Symmetry - Harmonic Oscillators to Quantum Dots

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1. **The Concept of Symmetry.**
2. **Groups and the Monster.**
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4. **Why do Dinosaurs have small heads?**
5. **The Classical 1-dimensional Harmonic Oscillator.**
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13. **Modelling Many-Particle States.**
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15. **Back to the Future.**

What an imperfect world it would be if every symmetry was perfect

Examples of Symmetry 1

Continuous Rotational Symmetry

Permutational Symmetry

Similarity Transformation

How would you like to live in Looking-Glass House, Kitty? I wonder if they'd give you milk in there? Perhaps looking-glass milk isn't good to drink?

Lewis Carroll — Through the Looking Glass

Examples of Symmetry 2

1. **Circular symmetry.**
2. **Spherical Symmetry.**
3. **Continuous and Discrete Symmetries.**
   a. **Additive and multiplicative quantum numbers.**
   b. **Reflections and Rotations.**

Symmetry and Conservation Laws

1. **Symmetry is always tentative and requiring experimental verification.**
2. **Every symmetry is associated with an impossible experiment. If you succeed with the experiment you fail the symmetry.**
3. **The wonderful discovery of Emmy Nöether (1905).**
4. **Every Conservation Law is associated with an Impossible Experiment.**

Anything that happens, happens.
Anything that, in happening, causes something else to happen, causes something else to happen.
Anything that, in happening, causes itself to happen again, happens again.
It doesn't necessarily do it in chronological order, though.
— Douglas Adams *Mostly Harmless* London: Heinemann (1992)

Types of Symmetry Groups

Finite Groups

a. **Number of group elements is finite.**

b. **Number of Representations is finite.**

c. **Dimension of representation matrices is finite.**

Examples

a. **Symmetric groups** $S(n)$.

b. **Finite sets of rotations.**

c. **Monster Group (smallest representation** $\approx 191,000$**) (number of elements** $\approx 10^{54}$**).**

Compact Lie Groups

a. **Number of elements infinite.**

b. **Number of unitary representations is infinite.**

c. **Dimension of representation matrices is finite.**

Examples

a. $SO(3) \sim SU(2)$ **in Quantum Theory of Angular Momentum.**

b. $SO(4)$ **Degeneracy group of hydrogen atom.**

c. $SU(3)$ **Degeneracy group of the isotropic 3-d harmonic oscillator.**

d. $E(8)$ **Coding theory, String theories.**

e. $SU(6), SO(6), SO(5)$ **Interacting Boson Model of nuclei.**

"Anyone who believes that exponential growth can continue indefinitely in a finite world is either a madman or an economist"

Prof. Kenneth Boulding (economist)

Non-Compact Lie Groups

a. **Number of elements is infinite.**

b. **Number of unitary representations is infinite.**

c. **Dimension of non-trivial unitary representations is infinite.**

Examples

a. $SU(1,1)$ **Interferometry.**

b. $Sp(6,R)$ **Dynamical group of the isotropic 3-d harmonic oscillator**

c. $SO(4,2)$ **Dynamical group of the hydrogen atom.**

d. $Sp(4,R)$ **Quantum Optics**

"When a thing was new, people said, 'It is not true'. Later, when its truth became obvious, people said, 'Anyhow, it is not important' and when its importance could no longer be denied, people said, 'Anyway, it is not new'". (William James, philosopher)

How big is the Monster group?

Number of elements of group $\approx 10^{54}$

Suppose a supercomputer counts at $10^9/sec$. **How long will it take to count to** 10^{54}**?**

Number of seconds required $= \frac{10^{54}}{10^9} = 10^{45}$

One year $\approx 3 \times 10^7$ **seconds.**

Number of years required $\approx 10^{38}$.

Age of the universe $\approx 10^{10}$ **years.**

Thus time required is $\approx 10^{28}$ **times the age of the universe!**

OR

Suppose each element of the group is associated with one molecule of a gas at NTP. How big a volume of gas is required?

ANSWER A sphere of radius ≈ 100 **light seconds.**

Recall light from SUN takes ≈ 500 **seconds to reach us. Thus sphere has a radius of** $\frac{1}{5}$ **of the distance between SUN and EARTH!**

Jonathan Swift and *Gulliver's Travels (1726)*. **Concerning the Liiliputians, Gulliver records "... his Majesty's mathematicians, having taken the height of my body ... and finding it to exceed theirs in the proportion of twelve to one, they concluded from the similarity of their bodies, that mine must contain at least 1724 of theirs..." [BUT 12 x 12 x 12 = 1728 NOT 1724]**

Changes of Scale in Physics

Area of a circle

$$\boxed{A_o = \pi r^2}$$

Area of the surface of a sphere

$$\boxed{A_s = 4\pi r^2}$$

Volume of a sphere

$$\boxed{A_v = \frac{4}{3}\pi r^3}$$

Why do small animals have a higher rate of metabolism than BIG animals?

small animal **BIG animal**

$$\boxed{\frac{\textbf{Heat Lost by a Sphere}}{\textbf{Heat content of a Sphere}} \propto \frac{1}{r}}$$

CONCLUSION: Small animals lose heat, relative to their size, faster than large animals and must metabolise food quicker as observed. Babies are more sensitive to thermal effects than adults.

Making a simple task look like an occult pursuit is a device used by professionals and experts in all walks of life
— Fred Plisner, Gravity is getting me down (1994)

The Simple One-Dimensional Harmonic Oscillator

1. The Symmetry Group of Newton's Equation of Motion

In appropriate units the Newton equation of motion is

$$\boxed{\boxed{\frac{d^2x}{dt^2} + x = 0}} \tag{1}$$

The Newton equation depends only on the two variables (x, t). What changes in x and t can we make while still leaving the form of the equation of motion unchanged?

What about $x \rightarrow ax$? Leaves Eq.(1) form invariant. Corresponds to a change of scale.

What about $t \rightarrow t + a$? Leaves Eq.(1) form invariant. Corresponds to a time translation.

What is the symmetry group of the complete set of such point transformations?

The Lie group associated with point transformations that leave Newton's equation of motion invariant is the non-compact group $SL(3, R)$.

Our question is, why snowflakes in their first falling before they are entangled in larger plumes, always fall with six corners and with six rods, tufted like feathers. There must be some definite cause why, whenever snow begins to fall its initial formations invariably display the shape of the six-cornered starlet

– J. Kepler The Six-Cornered Snowflake, (1611)

Group Problems

1. **In order to implement a group solution to a problem one must be able to compute the following :**
   - a. **Kronecker products of irreps.**
   - b. **Symmetrised powers of irreps.**
   - c. **Branching rules.**
   - d. **Dimensions of finite irreps.**
   - e. **Casimir invariants.**
   - f. **Matrix elements of group generators.**
2. **Computing with non-compact groups raises special problems, since the unitary irreps are usually of infinite dimension. This means that expansions must have a cutoff.**

What can be more like my hand or my ear than their reflections in a mirror?
And still the hand in the mirror cannot substitute for my real hand
– Immanuel Kant

States of an Isotropic Three-Dimensional HO
Infinite Mp(6) Tower
SO(2,1) Multiplets

States of a Hydrogen Atom
Infinite SO(4,2) Tower
SO(2,1) Multiplets

Infinite Sets of States

1. Quantum systems (as with Classical systems) are commonly associates with infinite sets of states.

   a. Single particle in an isotropic 3-d harmonic oscillator has an infinite set of discrete states.

   b. Hydrogen atom has an infinite set of discrete states as well as continuum states.

   c. These states span an infinite Hilbert space.

2. The discrete states in the above two examples span finite dimension unitary representations of their respective degeneracy groups, SU(3) and SO(4).

   a. The irreps $[n, 0]$ of SO(4) are of dimension n^2.

   b. The irreps $\{n, 0\}$ of SU(3) are of dimension $\frac{n}{2}(n+1)$.

Dynamical Groups

1. A Dynamical Group contains the degeneracy group as a subgroup and the complete set of states span a single irreducible representation.

2. Among the generators of the Degeneracy group are operators that allow one to ladder between the complete set of states associated with an arbitrary set of degenerate states BUT not between states belonging to different representations of the degeneracy group.

3. Among the generators of the Dynamical group are operators that allow one to ladder between states belonging to different representations of the degeneracy group.

4. The Dynamical group is necessarily a non-compact group.

5. For the H-atom the dynamical group is $SO(4,2) \sim SU(2,2)$.

6. For the isotropic 3-d harmonic oscillator the dynamical group is the metaplectic group $Mp(6)$ which is the covering group of $Sp(6, R)$.

7. Can we construct a dynamical group for a many-particle system?

Quantum Dots and Many-particle Systems

1. A quantum dot involves the confinement of N electrons in two or three dimensions, commonly by electrostatic fields, over a nano-metre scale.
2. The confining potential is parabolic to a good approximation.
3. The quantum dot behaves like a $N-$electron atom without a nuclear core.
4. In an atom the confinement length is small and kinetic energy tends to dominate over the potential energy.
5. In a quantum dot the confinement length is very much larger and the two contributions are roughly the same.
6. The quantum dot involves N identical particles (electrons) confined to d spatial dimensions ($d = 1, 2, 3$). Experimentally $N = 1, 2, \ldots, \sim 60$ electrons.
7. Electrons can be added or subtracted one at a time.

Other Confinement Problems

1. Confinement of Quarks in hadrons. c.f. MIT bag model - solve Dirac's equation in a spherical container. Bag pressure and radius enter as parameters.
2. Nucleons in a nucleus. Nucleons confined in an isotropic 3-d harmonic oscillator potential with nucleons having quantised motion about the centre-of-mass.

The Dynamical Group of Mesoscopic Systems

1. Assume the Hamiltonian of the $N-$particle system is a function of the coordinate and momentum operators of the individual particles.
2. The coordinate and momentum operators obey the usual Heisenberg commutation relations.
3. Bilinear combinations of these operators are constructed that close under commutation.
4. The dynamical group so constructed is the metaplectic group $Mp(2Nd)$, the covering group of the non-compact $Sp(2Nd, R)$ group.
5. The entire set of $N-$particle states span the fundamental irrep $\tilde{\Delta}$ of $Mp(2Nd)$.
6. Under the reduction $Mp(2Nd) \Rightarrow Sp(2Nd, R)$

$$\tilde{\Delta} \Rightarrow < \frac{1}{2}(0) > + < \frac{1}{2}(1) >$$

Subgroup Structures

1. The group $Mp(2Nd)$ contains an extensive subgroup structure as shown on the next transparency.
2. The various subgroup structures may be each given a physical interpretation.
3. Each subgroup structure is associated with a particular type of classification of the $N-$particle states.
4. A key problem is to be able to follow the decomposition of the $\tilde{\Delta}$ irrep as one proceeds through an appropriate chain of groups.
5. The irrep $\tilde{\Delta}$ includes spurious spin states that must be eliminated to maintain the Pauli exclusion principle.

Group-Subgroup structure appropriate to quantum dots

$Sp(18,R) \Rightarrow Sp(6,R) \times O(3)$ Branching Rules

$$< \frac{1}{2}(0) > \Rightarrow$$

$$\begin{array}{lll} < s1;(0) > [0] & + < s1;(1^2) > [1]\# & + < s1;(2) > [2] \\ + < s1;(31) > [3]\# & + < s1;(4) > [4] & + < s1;(51) > [5]\# \\ + < s1;(6) > [6] & + < s1;(71) > [7]\# & + < s1;(8) > [8] \\ + < s1;(91) > [9]\# & + < s1;(10\ ) > [10\ ] & \end{array} \quad (1)$$

$$< \frac{1}{2}(1) > \Rightarrow$$

$$\begin{array}{lll} < s1;(1) > [1] & + < s1;(1^3) > [0]\# & + < s1;(21) > [2]\# \\ + < s1;(3) > [3] & + < s1;(41) > [4]\# & + < s1;(5) > [5] \\ + < s1;(61) > [6]\# & + < s1;(7) > [7] & + < s1;(81) > [8]\# \\ + < s1;(9) > [9] & + < s1;(10\ 1) > [10\ ]\# & \end{array} \quad (2)$$

$O(3) \Rightarrow S(3)$ Branching Rules

$O(3)$	$\Rightarrow S(3)$		
$[0]$	$\{3\}$		
$[1]$	$\{3\}$	$+ \{21\}$	
$[2]$	$\{3\}$	$+ 2\{21\}$	
$[3]$	$2\{3\}$	$+ 2\{21\}$	$+ \{1^3\}$
$[4]$	$2\{3\}$	$+ 3\{21\}$	$+ \{1^3\}$
$[5]$	$2\{3\}$	$+ 4\{21\}$	$+ \{1^3\}$
$[6]$	$3\{3\}$	$+ 4\{21\}$	$+ 2\{1^3\}$

States transforming as $\{21\}$ under $S(3)$ have spin $\frac{1}{2}$ while those transforming as $\{1^3\}$ have spin $\frac{3}{2}$. States transforming as $\{3\}$ are spurious and must be eliminated.

$O(3)$ states $[n]\#$ branch to the conjugates of the $S(3)$ irreps found for the branching of $[n]$. e.g. $[0]\# \Rightarrow \{1^3\}$.

Table 1 Some $Sp(6,R)$ Plethysms

$<s;(0)>\otimes\{2\}=$	$<1;(0)>$	$+\ <1;(4)>$	$+\ <1;(8)>$
	$+\ <1;(12\ )>$		
$<s;(0)>\otimes\{1^2\}=$	$<1;(2)>$	$+\ <1;(6)>$	$+\ <1;(10)\}>$
$<s;(1)>\otimes\{2\}=$	$<1;(2)>$	$+\ <1;(6)>$	$+\ <1;(10\ )>$
$<s;(1)>\otimes\{1^2\}=$	$<1;(1^2)>$	$+\ <1;(4)>$	$+\ <1;(8)>$
	$+\ <1;(12\ )>$		
$<s;(0)>\otimes\{3\}=$	$<s1;(0)>$	$+\ <s1;(4)>$	$+\ <s1;(6)>$
	$+\ <s1;(8)>$	$+\ <s1;(91)>$	$+\ <s1;(10\ )>$
	$+\ 2<s1;(12\ )>$		
$<s;(0)>\otimes\{21\}=$	$<s1;(2)>$	$+\ <s1;(4)>$	$+\ <s1;(51)>$
	$+\ <s1;(6)>$	$+\ <s1;(71)>$	$+\ 2<s1;(8)>$
	$+\ <s1;(91)>$	$+\ 2<s1;(10\ )>$	
$<s;(0)>\otimes\{1^3\}=$	$<s1;(31)>$	$+\ <s1;(6)>$	$+\ <s1;(71)>$
	$+\ <s1;(91)>$	$+\ <s1;(10\ )>$	
$<s;(1)>\otimes\{3\}=$	$<s1;(3)>$	$+\ <s1;(61)>$	$+\ <s1;(7)>$
	$+\ <s1;(9)>$		
$<s;(1)>\otimes\{21\}=$	$<s1;(21)>$	$+\ <s1;(41)>$	$+\ <s1;(5)>$
	$+\ <s1;(61)>$	$+\ <s1;(7)>$	$+\ 2<s1;(81)>$
	$+\ <s1;(9)>$		
$<s;(1)>\otimes\{1^3\}=$	$<s1;(1^3)>$	$+\ <s1;(41)>$	$+\ <s1;(61)>$
	$+\ <s1;(81)>$	$+\ <s1;(9)>$	

Three-particle states

For three particles in an isotropic three-dimensional harmonic oscillator the dynamical group is $Mp(18)$ whose fundamental irrep $\tilde{\Delta}$ decomposes under restriction to $Sp(18,R)$ as

$$\tilde{\Delta} \rightarrow < \frac{1}{2}(0) > + < \frac{1}{2}(1) > \tag{11}$$

Then under $Sp(18,R) \rightarrow Sp(6,R) \times O(3)$

$$\begin{array}{lllll} < \frac{1}{2}(1) > \rightarrow & < s1;(0) > [0] & + & < s1;(1^2) > [1]\# & < s1;(2) > [2] \\ & + \ < s1;(31) > [3]\# & < s1;(4) > [4] & + & < s1;(51) > [5]\# \\ & + \ < s1;(6) > [6] & + & < s1;(71) > [7]\# & < s1;(8) > [8] \\ & + \ < s1;(91) > [9]\# & < s1;(10\ ) > [10\ ] & & \end{array} \tag{12}$$

$$\begin{array}{lllll} < \frac{1}{2}(1) > \rightarrow & < s1;(1) > [1] & + & < s1;(1^3) > [0]\# & < s1;(21) > [2]\# \\ & + \ < s1;(3) > [3] & + & < s1;(41) > [4]\# & < s1;(5) > [5] \\ & + \ < s1;(61) > [6]\# & < s1;(7) > [7] & + & < s1;(81) > [8]\# \\ & + \ < s1;(9) > [9] & + & < s1;(10\ 1) > [10\ ]\# & \end{array} \tag{13}$$

Once you enter the world of science or mathematics or philosophy, endless plains open around you. The more you learn, the more fascinating the whole thing becomes

— Colin Wilson, Voyage to a Beginning, 1968

Determination of Spin States

The spins associated with these representations can be found from a knowledge of the $O(3) \rightarrow S(3)$ branching rules. The relevant branching rules are given in Table 2. Note that to obtain the branching rule for $[n]\#$ one simply replaces the $S(3)$ irreps by their conjugates.

Table 2. Some $O(3) \rightarrow S(3)$ branching rules.

$[0] \rightarrow$	$\{3\}$		
$[1] \rightarrow$	$\{21\}$	$+\ \{3\}$	
$[2] \rightarrow$	$2\{21\}$	$+\ \{3\}$	
$[3] \rightarrow$	$\{1^3\}$	$+\ 2\{21\}$	$+\ 2\{3\}$
$[4] \rightarrow$	$\{1^3\}$	$+\ 3\{21\}$	$+\ 2\{3\}$
$[5] \rightarrow$	$\{1^3\}$	$+\ 4\{21\}$	$+\ 2\{3\}$
$[6] \rightarrow$	$2\{1^3\}$	$+\ 4\{21\}$	$+\ 3\{3\}$
$[7] \rightarrow$	$2\{1^3\}$	$+\ 5\{21\}$	$+\ 3\{3\}$
$[8] \rightarrow$	$2\{1^3\}$	$+\ 6\{21\}$	$+\ 3\{3\}$
$[9] \rightarrow$	$3\{1^3\}$	$+\ 6\{21\}$	$+\ 4\{3\}$
$[10\ ] \rightarrow$	$3\{1^3\}$	$+\ 7\{21\}$	$+\ 4\{3\}$

> *Every effort has been taken to present the mathematical developments in this chapter in a comprehensible logical sequence. But the nature of the developments simply does not allow a presentation that can be followed in detail with modest effort: the reductions that are necessary to go from one step to another are often very elaborate and, on occasion, may require as many as ten, twenty, or even fifty pages. In the event that some reader may wish to undertake a careful scrutiny of the entire development, the author's derivations (in some 600 legal-size pages and in six additional notebooks) have been deposited in the Joseph Regenstein Library of the University of Chicago.*
>
> — S. Chandrasekhar, The Mathematical Theory of Black Holes

Use of $Sp(6,R)$ Plethysms

The three-particle states can be equivalently found from the use of the $Sp(6,R)$ plethysms given in Table 1. The *even* parity states must arise from

$$(S=\frac{1}{2}) < \frac{1}{2}(0) > \otimes\{21\} + < \frac{1}{2}(1) > \otimes\{2\} < \frac{1}{2}(0) > \\ + < \frac{1}{2}(1) > \otimes\{1^2\} < \frac{1}{2}(0) > \tag{13}$$

$$(S=\frac{3}{2}) < \frac{1}{2}(0) > \otimes\{1^3\} + < \frac{1}{2}(1) \otimes \{1^2\} < \frac{1}{2}(0) > \tag{14}$$

while for the *odd* parity states they arise from

$$(S=\frac{1}{2}) < \frac{1}{2}(1) > \otimes\{21\} + < \frac{1}{2}(0) > \otimes\{2\} < \frac{1}{2}(1) > \\ + < \frac{1}{2}(1) >< \frac{1}{2}(0) > \otimes\{1^2\} \tag{15}$$

$$(S=\frac{3}{2}) < \frac{1}{2}(0) > \otimes\{1^3\} + < \frac{1}{2}(0) \otimes \{1^2\} < \frac{1}{2}(1) > \tag{16}$$

To weight 10 we obtain the following even parity states

$(S=\frac{1}{2})$
$$< s1;(1^2) > + 2 < s1;(2) > + 2 < s1;(31) > + 3 < s1;(4) > \\ + 4 < s1;(51) > + 4 < s1;(6) > + 5 < s1;(71) > + 6 < s1;(8) > \\ + 6 < s1;(91) > + 7 < s1;(10) >$$

$(S=\frac{3}{2})$
$$< s1;(1^2) > + 2 < s1;(31) > + < s1;(4) > + 2 < s1;(51) > \\ + 2 < s1;(6) > + 3 < s1;(71) > + 2 < s1;(8) > + 4 < s1;(91) > \\ + 3 < s1;(10) >$$

while for the odd parity states we obtain

$(S=\frac{1}{2})$
$$< s1;(1) > + 2 < s1;(21) > + 2 < s1;(3) > + 3 < s1;(41) > \\ + 4 < s1;(5) > + 4 < s1;(61) > + 5 < s1;(7) > + 6 < s1;(81) > \\ + 6 < s1;(9) >$$

$(S=\frac{3}{2})$
$$< s1;(1^3) > + < s1;(21) > + < s1;(3) > + 2 < s1;(41) > \\ + < s1;(5) > + 3 < s1;(61) > + 2 < s1;(7) > + 3 < s1;(81) > \\ + 3 < s1;(9) >$$

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People have now a-days, got a strange opinion that everything should be taught by lectures. Now, I cannot see that lectures can do as much good as reading the books from which the lectures are taken. I know nothing that can be best taught by lectures, except where experiments are to be shewn. You may teach chymistry by lectures. - You might teach making of shoes by lectures!

— Samuel Johnson, February 1766 : see Boswell *Life of Johnson, London (1791)*

Some Experimental Demonstrations Concerning Symmetry in Physics

1. **Chiral Symmetry and Apples**

   Two inequivalent halvings of apples

2. **Cubic and Hexagonal Packing**

   Which is the denser packing?

3. **Are Rotations of 2Π and 4Π equivalent?**

   a. **The Möbius Strip**

   b. **A Cup of Water**

   c. **Rotation of Triangle**

4. **Spontaneous Symmetry Breaking**

Lectures were once useful; but now, when all can read, and books are so numerous, lectures are unnecessary. If your attention fails, and you miss a part of a lecture, it is lost; you cannot go back as you do upon a book.
— Samuel Johnson, 15th April 1781 : see Boswell *Life of Johnson, London (1791)*

Tewnty Four and Twenty Five

Take 24 identical coins and arrange them in a 6 x 4 rectangle. Now rearrange them in such a way that 25 coins fill the same rectangle.

... face-centred-cubic packing, which is familiar to chemists and crystallographers; it fills just over 74 percent of the volume of space. As far as anyone knows, it is the densest packing that can be achieved.

I am sorry to report, however, that this density has never been mathematically proved to be maximal. The least upper bound on the density obtained so far in 1958 by C. A. Rogers ... proved that no packing of spheres can have a density greater than about .7796.

— N. J. A. Sloane, *The Packing of Spheres* Sci. Amer.

Handedness and Two ways of dividing an Apple

As an illustration of handedness consider the cutting of an apple into two equal halves. Our normal solution is to cut the apple with a single vertical cut to produce two symmetrical halves. There is another way to create two equal halves. First make a vertical cut to half way down, that is to the equator of the apple. Now turn the apple upside down and rotate it through 90^o and make a second vertical cut to the equator. Now make a horizontal cut to the centre of the apple along the equator starting at the point on the equator where one of the vertical cuts and cutting along the equatorial line for 90^o or, equivalently a quarter turn. You have a choice as to the direction you rotate your knife! Now rotate the apple to the position on the equator of the other vertical cut and make a second cut along the equator for 90^o moving in the same direction as the first horizontal cut. As long as you have been careful cutting to the centre of the apple your apple should separate into two equal halves. Now take a second apple and repeat the process but this time for the two horizontal equatorial cuts make the opposite choice you made when cutting the first apple. Now we have a further two halves of an apple. Can you fit a half from the first apple to a half of the second apple? You certainly could have if you had made the traditional cutting of the apples. What is the difference?

Question Time

If you look at the history of 20th century physics, you will find that the symmetry concept has emerged as a most fundamental theme, occupying center stage in today's theoretical physics. We cannot tell what the 21st century will bring us but I feel safe to say that for the next ten or twenty years many theoretical physicists will continue to try variations on the fundamental theme of symmetry at the very foundation of our theoretical understanding of the structure of the physical universe.
— C. N. Yang. *Chinese J. Phys.* **32**, 1437 (1994)

The only questions worth asking are the unanswerable ones
— John Ciardi, *Saturday Review (1973)*

To every complex question there is a simple answer and its wrong!
— H. L. Mencken