}^{\infty}\langle 2(4x+2)\rangle+\sum_{x,y=0}^{\infty}\langle 2(4x+2y+4,4x+2)\rangle, \tag{7.10b}$$

$$\tilde{\Box}_-\otimes\{2\}=\sum_{x=0}^{\infty}\langle 2(4x)\rangle^*+\sum_{x,y=0}^{\infty}\langle 2(4x+2y+2,4x)\rangle, \tag{7.10c}$$

$$\tilde{\Box}_-\otimes\{1^2\}=\sum_{x=0}^{\infty}\langle 2(4x+2)\rangle^*+\sum_{x,y=0}^{\infty}\langle 2(4x+2y+4,4x+2)\rangle, \tag{7.10d}$$

where it should be recalled that the irreducible representations $\langle 2(pq)\rangle$ of $\mathrm{Sp}(2n\mathbb{R})$ with $p\geqslant q \geqslant 1$ are self-associate, while $\langle 2(p)\rangle^*=\langle 2(p1^2)\rangle$ for $p\geqslant 1$, and $\langle 2(0)\rangle^*=\langle 2(1^4)\rangle$.

All this suggests as a strong possibility the existence of analogies between symmetrized tensor powers of the difference characters $\Box''$ of $\mathrm{SO}(2n)$ with those of $\tilde{\Box}''$ for $\mathrm{Sp}(2n,\mathbb{R})$ that are similar to those found between Δ'' and $\tilde{\Delta}$. Indeed, building on previously established results[1,16] for plethysms of $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n,\mathbb{R})$ it is not too difficult to show that

$$\Box''\otimes\{2\}=\Box''\left(\Box_+-\sum_{x=0}(-1)^x[1^{n-2x-2}]\right), \tag{7.11a}$$

$$\Box''\otimes\{1^2\}=\Box''\left(-\Box_-+\sum_{x=0}(-1)^x[1^{n-2x-2}]\right) \tag{7.11b}$$

and

$$\tilde{\Box}''\otimes\{2\}=\tilde{\Box}''\left(\tilde{\Box}_+-\sum_{x=0}(-1)^x\langle 1(2x+2)\rangle\right), \tag{7.12a}$$

$$\tilde{\Box}''\otimes\{1^2\}=\tilde{\Box}''\left(-\tilde{\Box}_-+\sum_{x=0}(-1)^x\langle 1(2x+2)\rangle\right). \tag{7.12b}$$

Comparison of (7.11) with (7.12) shows not only a striking analogy between the two pairs of plethysms, but also the existence of explicit factorization of these plethysms of $\Box''$ and $\tilde{\Box}''$ that are analogous to the factorizations of Δ'' and $\tilde{\Delta}$ given in (1.3) and (1.4). That this is not an accident may be seen by noting more generally that the factorization of the plethysms $\Box''\otimes\{\kappa\}$ and $\tilde{\Box}''\otimes\{\kappa\}$ essentially parallels that given in Sec. IV by changing Δ'' to $\Box''$ and $\tilde{\Delta}$ to $\tilde{\Box}''$ throughout and replacing $x_i^{1/2}$ by x_i for all i in (4.3), (4.5), (4.13), and (4.15). To present this formally, we note from (2.12) and the various definitions (1.1b), (1.1a), (7.5), and (7.6) that

$$\Box''=\Delta''\otimes p_2,\quad \tilde{\Box}''=\tilde{\Delta}\otimes p_2, \tag{7.13}$$

where p_2 is the elementary power sum function of degree 2. It then follows from the algebra of plethysm that

$$\begin{aligned}\Box''\otimes\{\kappa\}&=(\Delta''\otimes p_2)\otimes\{\kappa\}=\Delta''\otimes(p_2\otimes\{\kappa\})=\Delta''\otimes(\{\kappa\}\otimes p_2)=(\Delta''\otimes\{\kappa\})\otimes p_2\\&=((\Delta'')^{r(\kappa)}\Pi_\kappa)\otimes p_2=(\Delta''\otimes p_2)^{r(\kappa)}(\Pi_\kappa\otimes p_2)=(\Box'')^{r(\kappa)}(\Pi_\kappa\otimes p_2)\end{aligned} \tag{7.14}$$

and

$$\tilde{\Box}'' \otimes \{\kappa\} = (\tilde{\Delta} \otimes p_2) \otimes \{\kappa\} = \tilde{\Delta} \otimes (p_2 \otimes \{\kappa\}) = \tilde{\Delta} \otimes (\{\kappa\} \otimes p_2) = (\tilde{\Delta} \otimes \{\kappa\}) \otimes p_2$$
$$= ((\tilde{\Delta})^{r(\kappa)} \tilde{\Pi}_\kappa) \otimes p_2 = (\tilde{\Delta} \otimes p_2)^{r(\kappa)} (\tilde{\Pi}_\kappa \otimes p_2) = (\tilde{\Box}'')^{r(\kappa)} (\tilde{\Pi}_\kappa \otimes p_2). \quad (7.15)$$

This gives

$$\Box'' \otimes \{\kappa\} = (\Box'')^{r(\kappa)} \Sigma_\kappa, \quad \text{where } \Sigma_\kappa = \Pi_\kappa \otimes p_2 \quad (7.16)$$

and

$$\tilde{\Box}'' \otimes \{\kappa\} = (\tilde{\Box}'')^{r(\kappa)} \tilde{\Sigma}_\kappa, \quad \text{where } \tilde{\Sigma}_\kappa = \tilde{\Pi}_\kappa \otimes p_2. \quad (7.17)$$

Finally, from (4.6), (4.7), and (4.14) we have the determinantal expansions

$$\Sigma_\kappa = \Pi_\kappa \otimes p_2 = |\Pi_{\binom{a_s}{b_t}} \otimes p_2|_{r \times r} = |\Sigma_{\binom{a_s}{b_t}}|_{r \times r} \quad (7.18)$$

and

$$\tilde{\Sigma}_\kappa = \tilde{\Pi}_\kappa \otimes p_2 = |\tilde{\Pi}_{\binom{a_s}{b_t}} \otimes p_2|_{r \times r} = |\tilde{\Sigma}_{\binom{a_s}{b_t}}|_{r \times r}, \quad (7.19)$$

with

$$\dim \Sigma_\kappa = \dim \Pi_\kappa = (-1)^{b_1 + b_2 + \cdots + b_r} |(a_s + b_t + 1)^{n-1}|_{r \times r}. \quad (7.20)$$

To illustrate the outcome of calculations of $\tilde{\Sigma}_\kappa$ we quote the following two results calculated in the case of Sp(6,ℝ) up to terms of weight 12 in the first example and weight 10 in the second:

$$\tilde{\Sigma}_3 = \langle 2(0) \rangle - \langle 2(2) \rangle + \langle 2(2^2) \rangle - \langle 2(3^2) \rangle + \langle 2(4) \rangle - \langle 2(51) \rangle + \langle 2(53) \rangle - \langle 2(5^2) \rangle + \langle 2(61^2) \rangle$$
$$+ \langle 2(6^2) \rangle - \langle 2(73) \rangle - \langle 2(81^2) \rangle + \langle 2(82) \rangle + \langle 2(84) \rangle - \langle 2(86) \rangle - \langle 2(93) \rangle + \langle 2(10\ 1^2) \rangle$$
$$- \langle 2(11\ 1) \rangle + \langle 2(12) \rangle + \cdots, \quad (7.21)$$

and

$$\tilde{\Sigma}_{2^2} = \langle 2(2^2) \rangle - \langle 2(31) \rangle + 2\langle 2(3^2) \rangle + \langle 2(4) \rangle + \langle 2(41^2) \rangle - 2\langle 2(42) \rangle + 3\langle 2(4^2) \rangle + \langle 2(51) \rangle$$
$$- 3\langle 2(53) \rangle + 3\langle 2(5^2) \rangle - 2\langle 2(6) \rangle - 2\langle 2(61^2) \rangle + 4\langle 2(62) \rangle - 3\langle 2(64) \rangle - \langle 2(71) \rangle$$
$$+ 2\langle 2(73) \rangle + 3\langle 2(8) \rangle + 3\langle 2(81^2) \rangle - 5\langle 2(82) \rangle + \cdots. \quad (7.22)$$

From these, through multiplication by $\tilde{\Box}''$ and $(\tilde{\Box}'')^2$, respectively, we can recover:

$$
\begin{aligned}
\tilde{\square}''\otimes\{3\}=&\langle 3(0)\rangle-\langle 3(1^2)\rangle+\langle 3(21^2)\rangle+\langle 3(2^2)\rangle-\langle 3(2^3)\rangle-\langle 3(31)\rangle-2\langle 3(3^2)\rangle+\langle 3(3^22)\rangle\\
&+\langle 3(4)\rangle+2\langle 3(42)\rangle-\langle 3(431)\rangle-\langle 3(43^2)\rangle+3\langle 3(4^2)\rangle+\langle 3(4^3)\rangle-2\langle 3(51)\rangle\\
&-2\langle 3(53)\rangle+\langle 3(532)\rangle-4\langle 3(5^2)\rangle-\langle 3(5^22)\rangle+\langle 3(6)\rangle+\langle 3(61^2)\rangle+2\langle 3(62)\rangle\\
&-\langle 3(62^2)\rangle+3\langle 3(64)\rangle-\langle 3(642)\rangle+\langle 3(651)\rangle+4\langle 3(6^2)\rangle-2\langle 3(71)\rangle-3\langle 3(73)\rangle\\
&-5\langle 3(75)\rangle+\langle 3(8)\rangle+\langle 3(81^2)\rangle+3\langle 3(82)\rangle+4\langle 3(84)\rangle-2\langle 3(91)\rangle-3\langle 3(93)\rangle\\
&+\langle 3(10)\rangle+\langle 3(10\,1^2)\rangle+3\langle 3(10\,2)\rangle-2\langle 3(11\,1)\rangle+\langle 3(12)\rangle+\cdots
\end{aligned}
\tag{7.23}
$$

and

$$
\begin{aligned}
\tilde{\square}''\otimes\{2^2\}=&\langle 4(2^2)\rangle+\langle 4(2^3)\rangle-\langle 4(31)\rangle-2\langle 4(3^2)\rangle-\langle 4(3^22)\rangle+\langle 4(4)\rangle-\langle 4(41^2)\rangle+2\langle 4(42)\rangle\\
&+2\langle 4(42^2)\rangle-\langle 4(431)\rangle-\langle 4(43^2)\rangle+2\langle 4(4^2)\rangle+\langle 4(4^22)\rangle-2\langle 4(51)\rangle-3\langle 4(53)\rangle\\
&-\langle 4(532)\rangle-2\langle 4(5^2)\rangle+3\langle 4(62)\rangle+3\langle 4(62^2)\rangle+4\langle 4(64)\rangle-3\langle 4(71)\rangle-4\langle 4(73)\rangle\\
&-\langle 4(8)\rangle+\langle 4(81^2)\rangle-\langle 4(82)\rangle-4\langle 4(91)\rangle-2\langle 4(10)\rangle+\cdots .
\end{aligned}
\tag{7.24}
$$

VIII. CONCLUSION

In this pair of papers, KWI[1] and the present one, an attempt has been made to establish explicit analogies between character theoretic results for finite-dimensional irreducible representations of $SO(2n)$ and infinite-dimensional irreducible representations of $Sp(2n,\mathbb{R})$. This has involved spelling out in detail a range of corresponding results for these two groups: on their characters and products in part I, and on symmetrized powers or plethysms here in part II.

The most striking feature of these results is that the correspondence always involves, as in Propositions 3.1 and 3.2, for example, an infinite sequence of terms of the form $\langle m(\lambda)\rangle$ for $Sp(2n,\mathbb{R})$ and $[m^n/\lambda']$ for $SO(2n)$. In both cases the terms are indexed by partitions λ whose length $\ell(\lambda)$ is finite. In fact $\ell(\lambda)\leqslant m$, where m may be as large as one likes but is determined by the necessarily finite tensor power or degree of plethysm under consideration. However, their breadth $\ell(\lambda')$ is, in principle, unbounded in both cases. The fact that the $SO(2n)$ case is rendered finite dimensional, whereas the $Sp(2n,\mathbb{R})$ case is infinite dimensional, is a consequence of the dependence of the former on m^n/λ' rather than just λ. As a result all summations over λ in the $SO(2n)$ case are finite. This trivial looking distinction places an effective upper bound of n on $\ell(\lambda')$ in the case of $SO(2n)$. Fortunately, the unified approach adopted here allows both Propositions to be treated on an equal footing.

In deriving Propositions 3.1 and 3.2 a noteworthy theorem from the pure mathematics literature, due to Scharf and Thibon,[17] has been brought to bear in such a way as to provide a proof of a result of considerable significance in the study of symplectic models of nuclei that was first stated and indeed used by Carvalho.[8] By exposing and exploiting the analogy between $SO(2n)$ and $Sp(2n,\mathbb{R})$ the problem of decomposing tensor powers of both Δ'' and $\tilde{\Delta}$ has thus been reduced to that of evaluating the branching rule multiplicities associated with the group–subgroup restriction $O(k)\rightarrow S_k$.

However, it has proved possible to go further. The factorizations of the plethysms $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$, which have been identified in Propositions 4.1 and 4.3, were hitherto unexpected. Although of interest in their own right, it is perhaps more important that they contribute to the study in hand in two distinct ways. First, as indicated in (4.6) and (4.14), the resulting factors Π_κ and $\tilde{\Pi}_\kappa$ possess determinantal expansions which allow them to be calculated from special cases involving only partitions of the form $(a+1,1^b)$. Second, it has been shown not only that these factors Π_κ and $\tilde{\Pi}_\kappa$ belong to the rings of characters of $SO(2n)$ and $Sp(2n,\mathbb{R})$, respectively, but also that they possess explicit expansions in terms of such characters involving integer coefficients that are amenable to calculation through Corollaries 5.4 and 5.8. Both these properties of Π_κ and $\tilde{\Pi}_\kappa$ serve to make tractable the evaluation of the plethysms $\Delta''\otimes\{\kappa\}$ and $\tilde{\Delta}\otimes\{\kappa\}$ themselves, as illustrated here through the presentation of some substantial examples.

It is hoped, therefore, that the present work will have gone a long way toward dispelling any qualms researchers in the field might have quite naturally held regarding the difficulties of working with infinite-dimensional representations of the noncompact group Sp$(2n,\mathbb{R})$. They are really no more difficult to deal with than the finite-dimensional representations of SO$(2n)$.

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