STATISTICAL AND GROUP PROPERTIES OF THE FRACTIONAL QUANTUM HALL EFFECT

BRIAN G WYBOURNE
Instytut Fizyki, UMK, Toruń, POLAND
E-mail: bgw@phys.uni.torun.pl

We discuss, from a group point of view the angular momentum states that arise for N electrons on a Haldane sphere containing at the center a magnetic monopole. Angular momentum shells associated with large effective one electron angular momentum ℓ_n arise. For large N the multiplicities have a Wigner-type distribution. Many of the properties of such a distribution can be quite closely represented in terms of simple group properties. Furthermore simple relationships exist between certain totally symmetric boson states and totally antisymmetric fermion states.

1 Introduction

The theory of symmetric functions and of Lie groups plays an essential role in our understanding of the fractional quantum Hall effect. Here we first outline some properties of some very particular irreducible representations of the special unitary groups, SU_k, that are of direct significance in mappings of boson states to fermion states. We then discuss their statistical properties when k and the number of particles become large. Finally we review some combinatorial properties of the Laughlin wavefunctions.

2 Dimensions in SU_k

We shall focus on two types of irreducible representations of SU_k, the symmetric $\{p\}$ and antisymmetric $\{1^p\}$ representations that respectively can be associated with boson and fermion states for p particles. The dimensions of these irreducible representations can be written as:-

$$D_k(\{p\}) = \binom{k+p-1}{k-1} = \tfrac{(k+p-1)!}{(k-1)!p!} \tag{1}$$

$$D_k(\{1^p\}) = \binom{k}{p} = \tfrac{k!}{(k-p)!p!} \tag{2}$$

The following specialisations are to be noted:-

$$D(\{n\}) = \binom{m+1}{m-n+1} = \binom{m+1}{n}$$
$$(k=m-n+2, p=n) \quad or \ (k=n+1, p=m+1-n) \tag{3}$$

$$D(\{1^n\}) = \binom{m+1}{n}$$
$$(k=m+1, p=n) \quad or \ (k=m+1, p=m+1-n) \tag{4}$$

Suppose we have Bosons (B) and Fermions (F) with single particle angular momenta

$$\ell_B = \frac{m-n+1}{2}, \quad \ell'_B = \frac{n}{2}, \quad \ell_F = \frac{m}{2} \tag{5}$$

The many-boson states will be totally symmetric while those of the many-fermion states will be totally antisymmetric. Eqns (3) and (4) imply the Hilbert space dimensional equalities

$$\mathcal{D}im({\ell_B}^n)_{\mathcal{S}} = \mathcal{D}im({\ell'_B}^{m+1-n})_{\mathcal{S}} = \mathcal{D}im({\ell_F}^n)_{\mathcal{A}} = \mathcal{D}im({\ell_F}^{m+1-n})_{\mathcal{A}} \tag{6}$$

It is already clear that these are associated with very special irreducible representations. Do they have relevant properties that go beyond mere dimensional equalities?

3 The Second-order Casimir Invariants C_2 in SU_k

These may be calculated directly by specialisation of results given elsewhere[1] to give:-

$$C_2(\{p\}) = \tfrac{p(p+k)(k-1)}{2k^2} \tag{7}$$

$$C_2(\{1^p\}) = \tfrac{p(k+1)(k-p)}{2k^2} \tag{8}$$

These results can be specialised to give

$$C_2(\{n\}) = \tfrac{n(m+2)(m-n+1)}{2(m-n+2)^2} \quad k=m-n+2, p=n \tag{9}$$

$$C_2(\{n\}) = \tfrac{n(m+2)(m-n+1)}{2(n+1)^2} \quad k=n+1, p=m+1-n \tag{10}$$

$$C_2(\{1^n\}) = \tfrac{n(m+2)(m-n+1)}{2(m+1)^2} \quad k=m+1, p=n \tag{11}$$

$$C_2(\{1^n\}) = \tfrac{n(m+2)(m-n+1)}{2(m+1)^2} \quad k = m+1, p = m+1-n \tag{12}$$

Recall that the Casimir invariant is defined up to an overall normalisation.

4 The Second-order Dynkin Index I_2 for SU_k

We have[1]

$$I_2(\{\lambda\} = \mathcal{N}_k D_k(\{\lambda\}) \times C_2(\{\lambda\}) \tag{13}$$

where $\mathcal{N}_k$ is a normalisation constant and hence for SU_k

$$I_2(\{p\}) = \mathcal{N}_k \tfrac{(k+p-1)!}{(k-1)!p!} \times \tfrac{p(p+k)(k-1)}{2k^2} \tag{14}$$

$$I_2(\{1^p\}) = \mathcal{N}_k \tfrac{k!}{(k-p)!p!} \times \tfrac{p(k+1)(k-p)}{2k^2} \tag{15}$$

$$\tag{16}$$

Under the reduction $SU_k \to SO_3$ we consider for the vector irreducible representation $\{1\}$ of SU_k the decomposition

$$\{1\} \to [\frac{k-1}{2}] = [j] \tag{17}$$

Let us define the Dynkin index for a SO_3 irreducible representation $[j]$ as

$$I_2([j]) = j(j+1)(2j+1) \tag{18}$$

and fix the normalisation constant $\mathcal{N}_k$ so as to match the Dynkin index of the vector irreducible representation $\{1\}$ of SU_k. This gives

$$\mathcal{N}_k = \frac{k^2}{2} \tag{19}$$

and hence

$$I_2(\{p\}) = \tfrac{(k+p)!}{4(k-2)!(p-1)!} \tag{20}$$

$$I_2(\{1^p\}) = \tfrac{(k+1)!}{4(k-p-1)!(p-1)!} \tag{21}$$

leading to the specialisations:-

$$\begin{aligned} I_2(\{n\}) &= \tfrac{(m+2)!}{4(m-n)!(n-1)!} \\ &(k = m-n+2, p = n) or\ (k = n+1, p = m+1-n) \qquad (22) \\ I_2(\{1^n\}) &= \tfrac{(m+2)!}{4(m-n)!(n-1)!} \end{aligned}$$

$$(k = m + 1, p = n) or \ (k = m + 1, p = m + 1 - n) \tag{23}$$

Thus the Dynkin index is the same for both irreducible representations. Thus the dimensional equalities seen in Eq.(6) are also reflected in the equalities of their Dynkin index.

5 $SU_k \to SO_3$ branching rules and Gl_2 plethysms

The above observations then raise the question "Do the irreducible representations of SU_k that have the same dimension and Dynkin index have the same $SU_k \to SO_3$ decomposition when the embedding is defined as in Eq.(17)?"

For convenience let us introduce

$$j = \frac{k-1}{2} \tag{24}$$

as a single particle angular momentum that may be an integer or half-integer as k is respectively an odd or even integer. Then for an arbitrary irreducible representation $\{\lambda\}$ of SU_k the decomposition is given by the plethysm[2,3]

$$\{\lambda\} \to [j] \otimes \{\lambda\} = \sum_J g_{\lambda J}[J] \tag{25}$$

where $_{\lambda J}$ is the multiplicity, i.e. the number of times the SO_3 irreducible representation $[J]$ occurs in the decomposition.

Littlewood[4] has exploited the isomorphism between the representations $[\mu]$ of the ternary orthogonal group and the representations $\{2\mu\}$ of the binary full linear group to evaluate such plethysms. Each partition into two parts $\{\mu_1, \mu_2\}$, which arises in the binary analysis of the plethysm, may be converted into a partition into just one part by making use of the equivalence

$$\{\mu_1, \mu_2\} \equiv \{\mu_1 - \mu_2\} \tag{26}$$

Thus, the character decompositions under $SU_k \to SO_3$ may be found by simply replacing the characters $\{2\mu\}$ that arise in the plethysms $\{2j\} \otimes \{\lambda\}$ by the characters $[\mu]$ of SO_3.

Hermite's reciprocity law[3,5,6] plays a key role in what follows. Hermite's law states that[5] "the number of invariants and covariants of degree m of a binary form of degree n is the same as the number of invariants and covariants of degree n of a binary form of degree m." In terms of plethysm, this is equivalent to the statement that the binary analysis of the plethysms $\{m\} \otimes \{n\}$ and $\{n\} \otimes \{m\}$ for the linear group of any dimension coincide, thus in Gl_2,

$$\{m\} \otimes \{n\} = \{n\} \otimes \{m\} \tag{27}$$

The analysis of the plethysm $\{m\}\otimes\{n\}$ may be identified with the totally symmetric states of a system of identical particles, each of angular momentum $\frac{m}{2}$ while the plethysm $\{m\}\otimes\{1^n\}$ may be identified with the totally antisymmetric states of a system of identical particles, also each of angular momentum $\frac{m}{2}$.

Murnaghan[7] has shown that in the case of the binary full linear group, Hermite's reciprocity principle leads naturally to the identity

$$\{m\}\otimes\{1^n\} = \{m+1-n\}\otimes\{n\}, \quad m+1 \geq n \tag{28}$$

This result gives a direct link between the totally antisymmetric states of n identical particles of angular momentum $\frac{m}{2}$, and the totally symmetric states of n identical particles each having angular momentum $\frac{m+1-n}{2}$.

Use of Eq.(26) on the right-hand-side of Eq.(27) then gives

$$\{m\}\otimes\{1^n\} = \{n\}\otimes\{m+1-n\}, m+1 \geq n \tag{29}$$

from which we conclude that the totally antisymmetric states of n identical particles each of angular momentum $\frac{m}{2}$, are the same as for the totally symmetric states of $m+1-n$ identical particles each of angular momentum $\frac{n}{2}$.

Use of Eq.(26) again in Eq.(27) gives

$$\{m\}\otimes\{1^n\} = \{m\}\otimes\{1^{m+1-n}\} \tag{30}$$

corresponding to a particle-hole symmetry for fermions.

6 Equivalences among Bosonic and Fermionic States

The above results show that the dimensional equalities found in Eq.(6) extend not only to equalities of the Dynkin index of their associated special unitary group representations but also that there is a one-to-one correspondence between there angular momentum states at the SO_3 level. This also implies that the distribution of the states is the same for these particular irreducible representations independently of whether they are associated with bosons or fermions.

7 Statistical Distribution of States

Nearly thirty years ago[8] my second year physics student, John Cleary, and I considered the distribution of the number of states $D(L)$ as a function of the total angular momentum L associated with the states of maximal spin of a configuration ℓ^N of N electrons occupying equivalent orbitals of angular momentum ℓ. This was a natural outgrowth earlier work with Norbert Rosenzweig[9] which led me to study the statistical properties of the distribution energy levels of the lanthanides and actinides[10] where the Wigner distribution plays a key role[11,12].

Cleary and I found that indeed for sufficiently large ℓ and N the states are distributed according to a Wigner-type distribution

$$D(L) = \sum_i A_i(L + \frac{1}{2}) \exp[-(L + \frac{1}{2})^2/2\sigma_i^2] \qquad (31)$$

or, to a lesser approximation as a single Wigner distribution

$$D(L) = A(L + \frac{1}{2}) \exp[-(L + \frac{1}{2})^2/2\sigma^2] \qquad (32)$$

This study was later extended[13] but it seemed, at the time, to be largely a curiosity with no applications. Relevance has come with the need to understand certain aspects of the fractional quantum Hall effect where the size of the Hilbert space can become very large as the electron angular momentum can become very large[14]. In those cases rather than carrying out the explicit group subgroup decompositions we can take to a very good approximation the Wigner type distribution[13]

$$g_J = A(J + \frac{1}{2}) \exp[-(J + \frac{1}{2})^2/2\sigma^2] \qquad (33)$$

integrating we have

$$\sum_J g_J = A \int_0^\infty (J + \frac{1}{2}) \exp[-(J + \frac{1}{2})^2/2\sigma^2] dJ = A\sigma^2 \qquad (34)$$

which is simply the sum of the multiplicities.

Likewise

$$\sum_J (2J+1) g_J \qquad = A\sigma^3\sqrt{2\pi} \qquad (35)$$

$$\sum_J J(J+1)(2J+1) g_J = A\sigma^3\sqrt{2\pi}(3\sigma^2 - \tfrac{1}{4}) \qquad (36)$$

Eqns (34)-(36) are respectively, the sum of the multiplicities, the dimension and the second-order Dynkin index. Dividing Eq.(36) by Eq. (35) and noting earlier results leads to

$$\sigma^2 = \frac{2n(j+1)(2j+1-n)+1}{12} \qquad (37)$$

Similarly we have

$$A = \frac{\mathcal{D}im\{1^n\}}{\sigma^3\sqrt{2\pi}} \tag{38}$$

$$\langle 2J+1\rangle \;= \sigma\sqrt{2\pi} \tag{39}$$

$$g_{max} \;= A\sigma e^{-\frac{1}{2}} \tag{40}$$

$$g_0 \;= \frac{A}{2} \tag{41}$$

Eq.(39) gives the expectation value of J for the distribution, while Eq.(40) gives the maximal multiplicity which occurs when J attains the value $J_m \sim \sigma - \frac{1}{2}$

As an example, consider the case where there $j = 18$ and $n = 13$. Using the computer package SCHUR we can compute the complete set of branching multiplicities, g_J, for the decomposition of the $\{1^{13}\}$ irreducible representation of SU_{37} under $SU_{37} \to SO_3$ exactly and compare the results with those obtained using the Wigner-type distribution. It follows from Eqns (37) and (38) that

$$A = 45,758.4 \qquad \sigma = 31.434 \tag{42}$$

Using those two results leads, via Eqns (34) to (41) to

$$\sum_J g_J \qquad = 45,213,088 \quad (44,585,180) \tag{43}$$

$$\sum_J J(J+1)(2J+1)g_J = 10,560,043,704,670 \qquad (10,559,153,077,200) \tag{44}$$

$$g_{max} \qquad = 872,409 \qquad (848,521) \tag{45}$$

$$J_m \qquad = 31 \qquad (32) \tag{46}$$

$$g_0 \qquad = 22,879 \qquad (21,660) \tag{47}$$

The exact computed values from SCHUR are given in brackets.It is apparent that even a single Wigner type distribution gives the actual distribution to within $\sim 5\%$.Our problem is essentially related to corresponding problems in the theory of partitions of integers and generating functions[8,13].

8 Concluding Remarks

The fractional quantum Hall effect presents the combinatorist with many interesting and incompletely solved problems and in this presentation I have barely scratched the surface. We have identified certain irreducible representations of particular SU_k groups that can be used to describe the properties of bosons and fermions in Hilbert spaces of the same dimension with the same SO_3 multiplets. This has the consequence that any Hamiltonian that involves just the invariants of SU_k and SO_3 will yield the same energy spectrum, to within an overall constant as indeed recently noted by Quinn et al[15]. Note, however, that invariants for subgroups of SU_k that contain SO_3 as a subgroup do not conserve that property. Another combinatorial tool of great relevance to the fractional quantum Hall effect are the even powers of the Vandermonde determinant[16,17,18]. A remarkable conjecture concerning the number of admissible tableaux that arise in the expansion of the second power in terms of Schur functions was made by Di Francesco et al[18] which however failed for more than 7 particles. While clearly a close approximation it remains a combinatorial open problem to replace the conjecture by an exact result. Tellez and Forrester[19] have recently shown that in the results obtained[17] for the second power expansion lead to exact results in certain aspects of the statistical physics of two-dimensional one-component plasmas demonstrating again the richness of applications of combinatorial tools in physics.

9 Acknowledgements

My work has been partially supported by a Polish KBN Grant The initial part of this work was done during a visit to the Max-Planck Institut für Astrophysik. I am particularly appreciative of the support, hospitality and encouragement given by Professor Geerd Diercksen.

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