The Application of Symmetry Concepts to Physical Problems©

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■ Lecture One

I can well appreciate, Holy Father, that as soon as certain people realize that in these books which I have written about the Revolutions of the spheres of the universe I attribute certain motions to the globe of the Earth, they will at once clamour to me to be hooted off the stage with such an opinion.

Nicolaus Copernicus, *On the Revolutions*

■ 1.1 Introduction

In this series of lectures I want to introduce you to the idea of the concepts of symmetry and their applications to physical problems. These concepts are universal in their applicability to physical problems. They are profoundly unifying concepts finding applications in such apparently diverse topics as particle- , nuclear- atomic-, molecular- and solid state physics. They are also the key to understanding many of the problems involving linear, and more importantly, nonlinear differential equations. Symmetry also enters in many aspects art.

Throughout we will emphasise practical details rather than abstract theory often referring you to the literature for further details. Thus we will try to explains results but will not be going into formal proofs. I will be expecting you to apply these results in calculations for models of real systems. Note I say *models* of real systems. Every calculation is associated with a particular model or abstraction that can *never* correspond exactly to a real system in its full diversity. Nevertheless, we might hope that some of our calculations will be a reasonable approximation to some aspects of some system.

I shall assume that you have some knowledge of the quantum theory of angular momentum such as commonly associated with the angular momentum states $|JM\rangle$ and the use of ladder operators. The first few lectures will be devoted to examples of symmetry in a variety of areas of physics with a special emphasis on examples from nuclear and particle physics. Later we will develop applications to problems in atomic- and solid-state physics. Those lectures will be preceded by the development of aspects of the theory of symmetric functions and symmetry groups.

■ 1.2 Why Symmetry?

Symmetry is usually associated with an action or transformation of a system or object such that after carrying the operation the system or object is in a state indistinguishable from that which it had prior to carrying out the action or transformation. Thus there is a close relationship between symmetry and impossible experiments. The existence of a symmetry implies that it is impossible to devise an experiment to distinguish the before and after situation. If you succeed then the symmetry does not exist. All the great conservation laws are associated with the assertion that a particular experiment is impossible. Indeed in the early 1900's Emmy Nöether showed that *every conservation law is associated with a certain invariance which in turn is associated with the statement of an impossible experiment.* For example, the conservation of angular momentum is associated with the statement that no experimentalist has been able to determine a preferred direction in space. A partial list of impossible experiments is given in Table 1.1.

■ **Table 1.1** Impossible experiments and symmetries.

Immeasurable Quantity	*Implied Invariance*	*Conserved Quantity*	*Accuracy*
Absolute Position	Space Translation	Momentum	exact(?)
Absolute Time	Time Displacement	Energy	exact(?)
Absolute Direction	Rotational	Angular Momentum	exact(?)
Relative Phase of charged and neutral particles	Charge Gauge Transformations	Charge Q	exact(?)
Left and Right Indistinguishability	Space Inversion P	parity	violated in weak interactions
Direction of Time Flow Indistinguishability	Time Reversal T	-	violated
Particle-AntiParticle Distinction	Charge Conjugation C	Charge Parity	violated in weak interactions
Relative phase of baryons and other particles	Baryon Gauge Transformations	Baryon Number B	exact(?)
Relative phase of e^- & ν_e and other particles	Electron Number Gauge Transformations	Electron Number $\mathcal{L}_e$	exact(?)
Relative phase of μ^- & ν_μ and other particles	Muon Number Gauge Transformations	Muon Number $\mathcal{L}_e$	exact(?)

Thus the existence of a symmetry tells us what is NOT possible but does not tell us what IS possible. The existence of a symmetry rules out some possibilities.[1] [2] It leads to *selection rules.* The existence of a symmetry constrains the form of theories used to model the system possessing an observed symmetry. We must strongly emphasise that *the existence of a symmetry can only be determined by experiment and is always a tentative statement.* We can never be sure that some improvement in experimental technique or some experiment not hitherto contemplated will reveal an inexactitude in the symmetry. As examples consider the parity violation experiment or the CP violation experiments of kaons.

■ 1.3 Broken symmetry

In practice very few symmetries are 'exact' and in most cases we are led to consider 'approximate' symmetries. A symmetry need not be exact to be useful. Indeed I would assert the following:

Proposition: *We should always strive to construct theories with the highest possible symmetry even if these are not exact symmetries of nature. The physics comes in the process of breaking the symmetry.*

Consider the case of

$$Ce_2Mg_3(NO_3)_{12} \cdot 24H_2O : Nd^{3+}$$

What symmetry does the Nd^{3+} ion see in the rare earth double nitrates? In free space it sees spherical symmetry associated with the three-dimensional rotation group SO_3. The total electron angular momentum J has no preferred direction in free space and is a conserved quantity. [3] Associated with the conservation of the angular momentum J is a *degeneracy* of $(2J+1)$ since each component J_z occurs at the same energy. Switch on a magnetic field in the $z-$direction and create a locally preferred direction and the degeneracy is *lifted* and we observe $2J+1$ sublevels. [4]

Placing the Nd^{3+} ion in the crystal breaks the spherical symmetry so that J ceases to be a conserved quantity - there are local preferred directions imposed by the geometrical arrangement of the various ions clustering about the Nd^{3+} ion. To a first approximation we descend to the symmetry group of the icosahedron K_h and for $J \geq \frac{5}{2}$ there will be a partial lifting of the $(2J+1)-$fold degeneracy.

The nitrate ions cluster around the vertices of a slightly distorted tetrahedron and the symmetry is approximately that of the group of the tetrahedron, T_h. Again the degeneracy is further reduced. These reductions in degeneracy manifest themselves in the appearance of sublevels. The actual point symmetry observed by $X-$ray structural analysis is that of the trigonal point group, C_3.

Thus the entire breakdown of the symmetry could be described by the chain of nested subgroups [5] [6]

$$SO_3 \supset K_h \supset T_h \supset C_3$$

[1] Kepler in his beautiful Christmas essay *The Snowflake* is fascinated throughout by the existence of symmetry and cosmologically writes of the harmony of the spheres. Copernicus, prior to Kepler, recognises the approximate nature of symmetries - writing of the sphericity of the earth *Although it is not immediately apparent that it is a perfect sphere, because the mountains project so far and the valleys are so deep, they produce very little variation in the complete roundness of the Earth*

[2] Muslim theology sees only God as perfect and thus carpet chanters, recognising their own imperfection, will deliberately make the occasional error, so that such a carpet will contain imperfections which is then consistent with their theology.

[3] Of course if we recognise that Nd^{3+} possesses a nuclear magnetic moment which weakly couples the nuclear angular momentum I to the electronic angular momentum J such that the total angular momentum becomes $\mathbf{F} = \mathbf{I} + \mathbf{J}$. Thus strictly speaking the conserved quantity is F rather than J. This is manifested in the appearance of magnetic hyperfine structure at high enough resolution.

[4] N.B. Lowering the symmetry of a system usually results in a partial or total lifting of the degeneracy. Technologically this is *very* important.

[5] For further details see S.D.Devine, T_h *symmetry in rare earth double nitrates* J. Chem. Phys. **47**, 1844 (1967) and references therein.

[6] See also C. Görlier-Walrand *et al*, *Optical spectra and crystal field analysis of europium double nitrates* J. Chem. Phys. **96**, 5650 (1992). These authors seem unaware of the work of Devine but their data shows clear signs of higher symmetries.

■ 1.4 Global and local symmetries

A symmetry may be *global* or *local.* As already seen in the previous example a local symmetry need not be global. In most of this course we will be discussing local symmetries.

1.5 Types of symmetries

There are a wide range pf possible symmetries we might consider. Two major categories would be *discrete* and *continuous* symmetries. Discrete symmetries, such as reflections, inversions, time reversal, charge conjugation, parity, finite rotations, permutations etc. are associated with *multiplicative* or *phase-like* quantum numbers. Continuous symmetries such as translations and rotations are associated with *additive* quantum numbers (e.g. angular momentum $\mathbf{J}$ or linear momentum $\mathbf{p}$).

■ 1.6 **Symmetry and the Universe**

On a clear night, away from city lights, look up to the sky (A feat more readily accomplished in the time of Copernicus in old Toruń than in modern Toruń) and you arrive at two utterly amazing and deep conclusions concerning the nature of the Universe which are in accord with more detailed observations:-

1. *The universe is almost empty.*
2. *The universe is not empty.*

Matter in the universe is astonishingly rare. Radiation is in comparison superabundant there being about 10^{18} photons for every baryon[7]. Why is matter so rare? Why is there any matter in the universe? Or somewhat more anthropologically, Why can we ask these questions? Our ability to ask these questions hinges on their answer. Why is the matter in the universe predominantly of one type and does not appear in equal quantities of matter and antimatter? What is the origin of this broken symmetry between matter and antimatter? We shall return to these questions later.

■ **1.7 Some physical constants** We will require a number of physical constants for making calculations and estimates. The values of a number of useful physical constants are tabulated below. The precise values are given as well as rough values for back-of-the- envelope calculations. It is often desirable to be able to make quick estimates to get the order of magnitude of an effect in which case the rough values suffice. It is part of being a physicist to know when a precise calculation is required and when a rough calculation is sufficient and to know how many figures in a calculation are sufficient. In this course we do all our calculations in the SI units.

[7] Questions of the existence of dark matter are irrelevant. Even if, as some believe, that 95% of the matter in the universe is unseen the existence of matter in the universe remains very rare.

Table 1.2 Some physical constants.

Quantity	*Symbol*	*Precise Value*	*Rough Value*
speed of light	c	$299792458 ms^{-1}$	$3 \times 10^8 ms^{-1}$
Planck's constant	h	$6.6260755 \times 10^{-34} Js$	$6.6 \times 10^{-34} Js$
	$\hbar$	$1.05457266 \times 10^{-34} Js$	$10^{-34} Js$
Gravitational constant	G	$6.6725985 \times 10^{-11} m^3 kg^{-1} s^{-2}$	$10^{-10} m^3 kg^{-1} s^{-2}$
electric charge magnitude	e	$1.60217733 \times 10^{-19} C$	1.6×10^{-19}
electron mass	m_e	$9.10083897 \times 10^{-31} kg$	$9.1 \times 10^{-31} kg$
electron charge/mass	$\frac{e}{m_e}$		$18 \times 10^{10} Ckg^{-1}$
proton mass	m_p	$1.6726231 \times 10^{-27} kg$	$1.7 \times 10^{-27} kg$
proton charge/mass	$\frac{e}{m_p}$		$10^{18} Ckg^{-1}$
permittivity of free space	ϵ_0	$8.854187 \times 10^{-12} Fm^{-1}$	$9 \times 10^{-12} Fm^{-1}$
permeability of free space	μ_0	$4\pi \times 10^{-7} NA^{-2}$	$12.5 \times 10^{-7} NA^{-2}$
Bohr magneton μ_B	$\frac{e\hbar}{2m_e}$	$9.2740154 \times 10^{-24} JT^{-1}$	$9.3 \times 10^{-24} JT^{-1}$
Nucleon magneton μ_N	$\frac{e\hbar}{2m_p}$	$5.0507866 \times 10^{-27} JT^{-1}$	$5 \times 10^{-27} JT^{-1}$

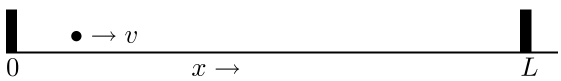
■ 1.8 The charge neutrality of matter

How neutral is matter? What would happen if we placed all the protons of a 68kg person in one box and a meter away put all the electrons? We could anticipate that an attractive electric force would develop between the two groups of charges. Application of the Coulomb force law readily leads to the magnitude of the force as $\sim 10^{30}$ Newtons! Matter is neutral to an astonishing extent. This neutrality of matter hides from us the strength of the Coulomb force. From our daily experience with forces we falsely conclude that gravity is the strongest force when in fact it is the weakest of all known forces[8]. The neutrality of matter hints at another conservation law - namely charge conservation. What is the origin of charge conservation and the neutrality of matter[9]?

■ 1.9 Classical mechanics and predictability

Before continuing further let us briefly consider the role of predictability in classical mechanics as ultimately we will be interested in predicting the behaviour of systems.

Aristotle is reputed to have boasted that given a suitable place and a lever he could move the earth. Laplace was even more ambitious and in his well known dictum said in effect *Give me the initial positions and velocities of all the particles in the Universe and I will predict the future forever.* Such a statement ignores the question of the *precision* to which the *initial* positions and velocities must be known. This may be illustrated by the following example. A perfectly elastic ball moves with a constant speed v on a frictionless surface between two infinitely rigid walls placed a distant L apart. Suppose the ball starts at a time $t_0 = 0$ from a position $x_0 = 0$ with constant speed v towards the second wall as illustrated below.



Can we predict the whereabouts of the ball between the walls at some later time t? This looks easy! The

[8] The separation of charges plays a key role in many human activities. A golfer propelling a golf ball by hitting it with a club involves *electrical forces* - gravity enters only in the subsequent motion of the struck ball

[9] In the 1950's R. A. Lyttleton suggested that the expansion of the universe could be explained if there was a slight difference in the charge on the proton from that on the electron. Subsequent precise measurements of the charge ratio have ruled out that possibility.

distance x travelled after t seconds is

$$x = vt \tag{1}$$

The ball will return to its starting position each time it has covered a distance $2L$. The number n_p of complete periods covered in time t will be the integer part of

$$\frac{vt}{2L}.$$

The remainder r

$$r = \left(\frac{vt}{2L} - n_p\right) \tag{2}$$

will be $0 < r < 1$. Thus evidently we know with certainty that after t seconds if $0 < r \leq 0.5$ then the ball is a distance $2Lr$ to the right of its initial position and moving from left-to-right whereas if $0.5 < r \leq 1$ the ball is moving from right-to-left and is a distance $2L(1-r)$ from its initial position. Is this really the case?

We have *assumed* that the initial values of x_0, t_0 and v were given with *infinite precision*, but even in classical physics such a precision is unattainable. It is not a question of simply doing a better experiment. The above system is most sensitive [10]to an error δv in the initial speed v as is evident from Eq. (1). The error δx in the position x is essentially given by

$$\delta x = \delta v t$$

and increases linearly with t. If $\delta x \geq L$ then we cannot say where the ball is between the walls. No matter how precise we measure v and how small we reduce the error δv after a time

$$t \sim \frac{L}{\delta v}$$

the only statement we can make is that the ball is somewhere between the walls. The above example constitutes a highly idealised model but illustrates a number of important points in physics which are often overlooked:-

1. In physics we are always dealing with *models* of real situations.
2. Physics is not just sets of mathematical expressions, the variables in these expressions relate experimentally measured quantities which carry with them uncertainties of measurement.
3. In assessing a model we must consider what to put in the model and what should be excluded.[11]
4. Physics involve both objective and subjective features. It is usually a mistake to overemphasise one or the other. [12]
5. Problems of predictability and initial conditions [13] exist in classical mechanics and persist into quantum mechanics where even the concept of a well-defined trajectory is lost.

■ 1.10 Symmetry and the classical one-dimensional harmonic oscillator

The classical one-dimensional harmonic oscillator gives a good example of the use of symmetry consider-

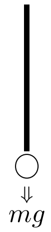
[10] In addition small errors in the angular alignment of the ball's trajectory will eventually become serious as the lapsed time increases.

[11] In the above case we left out such things as the influence of the moon, the effect of the interaction of measuring devices on the system etc. As we strive for a closer correspondence with the real world we must expand our model to encompass more features of the real world.

[12] For an interesting discussion on the question of objectivity and subjectivity in science see M. Polanyi *Personal Knowledge.*

[13] It was the objective of Hamilton-Jacobi mechanics to predict the time evolution of dynamical systems from a knowledge of the initial positions and velocities

ations in the differential equations of physical problems.[14]



$$\boxed{F = m\frac{d^2x}{dt^2} = -kx} \qquad \textit{or in suitable units} \qquad \boxed{\ddot{x} + x = 0} \tag{1}$$

A general solution is

$$x = A\cos t + B\sin t \tag{2}$$

We seek those infinitesimal transformations of x and t that leave the *form* of the equation of motion (1) invariant.

Consider a transformation that carries a point (x, t) into a point (x', t') such that

$$x' = \Phi(x, t, a_0 + \delta a) \qquad t' = \Psi(x, t, a_0 + \delta a) \tag{3}$$

where for the identity transformation $x = \Phi(x, t, a_0)$ and $t = \Psi(x, t, a_0)$. The infinitesimal change in x and t due to the infinitesimal variation δa of the parameter a is given by

$$\delta x = \xi\delta a \qquad \delta t = \eta\delta a \tag{4}$$

where

$$\xi = \xi(x, t) = (\partial\Phi/a)_{a_0} \qquad \eta = \eta(x, t) = (\partial\Psi/a)_{a_0} \tag{5}$$

If $f(x, t)$ is an analytic function of x, t then under an infinitesimal transformation

$$\delta f = Uf\delta a \tag{6}$$

where

$$U = \xi\partial/\partial x + \eta\partial/\partial t \tag{7}$$

One may show[15] that

$$U = \sum_{i=1}^{8} b_i X_i \tag{7}$$

where the b_i are constants of integration and the X_i are the following operators:-

$$X_1 = (1 + x^2)\sin t\partial/\partial x - x\cos t\partial/\partial t \tag{8a}$$
$$X_2 = (1 - x^2)\sin t\partial/\partial x + x\cos t\partial/\partial t \tag{8b}$$
$$X_3 = (1 + x^2)\cos t\partial/\partial x + x\sin t\partial/\partial t \tag{8c}$$
$$X_4 = (1 - x^2)\cos t\partial/\partial x - x\sin t\partial/\partial t \tag{8d}$$
$$X_5 = \partial/\partial t \tag{8e}$$
$$X_6 = x\partial/\partial x \tag{8f}$$
$$X_7 = x\cos 2t\partial/\partial x + \sin 2t\partial/\partial t \tag{8g}$$
$$X_8 = -x\sin 2t\partial/\partial x + \cos 2t\partial/\partial t \tag{8h}$$

The above set of eight operators close under commutation in the sense that

$$[X_i, X_j] = c_{ij}^k X_k \tag{9}$$

[14] For a detailed account see the book G. W. Bluman and S. Kumei, *Lie Symmetries of Differential Equations* and recent papers by G. W. Bluman and G. J. Reid.

[15] see C. E. Wulfman and B. G. Wybourne, *The Lie group of Newton's and Lagrange's equations for the harmonic oscillator*, J. Phys. A: Math.Gen. **9** 507-18 (1976)

where the c_{ij}^k are the so-called *structure constants* of a Lie algebra A_2. Indeed one may show that it is the Lie algebra associated with the global Lie group of Newton's or Lagrange's equation for the oscillator $SL(3, R)$.

Each of the operators X_i has a physical interpretation. Thus X_5 is the time translation generator $\partial/\partial t$ and leads to the observation that if $f(t)$ is any solution of the equation of motion then

$$f(t-\pi) = f(t+\pi) \tag{10}$$

The motion is cyclic with a period 2π. In so far as the oscillator is concerned, $t + 2\pi = t$, which is of course why oscillators are used as clocks!

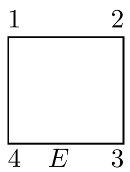
The operator X_6 shows that the equation of motion of the oscillator is unchanged under a spatial change of scale - viewing the oscillator through a magnifying glass does not change the equation of motion of the oscillator! The other operators admit a more complex interpretation associated with so-called conformal transformations. If we used a lens to project the image of the oscillator onto a curved screen we would see motion that could still be described by an equation of motion of the same form as originally but in conformally transformed variables. For example, the operator X_8 would correspond to transforming to the variables

$$x'(x,t,a) = \frac{xe^a \mathit{cosec}\ t}{\sqrt{1 + e^{4a}\cot\ t}} \qquad t'(x,t,a) = \cot^{-1}(e^{2a}\cot\ t) \tag{11}$$

■ 1.11 Symmetry of a square

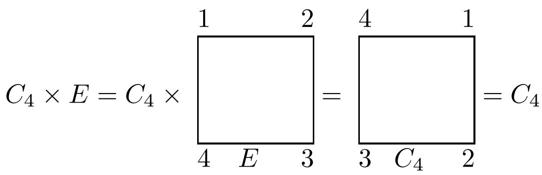
■ Rotational point symmetry

What can we say about the symmetry of an unadorned square? Let us assume that our square is completely unmarked and is lying on a horizontal plane. We wish to carry out *point* symmetry transformations that carry the square into a position indistinguishable from its initial position and which leave the centre point of the square fixed.[16] As an aid to visualising the symmetry transformations let us label the corners of the square with the integers $1, 2, 3, 4$ as below:-



We have used the letter E to indicate the symmetry transformation that amounts to doing nothing - the so-called *identity transformation.*

Performing a *clockwise* rotation C_4, about the centre point of the square, through $\frac{\pi}{2} = 90^o$ puts the square into a position indistinguishable from its initial position as can be seen below:-



The *inverse* transformation, C_4^{-1} corresponds to performing a *counterclockwise* rotation through 90^o, which would be equivalent to producing a clockwise rotation through 270^o. Clearly we must have

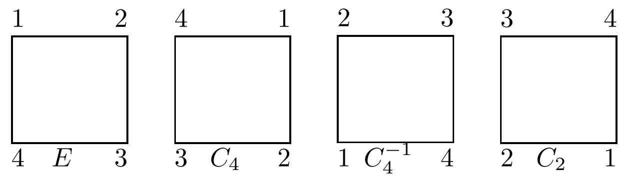
$$C_4 \times C_4^{-1} = C_4^{-1} \times C_4 = E$$

However, do not rush to the false conclusion that the successive application of two symmetry transformations is commutative - we shall shortly produce a counter example!

Performing the symmetry operation C_4 twice corresponds to rotating the square through 180^o. The same effect would be produced by performing the symmetry operation C_4^{-1} twice or a single rotation, C_2, through $\pm 180^o$. The operation C_2 is *self-inverse.* The complete set of rotational point symmetry

[16] Keeping the centre point fixed removes translational invariance from our considerations.

transformations of the square may be designated by the set of four operators (E, C_4, C_4^{-1}, C_2) and their action may be visualised as below:-



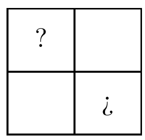
These four operators describe the elements of a *group* $\mathcal{G}$ in the sense that:-

1. There exists an identity element E.
2. There exists a law of combination of the elements g_i such that if $g_i, g_j \in \mathcal{G}$ then so is the element $g_i g_j$ (The *closure* property).
3. For every element $g_i \in \mathcal{G}$ there exists an inverse element g_i^{-1} such that $g_i^{-1} \in \mathcal{G}$ and $g_i g_i^{-1} = E$.

We may construct a *multiplication* table, *a Cayley table*, that summarises the combination of the group elements g_i. Thus for the four symmetry operators (E, C_4, C_4^{-1}, C_2) we have the Cayley table:-

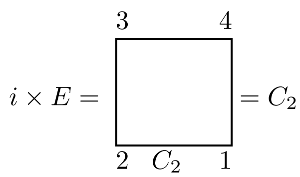
	E	C_4	C_4^{-1}	C_2
E	E	C_4	C_4^{-1}	C_2
C_4	C_4	C_2	E	C_4^{-1}
C_4^{-1}	C_4^{-1}	E	C_2	C_4
C_2	C_2	C_4^{-1}	C_4	E

Notice that the table indicates the existence of a *subgroup* $\mathcal{H} \in \mathcal{G}$ involving just the two elements E, C_2. This would be the rotational point symmetry group of an object like:-



■ **Inversion symmetry**

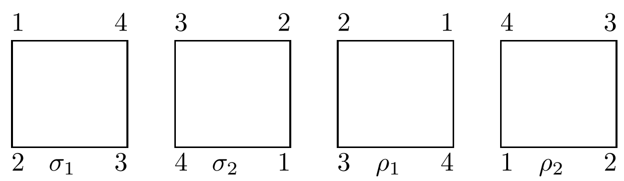
Are there other symmetry operations that we can perform on the square? Clearly there is symmetry with respect to an *inversion* i through the centre of the square but



Thus in two dimensions an inversion is equivalent to a point symmetry rotation, this is certainly *not* the case for higher dimensions.

■ **Reflection symmetry**

A larger symmetry group for the square can be obtained by considering reflections perpendicular to the plane of the square. By considering their action on the identity element E we see there are four distinct reflection symmetry operators which we shall label as $(\sigma_1, \sigma_2, \rho_1, \rho_2)$ leading to:-



The reflection operators are each self-inverse. However, they do not necessarily commute. For example, $\sigma_1 \times \rho_1 = C_4^{-1}$ whereas $\rho_1 \times \sigma_1 = C_4$. The complete Cayley table can be constructed by looking at the products of pairs of group elements to yield:-

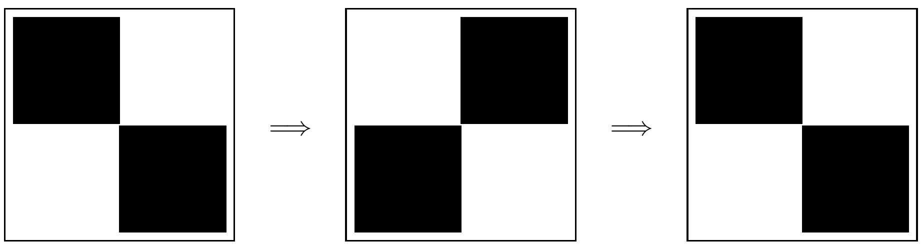
	E	C_4	C_4^{-1}	C_2	σ_1	σ_2	ρ_1	ρ_2
E	E	C_4	C_4^{-1}	C_2	σ_1	σ_2	ρ_1	ρ_2
C_4	C_4	C_2	E	C_4^{-1}	ρ_1	ρ_2	σ_2	σ_1
C_4^{-1}	C_4^{-1}	E	C_2	C_4	ρ_2	ρ_1	σ_1	σ_2
C_2	C_2	C_4^{-1}	C_4	E	σ_2	σ_1	ρ_2	ρ_1
σ_1	σ_1	ρ_2	ρ_1	σ_2	E	C_2	C_4^{-1}	C_4
σ_2	σ_2	ρ_1	ρ_2	σ_1	C_2	E	C_4	C_4^{-1}
ρ_1	ρ_1	σ_1	σ_2	ρ_2	C_4	C_4^{-1}	E	C_2
ρ_2	ρ_2	σ_2	σ_1	ρ_1	C_4^{-1}	C_4	C_2	E

Inspection of the Cayley table permits a number of interesting observations:-

1. The rotations (E, C_4, C_4^{-1}, C_2) form a subgroup.
2. The product of two reflections is equivalent to a rotation.
3. The product of a rotation and a reflection is equivalent to a reflection.
4. In a given row or column of the table every group element occurs once and only once.

■ Black and white symmetry

In the textile industry there exists a symmetry operation known as the *counterchange* where the black parts of a pattern are changed to white and the white parts to black. This clearly breaks the symmetry but a combination of a rotation *and* the counterchange operation can restore the symmetry leading to the so-called *black and white* groups[17]. The combination of a rotation followed by a counterchange can be noted in the example below:-



rotation *counterchange*

I will not pursue this subject further in these lectures save to emphasise the need for imaginative thinking to dream up new structures.

[17] A fascinating series of four articles by H. J. Woods, working in the Textile Physics section of Leeds University, appeared in the 1930's in the British Journal of the Textile Institute. Therein Woods classified all the black and white groups of the plane as well as considering the symmetry groups of braids. One can also consider polychromatic groups. The subject was taken up by A. V. Shubnikov and the black and white groups became known as Shubnikov groups. It is interesting to recall that Shubnikov was a member of a Soviet delegation that visited Leeds in the 1930's. Shubnikov, a crystallographer, failed to see the physical significance of the black and white groups. It was L. Landau who noted the equivalence of the counterchange operation and spin flipping and hence their extensive application in magnetism.

The Application of Symmetry Concepts
to
Physical Problems©

Lecture Two

You boil it in sawdust: you salt it in glue;
You condense it with locusts and tape
Still keeping one principal object in view
To preserve its symmetrical shape.
— Lewis Carroll *The Hunting of the Snark*

2.1 Introduction

In this lecture I want to introduce you to the role of symmetry in the description of particles and isospin as an approximate symmetry in nuclear and particle physics.

2.2 The forces of physics

The first attempt to unify the forces of physics was Maxwell's development of electromagnetic theory which unified the seemingly separate electric and magnetic forces into a single coherent theory - electromagnetism. At that time the only other known force was Newton's gravitational force. Both forces are *long range* forces. The electromagnetic force arises from spin 1 photon exchange while the gravitational force is believed to be associated with spin 2 graviton exchange. As we have noted earlier the electromagnetic force is vastly stronger than the gravitational force ($F_{em} \sim 10^{35} F_G$). There is also the important difference that electromagnetic forces may be attractive or repulsive whereas the gravitational force appears to be purely attractive. Thus the gravitational force accumulates as matter comes together whereas the neutrality of matter conceals the much stronger electromagnetic forces.

The discovery of radioactivity and subsequently $\beta-$decay eventually led to the recognition of a third force - the *weak* force. The weak force is a *short range* force (range $\sim 10^{-15}m$) mediated by the exchange of the vector bosons $W^{\pm}, Z^0$. The electro-weak theory developed by Ward, Glashow, Weinberg, Salam and others in the late 1960's led to a prediction of the masses of the vector bosons and later at CERN to their discovery at $m_{W^{\pm}} = 80.22GeV$ and $m_{Z^0} = 91.173GeV$. The electro-weak theory gives a unified theory of electromagnetic and weak forces.

The study of nuclear reactions led to the recognition of a fourth force - the *strong* force. The strong force is a short range force felt only over nuclear distances. Yukawa, in the 1930's, attributed the strong force to the exchange of a meson that was subsequently identified with the pions $\pi^{\pm}, \pi^0$. With the development of the quark model in the early 1960's the pions were regarded as composite particles comprising quark-antiquark pairs. The forces between quarks are believed to be associated with the exchange of an octet of particles known as *gluons* which are evidently massless particles that engage in strong short range interactions with the observed nucleon- nucleon interactions being analogous to the van der Waals forces of molecules.

2.3 Lifetimes and interactions

The strong, electromagnetic and weak interactions are associated with very different lifetimes. Typical lifetimes for decays by strong interactions are $\sim 10^{-23}s$. A particle travelling at the speed of light would travel a mere $3 \times 10^{-15}m$ which is of the order of a nuclear diameter. Particles decaying via the weak interation have typical lifetimes of $\sim 10^{-10}s$[1]. Electromagnetic decays are usually shorter than those associated with the weak interaction. Thus the $\pi^{\pm}$ pions decay via the weak interaction with a mean-life of $2.6 \times 10^{-8}s$ whereas the neutral pion π^0 decays via the electromagnetic interaction with the much shorter mean-life of $8.4 \times 10^{-17}s$. In the former case the predominant decay is

$$\pi^+ \to \mu^+ + \nu_\mu$$

whereas in the latter case

$$\pi^0 \to 2\gamma$$

[1] The notable exception is the neutron n^0 which in free space decays with a mean-life of 889.1 s. Within a nucleus the neutron is as stable as the proton, a consequence of the Pauli exclusion principle.

In general it is found that

1. *All reactions involving photons (γ) are electromagnetic.*
2. *All reactions involving neutrinos (ν) are weak.*
3. *All reactions involving electrons ($e^{\pm}$) or muons ($\mu^{\pm}$) are electromagnetic or weak.*

Thus the decay

$$\Lambda^0 \to p^+ + e^- + \nu_e$$

proceeds by the weak interaction.

The decay

$$\Lambda^0 \to p^+ + \pi^0$$

involves no neutrino but the Λ^0 has a mean-life $\sim 2.6 \times 10^{-10}s$ leading to the conclusion that it is sometimes possible to have decay via the weak interaction without neutrino production.

■ 2.4 Bosons and Fermions

The particles we commonly encounter in physics can be divided into two classes *bosons* and *fermions*. Bosons are associated with *integer* spin, examples being photons, gluons and the weak interaction bosons Z^0 and $W^{\pm}$. Fermions are associated with *half-integer* spin, examples being electrons, neutrinos and quarks. Bosons establish the *interactions* between fermions. Thus the *photon*, a massless spin 1 particle, is the exchange particle associated with electromagnetic interactions. In most of atomic and molecular physics we can restrict our attention to quantum electrodynamics (QED). The weak interactions manifest themselves in atomic and molecular physics in very small parity violations. Bosons and fermions obey different statistics, namely Bose-Einstein and Fermi-Dirac, respectively. That requires us to construct totally symmetric wavefunctions for many-boson systems and totally antisymmetric wavefunctions for many-fermion systems.

■ 2.5 Permutational symmetry

Bosons and fermions differ with respect to their behaviour under an interchange of their position, or equivalently with respect to a rotation through 2π or 360^o. We shall designate the wavefunction for a single fermion or boson as $\phi(\alpha)$ where α is an appropriate set of single particle quantum numbers associated with some single particle solution of , for example, some central field potential. Thus for a hydrogen atom we might use $\alpha = \{ns\ell m_s m_\ell\}$ or $\alpha = \{ns\ell j m_j\}$.

A $N-$particle system will involve $N-$single particle wavefunctions ($\phi_i \quad i = 1, 2, \ldots, N$) and $N-$sets of single particle quantum numbers ($\alpha_k \quad k = 1, 2, \ldots, N$). The wavefunction, Ψ, for the $N-$particle system will be such that

$$\Psi = \Psi(\phi_1, \phi_2, \ldots, \phi_N) \tag{2.1}$$

For a two-particle system we could write

$$\Psi(\phi_1, \phi_2) = \frac{1}{\sqrt{2}}\{\phi_1(\alpha_1)\phi_2(\alpha_2) \pm \phi_1(\alpha_2)\phi_2(\alpha_1)\} \tag{2.2}$$

The *positive* sign corresponds to a *symmetric* wavefunction and the *minus* sign corresponds to an *antisymmetric* wavefunction. Note that we have permuted the quantum numbers with respect to the coordinates of the particles. The wavefunction of a pair of fermions, unlike a pair of bosons, undergoes a change of sign. If $\alpha_1 = \alpha_2$ then for identical fermions Eq.(2.2) vanishes though not for bosons. That is consistent with the Pauli exclusion principle for identical fermions.

Thus permutational symmetry, required by the indistinguishability of identical particles, leads for $N-$fermions to the construction of of *determinantal states* to give totally *antisymmetric* states while for $N-$bosons to the construction of *permanental states* to give totally *symmetric* states. Hence for an $N-$fermion system we have the totally antisymmetric wavefunction

$$\Psi(\phi_1, \phi_2, \ldots, \phi_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(\alpha_1) & \phi_1(\alpha_2) & \ldots & \phi_1(\alpha_N) \\ \phi_2(\alpha_1) & \phi_2(\alpha_2) & \ldots & \phi_2(\alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(\alpha_1) & \phi_N(\alpha_2) & \ldots & \phi_N(\alpha_N) \end{vmatrix}^{\{1^N\}} \tag{2.3}$$

In $LS-$coupling basis we use $\alpha = \{ns\ell m_s m_\ell\}$ whereas in $jj-$coupling we would use $\alpha = \{ns\ell jm_j\}$. The information content of the determinantal state may be fully specified by the abbreviated form

$$\{\alpha_1\alpha_2\dots\alpha_N\} \tag{2.4}$$

In the case of bosons we are required to construct permanental states to yield totally symmetric wave-functions,

$$\Psi(\phi_1, \phi_2, \dots, \phi_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_1(\alpha_1) & \phi_1(\alpha_2) & \dots & \phi_1(\alpha_N) \\ \phi_2(\alpha_1) & \phi_2(\alpha_2) & \dots & \phi_2(\alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(\alpha_1) & \phi_N(\alpha_2) & \dots & \phi_N(\alpha_N) \end{vmatrix}^{\{N\}} \tag{2.5}$$

The information content of the permanental state may be fully specified by the abbreviated form

$$[\alpha_1\alpha_2\dots\alpha_N] \tag{2.6}$$

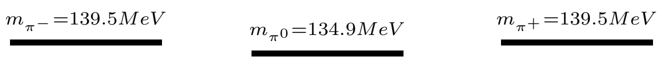
■ 2.7 Classification of particles

Faced with a diversity of interactions and particles it is natural to attempt to give a systematic order to their description. Two broad categories immediately suggest themselves, *bosons* and *fermions*. That classifies particles according to their statistics. A somewhat finer classification comes by recognising that the particles $e^\pm, \mu^\pm, \nu_e, \nu_\mu$ and the photon γ do not engage in strong interactions. The photon is associated with electromagnetic interactions alone so can be put into a class of its own. The photon is a massless spin one particle and hence is a boson. The remaining particles that do not experience the strong interaction are known as *leptons* and are all fermions with spin $= \frac{1}{2}$.

Particles that engage in strong interactions are called *hadrons* Hadrons with integer spin are bosons and are called *mesons* whereas those with half-integer spin are fermions and are called *baryons*.

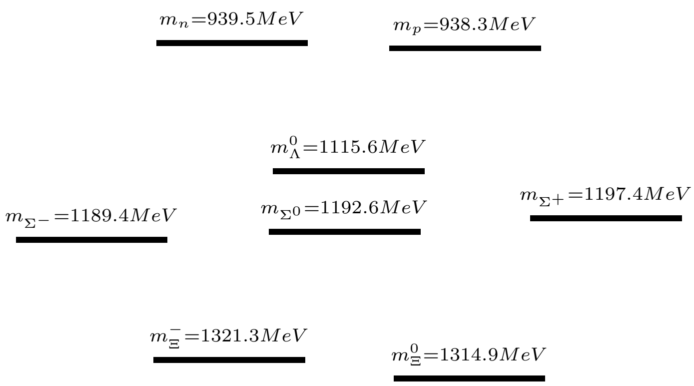
■ 2.8 Mass plots for baryons and mesons

One cannot be but struck by the remarkably small difference in the mass of the proton and neutron $(\sim 1.29MeV)$[2]. They are the same to better than 1%. It is interesting to plot the masses of the low mass mesons and baryons as below

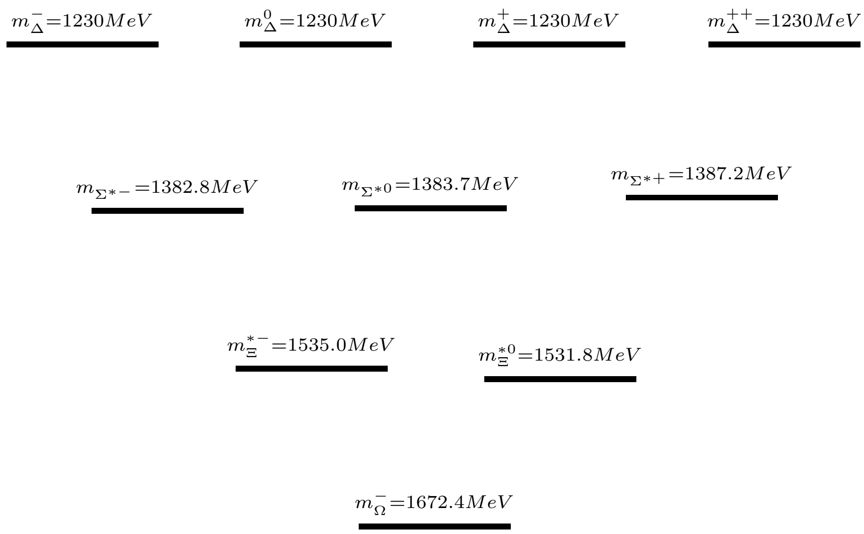


The $\pi-$mesons with $J^p = 0^-$

2 Throughout these lectures I shall put $c = 1$ and then express the masses of particles simply in MeV rather than MeV/c^2.



Baryons with $J^p = \frac{1}{2}^+$



Baryons with $J^p = \frac{3}{2}^+$

Perhaps the most striking feature of the above plots is the appearance of nearly degenerate levels similar to the multiplets of atomic energy levels deduced from atomic spectra. It appears that we are encountering an example of an approximate symmetry. Indeed it appears that if we could switch off the electromagnetic interaction we would attain actual degeneracy. Perhaps even more startling is the manner in which the 'levels' of the eight baryons with $J^p = \frac{1}{2}^+$ form an *octet* and those with $J^p = \frac{3}{2}^+$ form a *decuplet.* In the latter case note how the successive levels are almost equally spaced in mass. Why?

In the case of the pions note that $m_{\pi^+} = m_{\pi^-}$ as expected for a particle-antiparticle pair whereas the π^0 is less massive.

■ 2.9 Isospin multiplets

Heisenberg suggested that the proton and neutron could be viewed as two states of a single particle, the *nucleon.* By analogy with spin, the nucleon was said to be a particle with *isospin* $I = \frac{1}{2}$. The projection I_3 on the third axis has the values $\pm\frac{1}{2}$. We *choose* to associate the proton with the isospin projection $I_3 = +\frac{1}{2}$ which then *requires* that we associate the neutron with the isospin projection $I_3 = -\frac{1}{2}$.

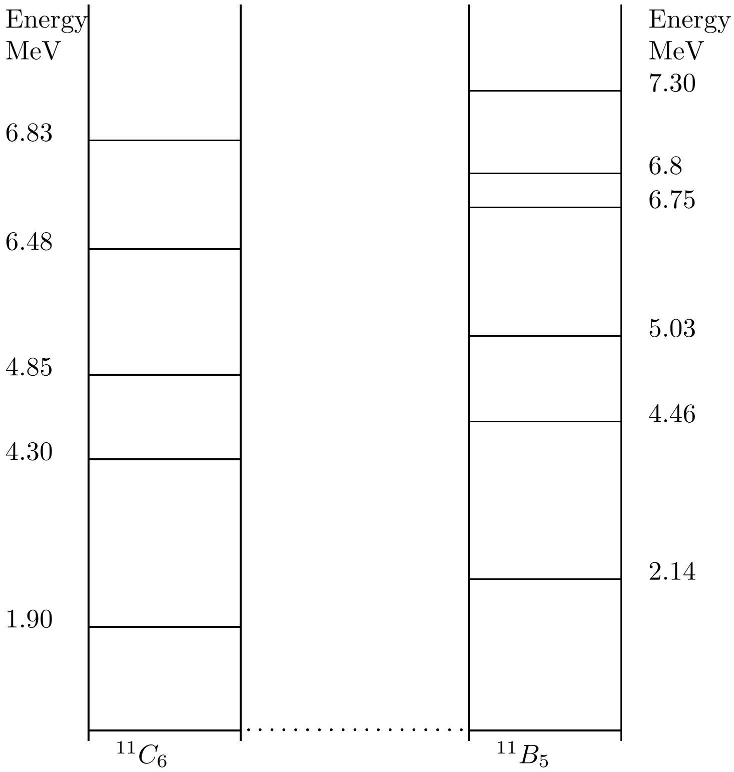
Having made this *choice* [3] we determine the isospin and its projection of all other hadrons in relationship to the choice made for the proton. We shall return to this point shortly.

We note that we can write a *charge equation* for the nucleon

$$Q_N = I_3 + \frac{1}{2} \tag{2.7}$$

where we take the absolute magintude of e as unity. We note that the neutron, while uncharged has a nuclear magnetic moment $\sim -\frac{2}{3}$ that of the proton. This demands an explanation!

There is convincing experimental evidence for approximate isospin conservation for strong interactions. If the strong interaction were charge independent then we would expect mirror nuclei [4] to exhibit almost identical nuclear energy level structure with any difference arising from the effect of the much weaker Coulomb force. We show below the low energy levels of the mirror pair $^{11}C_6$ and $^{11}B_5$ where the similarity in the energy level structure should be self-evident.



Energy levels of the mirror nuclei $^{11}C_6, ^{11}B_5$

Such a structure hints at the approximate equality of $n-p \sim n-n \sim p-p$ forces, or equivalently of the charge-independence of the strong nuclear force. Nucleon-nucleon scattering experiments supply further evidence. Thus we reach the conclusion that

Isospin I is conserved in strong interactions. An *isospin multiplet* is a set of $2I+1$ hadrons each

[3] I emphasise the word *choice* because we are free to make either choice. The important point is that having made an assignment of the isospin I and its projection I_3 to the proton we must then make *all* assignments of I, I_3 for *all* other particles relative to the proton. This is precisely the same as assigning negative charge $-e$ to the electron. The charge of all other particles are then measured relative to that of the electron. We could have equally well established a convention that gave the charge of the electron as $+e$. It is important in physics to clearly identify where a freedom of choice exists. The freedom of choice usually implies a conservation law. In the case of the electron charge it leads to charge conservation which to date appears rigorous.

[4] *mirror nuclei* are pairs of nuclei that have the same total number of nucleons A but with the number of protons Z of one being equal to the number of neutrons $A-Z$ of the other. Examples are $^{11}C_6$ and $^{11}B_5$ or $^3H_1$ and $^3He_2$.

of which is labelled by the quantum numbers I, I_3 and exhibiting an *approximate* $2I+1$-fold degeneracy.

■ 2.10 Isospin for pions

We have already noted that the pions appear to form a mass triplet. This suggests that they are members of an isospin triplet (i.e. $I=1$). Convincing evidence comes from nucleon-nucleon scattering experiments. For example, the reaction

$$p+n \to n+n+\pi^+$$

conserves isospin if the π^+ has $I=0$ or $I=1$, however the charge- independence of the strong interactions requires that the projections I_3 have the same sum on both sides of the reaction and hence we expect $I_3=1$ for the π^+ and hence must have $I=1$. Similar experiments lead to the conviction that the three pions are different charge states of the pion which together form an isospin triplet.

■ 2.11 Isospin transformations

We may develop a two-component isospin formalism for describing the nucleon in almost the same way as for two-component electron spin. We shall use the letters N, p, n for the nucleon, proton and neutron respectively. and introduce the two-component spinors

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.8}$$

and for the nucleon wave function we write[5]

$$\psi_N = \begin{pmatrix} a \\ b \end{pmatrix} = ap + bn \tag{2.9}$$

Recall the Pauli spin matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}$$

We readily find that

$$\tau_3 p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p$$

and

$$\tau_3 n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -n$$

The charge equation (Eq. 2.7) requires that

$$I_3 p = \frac{1}{2} p \quad \text{and} \quad I_3 n = -\frac{1}{2} n$$

and hence we make the identification

$$I_3 = \frac{1}{2}\tau_3 \tag{2.11}$$

and similarly

$$I_1 = \frac{1}{2}\tau_1 \quad \text{and} \quad I_2 = \frac{1}{2}\tau_2 \tag{2.12}$$

Let us now introduce the *isospin ladder operators* by defining

$$I_+ = I_1 + iI_2 = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.13}$$

and

$$I_- = I_1 - iI_2 = \frac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.14}$$

Acting on the p and n states we find

$$I_+ p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \tag{2.15}$$

[5] The complete wave function will be of the form $\psi = \psi_{space}\psi_{spin}\psi_{isospin}$

$$I_+n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p \tag{2.16}$$

$$I_+p = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = n \tag{2.17}$$

$$I_+n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{2.18}$$

The operators $I_\pm, I_3$ form the elements of the *Lie algebra* associated with the group SU_2.

■ 2.12 Isospin for pion-nucleon systems

The pions form the components of an isospin triplet whereas the nucleon forms the components of an isospin doublet. We may designate an isospin state by $|I, M\rangle$. Thus

$$|1,1\rangle \sim \pi^+, \quad |1,0\rangle \sim \pi^0, \quad |1,-1\rangle \sim \pi^-$$

Recalling the properties of angular momentum ladder operators we have

$$I_\pm|I, M\rangle = \sqrt{I(I+1) - M(M \pm 1)}|I, M \pm 1\rangle \tag{2.19}$$

and

$$I_3|I, M\rangle = M|I, M\rangle \tag{2.20}$$

Thus

$$I_+\pi^- = \sqrt{2}\pi^0, \quad I_+\pi^0 = \sqrt{2}\pi^+, \quad I_-\pi^+ = \sqrt{2}\pi^0, \quad I_-\pi^0 = \sqrt{2}\pi^- \tag{2.21}$$

and

$$I_-p^+ = n^0, \quad I_+n^0 = p^+ \tag{2.22}$$

The usual properties of angular momentum addition hold. Thus

$$\mathbf{I}_1 + \mathbf{I}_2 = (I_1 + I_2) + (I_1 + I_2 - 1) + \ldots + |I_1 - I_2| \tag{2.23}$$

and

$$M_1 + M_2 = M \tag{2.24}$$

Combining a nucleon, N, with a pion amounts, in isospin space combining $I_N = \frac{1}{2}$ with $I_\pi = 1$ and hence leads to isospin multiplets with $I = \frac{1}{2}, \frac{3}{2}$. The fully stretched state $|\frac{3}{2}, \frac{3}{2}\rangle$ must correspond to π^+p and hence we make the assignment

$$|\frac{3}{2}, \frac{3}{2}\rangle = \pi^+p \tag{2.25}$$

From (2.19)

$$I_-|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{3}|\frac{3}{2}, \frac{1}{2}\rangle \tag{2.26}$$

but $I_- = I_-^\pi + I_-^p$ and hence

$$\begin{aligned} I_-|\pi^+p\rangle &= I_-^\pi|\pi^+p\rangle + I_-^p|\pi^+p\rangle \\ &= \sqrt{2}|\pi^0p\rangle + |\pi^+n\rangle \end{aligned} \tag{2.27}$$

Comparison of (2.26) with (2.27) leads to

$$|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\pi^0p\rangle + \sqrt{\frac{1}{3}}|\pi^+n\rangle \tag{2.28}$$

Further application of the step-down isospin operator leads to:-

$$|\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\pi^-p\rangle + \sqrt{\frac{2}{3}}|\pi^0n\rangle \tag{2.29}$$

and

$$|\frac{3}{2}, -\frac{3}{2}\rangle = |\pi^-n\rangle \tag{2.30}$$

Orthogonality then leads to

$$|\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\pi^0p\rangle - \sqrt{\frac{2}{3}}|\pi^+n\rangle \tag{2.31}$$

Let us now consider a practical application of this last result to the proton-deuteron reaction.

■ 2.13 A practical application of isospin symmetry

Consider the interaction of a proton with a deuteron. The deuteron has a ${}^3S_1$ ground state and thus the spin space is symmetric. With respect to isospin the deuteron is an isospin singlet with $I = M = 0$. Since the proton has isospin $|\frac{1}{2}, \frac{1}{2}\rangle$ and we assume strong interactions conserve isospin the product of the reaction must be a member of an isospin doublet. Thus we could anticipate two outcomes

$$p + d \rightarrow \pi^0 +^3 He_2 \tag{2.32}$$

and

$$p + d \rightarrow \pi^+ +^3 H_1 \tag{2.33}$$

with the mirror nuclei $({}^3He_2, {}^3H_1)$ forming an isospin doublet. We can write by direct analogy with (2.31)

$$|\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\pi^0\ {}^3He_2\rangle - \sqrt{\frac{2}{3}}|\pi^+\ {}^3H_1\rangle \tag{2.34}$$

from which it follows that

$$\frac{amplitude(p + d \rightarrow \pi^+ +^3 H_1)}{amplitude(p + d \rightarrow \pi^0 +^3 He_2)} = -\sqrt{2} \tag{2.35}$$

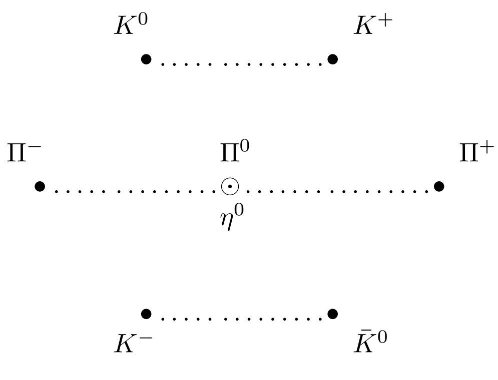
But the cross-section is proportional to the square of the amplitude and hence if isospin is conserved we predict the ratio for the cross-sections for the two reactions is

$$\frac{\sigma(p + d \rightarrow \pi^+ +^3 H_1)}{\sigma(p + d \rightarrow \pi^0 +^3 He_2)} = 2 \tag{2.36}$$

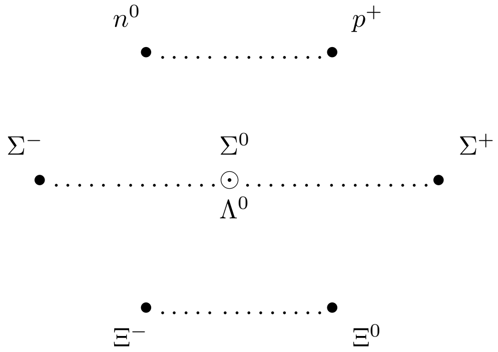
which agrees with experiment to better than 10%.

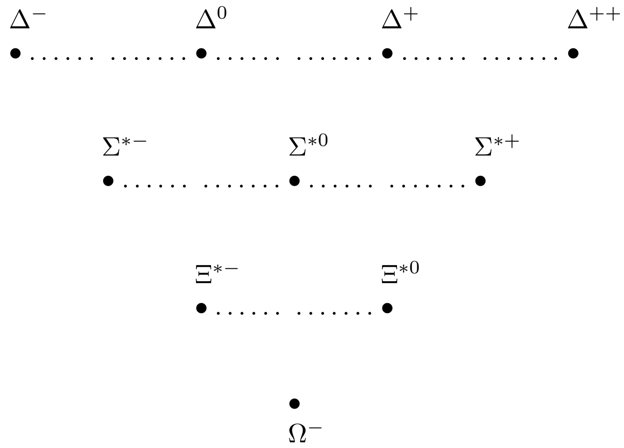
■ 2.14 Isospin multiplets and low mass baryons and mesons

Remarkably the low mass baryons and mesons may be readily organised into isospin multiplets as shown below:-



Meson Octet $J^p = 0^-$





Baryon Octet $J^p = \frac{1}{2}^+$ Baryon Decuplet $J^p = \frac{3}{2}^+$

At this stage we may well ask "Are there other quantum numbers that distinguish between the different isospin multiplets?" The appearance of octets and decuplets hints at the existence of a higher symmetry. The anti-baryons arrange themselves also in octets and decuplets while the mesons, particles and anti-particles, occur in common octets and singlets but *not* in decuplets.

In this lecture we have encountered the four forces of nature. We noted that Maxwell had unified the separate theories of electricity and magnetism into a single coherent theory known as electromagnetism and in our own time electromagnetism and the weak interaction have been unified in the electro-weak theory. Will it be possible to produce a grand unified theory that unifies the electro-weak and strong interaction? Perhaps in some very distant past these forces were on an equal footing and as the universe cooled symmetry breaking occurred with the different forces taking on the characteristics we know them by today. Finally, can a super unified theory including gravity be constructed? For the moment these must remain as unanswered questions.[6]

Before continuing our exploration of particles we should first examine the properties of the symmetries associated with Charge conjugation, $\mathcal{C}$, Parity, $\mathcal{P}$, and Time reversal, $\mathcal{T}$, the subject of the next lecture.

[6] Wigner has suggested that perhaps the answers to these ultimate questions lie beyond human possibilities. A mouse is unlikely to ever reach even modest conclusions about the nature of the universe. Perhaps it is a distinguishing feature of humankind that we ask such questions. Pauli's reaction to suggestions of unified theories was "What God has put asunder no man shall unite".

The Application of Symmetry Concepts
to
Physical Problems©

Lecture Three

For every complex question there is a simple answer
— and it's wrong.
— H. L. Mencken

3.1 Introduction

In this lecture I want to introduce three fundamental symmetries that have wide ranging implications for all areas of physics, namely Charge conjugation, $\mathcal{C}$, Parity, $\mathcal{P}$, and Time reversal, $\mathcal{T}$, symmetries. These find there ultimate expression in the CPT theorem. Before proceeding we first consider some other quantum numbers that assist in labelling the various particles.

3.2 Charge conservation

Electrons do not appear to disappear. The experimental limit for the decay

$$e \not\rightarrow \nu + \gamma$$

is $> 1.5 \times 10^{25} yr$. Within these limits we know of no exception to the statement that *charge is conserved in all reactions* and hence we may label particles by their electric charge Q. We note that the difference in the absolute charge of the electron and proton is $< 10^{-21} e$.

3.3 Baryon number $\mathcal{B}$

The proton appears to be a remarkably stable particle. Considering all possible modes of decay the mean life of the proton is $> 1.6 \times 10^{25} yr$ while for specific modes of decay the mean life of the proton is $> 10^{31} yr$. One might have expected decays of the type

$$p \not\rightarrow e^{+} + \gamma$$

which is certainly energetically possible. Likewise the decay of protons into mesons is not observed. These experimental observations strongly suggest that baryons carry a conserved quantum number, the *Baryon number B*, sometimes termed the baryonic charge. To date there is no evidence for any reactions that violate conservation of baryon number. Note anti-baryons carry the *opposite* baryon number to their baryon partner just like for the charge quantum number[1]. We make the arbitrary assignment of $\mathcal{B} = 1$ to baryons and $\mathcal{B} = 0$ for all leptons, mesons and the photon γ.

3.4 Lepton numbers

Many other reactions that satisfy all known conservation laws do not appear to occur. For example

$$e^{-} + e^{-} \not\rightarrow \pi^{-} + \pi^{-}$$

The absence of such reactions suggests the conservation of a *Lepton number L* with $\mathcal{L}_e = 1$ for the electron and $\mathcal{L}_e = 0$ for baryons, mesons and the photon. However, the reactions

$$\mu^{\pm} \not\rightarrow e^{\pm} + \gamma \quad \text{and} \quad \mu^{\pm} \not\rightarrow e^{\pm} + e^{+} + e^{-}$$

are not observed. The reactions

$$\pi^{-} \longrightarrow \mu^{-} + \bar{\nu}_{\mu} \quad \text{and} \quad \pi^{-} \longrightarrow e^{-} + \bar{\nu}_{e}$$

are observed the reactions whereas the reactions

$$\pi^{-} \not\rightarrow \mu^{-} + \nu_{\mu} \quad \text{and} \quad \pi^{-} \not\rightarrow e^{-} + \nu_{e}$$

are not observed.

[1] Our existence hints that the baryon number cannot be absolutely conserved.

Experimentally the neutrinos ν_e and ν_μ are found to be distinct particles. Furthermore the neutrino ν and the antineutrino $\bar{\nu}$ are distinct[2]. This is seen in neutron decay, the decay

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

occurs whereas the decay

$$n \not\longrightarrow \ p + e^- + \nu_e$$

does not occur.

There is one other known massive[3] lepton, the tauon τ occurs with an associated neutrino ν_τ. The reactions involving the three types of neutrinos ν_e, ν_μ and ν_τ and the corresponding massive leptons e, μ and τ are consistent with three separate lepton quantum numbers, L_e, L_μ and L_τ which assume the values given in Table 3.1. For all non-leptonic particles the lepton numbers are zero.

Lepton	L_e	L_μ	L_τ
e^-	1	0	0
e^+	-1	0	0
ν_e	1	0	0
$\bar{\nu}_e$	-1	0	0
μ^-	0	1	0
μ^+	0	-1	0
ν_μ	0	1	0
$\bar{\nu}_\mu$	0	-1	0
τ^-	0	0	1
τ^+	0	0	-1
ν_τ	0	0	1
$\bar{\nu}_\tau$	0	0	-1

Table 3.1 Lepton numbers

All experimental data to date indicates that the lepton numbers, individually and collectively, are conserved. There are three families of leptons based on the electron, muon and tauon. The masses are:-

$$m_e = 0.511MeV, \quad m_\mu = 105.66MeV, \quad m_\tau = 1784.1MeV$$

The electron appears to be stable (mean life $> 1.9 \times 10^{23} yr$) whereas the muon has a mean life of $2.2 \times 10^{-6} s$ and the tauon a mean life of $3 \times 10^{-11} s$. Remarkably, the magnetic moments of the electron and muon are identical[4] to within 1 part in 10^6 .

■ 3.5 Particles and antiparticles

Dirac's 1929 relativistic wave function for the electron admitted a 4-component spinor solution corresponding to left- and right-handed electron states together with left- and right-handed positron states, the positron being the antiparticle of the electron[5]. Experiments indicate no difference in mass between a particle and its antiparticle. Particle and antiparticles have the same spin quantum number but the values of all charge quantum numbers such as $Q, B, L_e, L_\mu, L_\tau,$ and $\mathcal{S}$ are reversed in sign[6]. Consequently

[2] Remarkably the neutrino appears only as a left-handed particle ν_L whereas the antineutrino occurs only as a right-handed particle $\bar{\nu}_R$. Technically, and we shall expand on this point later, the particle and antiparticle have opposite helicity and leads to a gross violation of parity conservation in weak interactions.

[3] Here I use the word *massive* as opposed to *massless.* The photon γ is a massless particle (certainly $m_\gamma < 3 \times 10^{-33} MeV$, the neutrino ν_e is close to, if not actually, massless (certainly $m_{\nu_e} < 8eV$). The neutrinos e, μ and τ are massive (i.e. have nonzero rest mass).

[4] The remarkable similarity of the electron and muon led I. Rabi to exclaim "Who ordered this?".

[5] Again the choice of particle and antiparticle is arbitrary. We make the choice by *assuming* we are matter and that what we call an electron is a particle. The object that annihilates the electron we take as the positron. Antimatter appears to be very rare in the universe.

[6] $\mathcal{S}$ is the *strangeness* quantum number which we shall shortly introduce.

the magnetic moments of a particle and its antiparticle are of the opposite sign. Particles whose charge quantum numbers are all zero are their own antiparticles as is the case for the neutral pion π^0 and the photon γ.[7]

■ 3.6 Parity and spatial inversion symmetry

In Lecture one we mentioned briefly inversion symmetry in regard to a square in two-dimensions. There we saw it was equivalent to a point symmetry rotation through 180^o. In three-dimensions the situation is quite different. A spatial inversion cannot be reduced to a set of rotations in 3-space. By a spatial inversion $\mathcal{I}$ we mean a symmetry transformation $\mathcal{P}$ such that

$$(\mathbf{r}, t) \xrightarrow{\mathcal{P}} (-\mathbf{r}, t) \tag{3.1}$$

The operator P is commonly referred to as the *Parity* operator [8].

■ 3.7 Parity and spherical harmonics

The spherical harmonics $Y_{\ell m}(\theta, \phi)$ assume particular importance in the theory of angular momentum wave functions. In the usual spherical coordinates we have

$$\theta \xrightarrow{\mathcal{P}} \pi - \theta, \quad \text{and} \quad \phi \xrightarrow{\mathcal{P}} \pi + \phi \tag{3.2}$$

The spherical harmonics are proportional to the Legendre polynomials

$$\sin^{|m|} \theta P_\ell^m(\cos\theta) e^{im\phi} \tag{3.3}$$

Under the parity operation we then have

$$Y_{\ell m}(\theta, \phi) \xrightarrow{\mathcal{P}} Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi) \tag{3.4}$$

and hence we conclude that under the parity operation $\mathcal{P}$ the orbital part of the wave function is *odd* or *even* as is the orbital quantum number ℓ. For an $n-$particle state the parity has the eigenvalue

$$(-1)^{\sum_{i=1}^{n} \ell_i} \tag{3.5}$$

Parity conservation amounts to asserting that the Hamiltonian commutes with the parity operator $\mathcal{P}$. Acting on a state of well-defined parity p twice gives

$$\mathcal{P}^2 |\psi\rangle = p^2 |\psi\rangle = |\psi\rangle \tag{3.6}$$

and hence we may take the eigenvalues of the parity operator $\mathcal{P}$ as ± 1. Note carefully the distinction between states and operators. The parity of angular momentum *states* can be *even*, $p = +1$ or *odd*, $p = -1$ whereas the angular momentum operator $\ell = \mathbf{r}\ \mathbf{x}\ \mathbf{p}$ is of *even* parity since under $\mathcal{P}$

$$\mathbf{r} \xrightarrow{\mathcal{P}} -\mathbf{r} \quad \text{and} \quad \mathbf{p} \xrightarrow{\mathcal{P}} -\mathbf{p} \tag{3.7}$$

---

[7] It is interesting to note that antimatter, in the form of positrons, is produced in the human body. The average human body contains about 100gm of potassium of which 0.0117% is in the form of the radioactive isotope $^{40}K_{19}$ which has a half-life of $1.3 \times 10^9 yr$ 89.3% of the decays involve the β^- process

$$^{40}K_{19} \rightarrow ^{40} Ca_{20} + e^- + \bar{\nu}_e$$

while the rest involve either the electron capture EC process

$$^{40}K_{19} \stackrel{\text{EC}}{\rightarrow} {}^{40}Ar_{18} + \nu_e$$

or the β^+ process

$$^{40}K_{19} \rightarrow ^{40} Ar_{18} + e^+ + \nu_e$$

The positron annihilates with an electron to produce two $0.511 MeV$ gammas. This latter phenomena has been used in agricultural research to determine the fat content of live pigs. Radioactive potassium is one of the principal sources of radioactive decay in the human body producing about $5000 Bq$.

[8] NB. The parity operator is very different from that of the angular momentum operator. The former is associated with multiplicative eigenvalues while the latter is associated with additive eigenvalues. The parity operator is always a discrete operator whereas the angular momentum can be a continuous operator. Angular momentum conservation arises from the assumption that there is no preferred direction in space. Spatial inversion is a less obvious property of space and indeed less fundamental.

Likewise, spin and charge are even parity operators whereas the electric field $\mathcal{E}$ is of *odd* parity. Recalling that

$$\mathcal{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathcal{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{3.8}$$

we may conclude that the magnetic field vector $\mathcal{B}$ is of *even* parity.

■ 3.8 Intrinsic parity of particles

We have deduced an orbital parity for angular momentum states but have not considered the possibility that some particles may possess an *intrinsic* parity. Suppose we have a two-particle system such that

$$\Psi_{12} = \psi_1\psi_2 \tag{3.9}$$

then if

$$\mathcal{P}\psi_1 = p_1\psi_1 \quad \text{and} \quad \mathcal{P}\psi_2 = p_2\psi_2 \tag{3.10}$$

then

$$P\Psi_{12} = p_1p_2(-1)^{\ell_{12}}\Psi_{12} \tag{3.11}$$

where ℓ_{12} is the orbital angular momentum between particles 1 and 2, and p_1 and p_2 are the intrinsic parities.

The intrinsic parity of a third particle, say Z_3, may be determined by studying a scattering process

$$X_1 + Y_2 \to X_1 + Y_2 + Z_3 \tag{3.12}$$

Then if parity is conserved

$$\Psi^*_{12}\mathcal{P}\Psi_{12} = \Psi^*_{123}\mathcal{P}\Psi_{123} \tag{3.13}$$

and

$$p_1p_2(-1)^{\ell_{12}} = p_1p_2p_3(-1)^{L_{123}}$$

and hence

$$p_3 = (-1)^{L_{123}+\ell_{12}} \tag{3.14}$$

Such a process for determining the intrinsic parity of a particle is only possible if the particle Z_3 can be created *singly*. This rules out the possibility of defining, for example, an intrinsic parity for any charged particle since due to charge conservation one cannot create a single charged particle.

A more useful concept is that of the *relative parity* of particles. Consider the process

$$X_1 + Y_2^+ \to X_1 + Z_3^+ \tag{3.15}$$

where Y and Z carry a positive charge. In that case we have for parity conservation

$$p_1p_2(-1)^{\ell_{12}} = p_1p_3(-1)^{\ell_{13}}$$

and hence the *parity* of Y (p_2) relative to that of Z (p_3) is

$$p_2 = (-1)^{\ell_{13}-\ell_{12}}p_3$$

Again we are free to make a choice and then define the relative parities of other particles. If we choose to take the proton (p), neutron (n) and lambda (Λ) as having even parity then we can make assignments to other particles by detailed analysis of reactions typified by Eq.(3.15). In general it may be shown that for all bosons the particle and antiparticle have the *same* parity whereas for fermions the particle and antiparticle are of opposite parity. This means that for bosons the particle and antiparticle may appear in the same isospin multiplet whereas for fermions the particle and antiparticle must appear in different isospin multiplets.

The photon has *negative* intrinsic parity as follows from the fact that the photon is associated with the 4−vector potential $(\mathbf{A}, \phi)$ and under a spatial inversion

$$\mathbf{A} \xrightarrow{\mathcal{P}} -\mathbf{A} \quad \text{and} \quad \phi \xrightarrow{\mathcal{P}} \phi \tag{3.16}$$

Recall that the photon is a massless spin 1 particle that can occur in both right-handed and left-handed polarisations.

The pions are massive spin 0 particles with negative intrinsic parity. This would be consistent with regarding the pions as composite particles made up of fermion-antifermion pairs.

■ 3. 9 Parity selection rules

Parity appears to be conserved in all electromagnetic and strong interactions but not in weak interactions. In almost all of solid state, atomic and molecular physics weak interactions are of no significance and *if a centre of inversion exists* the states will have well defined parity. This is certainly the case for atoms and for crystals with cubic symmetry. In the case of electric dipole transitions the relevant interaction operator is of the form $e\mathbf{r}$ and is thus an *odd* parity operator. Parity conservation for electromagnetic interactions then forbids[9] electric dipole transitions between states of the same parity. Thus, for example, in an atom it is not possible to have electric dipole transitions between states of the f^n configurations of the lanthanides or actinides. Place the corresponding ion in a crystalline environment lacking a centre of inversion and it is possible for the crystal field to mix states of opposite parity and previously forbidden transitions become allowed, albeit weakly.

The electric quadrupole is quadratic in $\mathbf{r}^2$ and z^2 and hence is a parity *even* operator and can only couple between states of the *same* parity. The magnetic dipole operator involves the angular momentum operators $\mathbf{L}$ and $\mathbf{S}$ which are of even parity so also can only couple between states of the same parity.

■ 3.10 Time reversal invariance

Under time reversal $\mathcal{T}$

$$(\mathbf{r}, t) \xrightarrow{\mathcal{T}} (\mathbf{r}, -t) \tag{3.17}$$

Consider a system with a Hamiltonian $\mathcal{H}$ and satisfying the Schrödinger equation

$$\mathcal{H}\psi(\mathbf{r}, t) = i\frac{\partial\psi(\mathbf{r}, t)}{\partial t} \tag{3.18}$$

If under time reversal $\mathcal{H} \xrightarrow{\mathcal{T}} \mathcal{H}$ for the time reversed system

$$\mathcal{H}\psi(\mathbf{r}, -t) = -i\frac{\partial\psi(\mathbf{r}, -t)}{\partial t} \tag{3.19}$$

It is apparent from inspection of Eqs. (3.18) and (3.19) that $\psi(\mathbf{r}, t)$ and $\psi(\mathbf{r}, -t)$ satisfy different equations. It is the complex conjugate wave function $\psi^*(\mathbf{r}, t)$ that satisfies (3.19). Indeed

$$\mathcal{H}^*\psi^*(\mathbf{r}, t) = -i\frac{\partial\psi^*(\mathbf{r}, t)}{\partial t} \tag{3.20}$$

There is a fundamental difference between time inversion and spatial inversion. In the latter case we can construct states that are eigenstates of the parity operator $\mathcal{P}$ and in appropriate situations there is a well defined parity quantum number $p = \pm 1$. Under time inversion $\mathcal{T}$ a wave function is changed into its complex conjugate and hence a state cannot be an eigenstate of $\mathcal{T}$ and there cannot be an associated conserved quantum number.

Under time reversal

$$\mathbf{p} \xrightarrow{\mathcal{T}} -\mathbf{p}$$

and hence the motion of objects is reversed. As a consequence the time reversal operation changes the sign of angular momentum operators and since the directions of currents are reversed so is the direction of the magnetic field $\mathbf{B}$.

■ 3.11 Charge conjugation

The operation of *Charge conjugation C* replaces a given particle by its antiparticle. Thus $\mathcal{C}$ changes the sign of all charges $(\mathcal{Q}, \mathcal{B}, \mathcal{L}, \mathcal{S})$. The electric field $\mathcal{E}$ is associated with static charges while the magnetic field $\mathcal{B}$ is associated with currents and hence

$$\mathcal{E} \xrightarrow{\mathcal{C}} -\mathcal{E} \quad \text{and} \quad \mathcal{B} \xrightarrow{\mathcal{C}} \mathcal{B} \tag{3.21}$$

We summarise the operations of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ in Table 3.2.

Table 3.2 Transformations under $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$.

[9] Note we use the word *forbids* since a selection rule tells what is forbidden but *not* what is permitted.

Quantity	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$
$\mathbf{r}$	$\mathbf{r}$	$-\mathbf{r}$	$\mathbf{r}$
t	t	t	$-t$
$\mathbf{p}$	$\mathbf{p}$	$-\ \mathbf{p}$	$-\ \mathbf{p}$
$\mathbf{L} = \mathbf{r}\ \mathbf{x}\ \mathbf{p}$	$\mathbf{L}$	$\mathbf{L}$	$-\ \mathbf{L}$
$\mathbf{S}$	$\mathbf{S}$	$\mathbf{S}$	$-\mathbf{S}$
$\mathcal{E}$	$-\mathcal{E}$	$-\mathcal{E}$	$\mathcal{E}$
$\mathcal{B}$	$-\mathcal{B}$	$\mathcal{B}$	$-\mathcal{B}$
$\mathcal{Q}$	$-\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$

■ 3.12 The $\mathcal{CPT}$ theorem

Schwinger, Lüders and Pauli have established a remarkable theorem known as the $\mathcal{CPT}$ theorem which states that any quantum field theory *compatible with special relativity* and which *assumes only local interactions* is invariant under the combined action of $\mathcal{CPT}$ including all orderings of the three operators. This means that while there may be noninvariance with with any of the individual operators there cannot be, within the assumptions of the theorem, noninvariance with the triple product. Thus if there is noninvariance with respect to $\mathcal{CP}$ then there will consequently be noninvariance with respect to $\mathcal{T}$ but not for $\mathcal{CPT}$.

As an example consider the action of $\mathcal{C}$ and $\mathcal{P}$ on the left-handed neutrino ν_L as illustrated below. Under the action of $\mathcal{P}$ the left-handed neutrino is turned into the non-existent right-handed neutrino ν_R while under charge conjugation it is turned into the non-existent left-handed antineutrino $\bar{\nu}_L$. However, under the joint action of $\mathcal{CP}$ we obtain the observed right-handed antineutrino $\bar{\nu}_R$. Since $\mathcal{CP}$ invariance is maintained in weak interactions then if the $\mathcal{CPT}$ theorem holds then weak interactions would also be time reversal $\mathcal{T}$ invariant.

$$\begin{tikzcd}
\nu_L \arrow[r, "\mathcal{P}"] \arrow[d, "\mathcal{C}"'] & \nu_R \arrow[d, "\mathcal{C}"'] \\
\bar{\nu}_L \arrow[r, "\mathcal{P}"] & \bar{\nu}_R
\end{tikzcd}$$

3.13 Time reversal invariance and Kramer's theorem

Let us now expand on the idea of time reversal to obtain Kramer's theorem that plays an important role in solid state physics. In our earlier discussion we made not explicit reference to spin and found that the action of $\mathcal{T}$ was simply to transform the Schrödinger wave function ψ into its complex conjugate ψ^*. Assuming $\mathcal{H}$ to be real then

$$\mathcal{H}\psi = E\psi \xrightarrow{\mathcal{T}} \mathcal{H}\psi^* = E\psi^* \tag{3.22}$$

The decision as to whether we obtain degeneracy or not will rest on whether the set ψ^* is, or is not, linearly independent of the set ψ for a particular energy. In carrying out the action of $\mathcal{T}$ we must remember to reverse the direction of any magnetic field in $\mathcal{H}$. This will be automatically satisfied *if the field is produced by part of the quantum system under consideration* - the reversal of momenta will automatically lead to a reversal of the fields. Suppose, however, that the field $\mathcal{B}$ is an *external* field. In that case the field must be explicitly reversed. The wave function $\psi \sim e^{im_\ell\phi}$ goes into $\psi^* \sim e^{-im_\ell\phi}$. In the absence of a magnetic field these two solutions should yield the same energy. In the case of an external magnetic field this need not be the case.

If we consider the inclusion of spin the situation is somewhat changed. First we note that $\mathcal{T}$ is *not* a linear or unitary operator. If c is a complex number then $\mathcal{T}(c\psi) = c^*\mathcal{T}\psi$ whereas for a unitary transformation linearity requires that $\mathcal{T}(c\psi) = c\mathcal{T}\psi$ which is not the case here. Rather $\mathcal{T}$ is an *antilinear operator.* If the operator also preserves the magnitude of the scalar product such that $|(\mathcal{T}\phi, \mathcal{T}\phi)| = |(\phi, \phi)|$ then the operator $\mathcal{T}$ is an *antiunitary* operator. Thus we should not treat the time reversal operator as simply having the action of complex conjugation. Let us use $\mathcal{K}$ to denote the operation of complex conjugation. Clearly $\mathcal{K}^2 = \mathcal{I}$ where now $\mathcal{I}$ is the identity operator.

The product of any two antiunitary operators is a unitary operator, say $\mathcal{U}$. Let us write

$$\mathcal{T}\mathcal{K} = \mathcal{U} \tag{3.23}$$

Left multiplying both sides by $\mathcal{K}$ yields

$$\mathcal{T} = \mathcal{U}\mathcal{K} \tag{3.24}$$

from which we may conclude that the most general form for $\mathcal{T}$ is to take it as the product of a unitary operator $\mathcal{U}$ and the antiunitary operator of complex conjugation $\mathcal{K}$. Consider the following matrix operations

$$\mathcal{T}^2 = \mathcal{U}\mathcal{K}\mathcal{U}\mathcal{K} = \mathcal{U}\mathcal{U}^* = c\mathbf{E} \tag{3.25}$$

with $\mathbf{E}$ being the unit matrix. Noting the unitarity of $\mathcal{U}$ we have

$$\mathcal{U} = c(\mathcal{U}^*)^{-1} = c\tilde{U}$$

Now take the transpose of the above

$$\tilde{\mathcal{U}} = c\mathcal{U} = c^2\tilde{\mathcal{U}}$$

Thus $c^2 = 1$ and hence $c = \pm 1$. Therefore the matrix $\mathcal{U}$ must be either totally symmetric or antisymmetric in taking its transpose. Thus Eq. (3.25) becomes

$$\mathcal{T}^2 = \pm\mathbf{E} \tag{3.26}$$

and hence we are restricted to

$$\mathcal{T}^2\psi = \pm\psi \tag{3.27}$$

For a spinless Schrödinger equation we have $\mathcal{T} = \mathcal{K}$ and only the positive solution is possible.

We can distinguish two classes of operators, those $\mathcal{O}_e$ involving only *even* powers of time and those $\mathcal{O}_o$ involving only *odd* powers of time. The first class would include ordinary coordinate operators and accelerations while the second class would include linear momentum and angular momentum operators. Clearly

$$\mathcal{T}\mathcal{O}_e = \mathcal{O}_e\mathcal{T} \quad \text{and} \quad \mathcal{T}\mathcal{O}_o = -\mathcal{O}_o\mathcal{T} \tag{3.28}$$

Recalling that $\mathcal{T} = \mathcal{U}\mathcal{K}$ and noting, for example, that for the linear momentum $\mathbf{p}= -i\nabla$ we see from Eq. (3.28) that $\mathcal{U}$ commutes with any multiplication by and differentiation with respect to the spatial coordinates and hence at most $\mathcal{U}$ has an effect on spin variables.

Recall the Pauli spin matrices, Eq. (2.10),

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}$$

τ_1 and τ_3 are real while τ_2 is pure imaginary. We have already noted that the time reversal operator $\mathcal{T}$ *anticommutes* with the spin $\mathbf{s}$ and hence its three components s_i i.e.

$$\mathcal{T} s_i = -s_i \mathcal{T} \quad i = 1, 2, 3 \tag{3.29}$$

but

$$\mathcal{T} s_k = \mathcal{U}\mathcal{K} s_k = \mathcal{U} s_k \mathcal{K} = -s_k \mathcal{U}\mathcal{K} \quad k = 1, 3 \tag{3.30}$$

and

$$\mathcal{T} s_2 = \mathcal{U}\mathcal{K} s_2 = -\mathcal{U} s_2 \mathcal{K} = -s_2 \mathcal{U}\mathcal{K} \tag{3.31}$$

Thus $\mathcal{U}$ commutes with s_2 and anticommutes with s_1 and s_3. Hence we may obtain a representation of the operator $\mathcal{U}$ for a system of n spins with the correct commutation properties by choosing

$$\mathcal{U} = \tau_{12}\tau_{22} \ldots \tau_{n2} \tag{3.32}$$

to give

$$\mathcal{T} = \tau_{12}\tau_{22} \ldots \tau_{n2}\mathcal{K} \tag{3.33}$$

Remembering that

$$s_k = \frac{1}{2}\tau_k$$

and writing the third component of the spin of the $i-$th particle as $s_{i3} = \pm\frac{1}{2}$ we have for an $n-$particle system

$$\mathcal{T}\psi(\mathbf{r}_1, s_{13}, \ldots, \mathbf{r}_n, s_{n3}) = (-i)^{2\sum_i s_{i3}}\psi(\mathbf{r}_1, -s_{13}, \ldots, \mathbf{r}_n, -s_{n3}) \tag{3.34}$$

We thus obtain two possibilities

$$\mathcal{T}^2\psi = \psi \quad n \textit{ even} \quad \text{or} \quad \mathcal{T}^2\psi = -\psi \quad n \textit{ odd} \tag{3.35}$$

We can now almost immediately obtain Kramer's theorem[10] which states that

> *In the absence of a magnetic field all energy levels of a system containing an odd number of electrons must be at least doubly degenerate regardless of how low the symmetry is.*

Thus a pure electric field cannot remove all the degeneracy for a system involving an odd number of electrons.

■ **Proof of Kramer's Theorem**

For an *odd* number of electrons $\mathcal{T}^2\psi = -\psi$ and consequently $\mathcal{T}\psi$ is orthogonal to ψ and by time-reversal symmetry $\mathcal{T}\psi$ must have the same energy as ψ and hence there must be at least two-fold degeneracy [11].

[10] Kramer's theorem has important technological consequences as it makes a prediction of a situation where 2-fold degeneracy is assured. Degeneracy can be lifted in a controlled manner by application of an external magnetic field. A second oscillatory magnetic field can be applied to induce transitions between the split levels. This is of importance in microwave communications and nuclear magnetic resonance imaging techniques in medicine.

[11] The orthogonality of $\mathcal{T}\psi$ and ψ may be proven by first considering two functions ψ and ϕ.

$$(\mathcal{T}\psi, \mathcal{T}\phi) = (\mathcal{U}\mathcal{K}\psi, \mathcal{U}\mathcal{K}\phi) = (\mathcal{K}\psi, \mathcal{K}\phi) = (\psi, \phi)^* = (\phi, \psi)$$

Therefore

$$(\mathcal{T}\psi, \psi) = (\mathcal{T}\psi, \mathcal{T}^2\psi) = (\mathcal{T}\psi, -\psi) = -(\mathcal{T}\psi, \psi)$$

and hence $(\mathcal{T}\psi, \psi) = 0$.

The Application of Symmetry Concepts
to
Physical Problems©

■ Lecture Four

■ 4.1 Introduction

In this lecture I want to introduce you to the elements of the quark model of hadrons (i.e. mesons and baryons). Ultimately we seek to describe the properties of hadrons in terms of sub structures based on more fundamental entities. Scattering experiments involving the collision of electrons with protons probe the substructure of the proton and in the late 1950's it was clear that the nucleon was composite made up of *partons*, a phrase introduced by R. P. Feynman. What should we take as these fundamental entities or partons? Remember mesons have integer spin and are bosons whereas baryons have half-integer spin and are fermions. If we wish to build *both* mesons and baryons out of the same entities and if these entities are not themselves mesons or baryons then we should take the basic entities as fermions since we may combine fermions to form bosons but not the converse.[1] Let us first see if we can describe the properties of the pions and nucleon in terms of some basic entities.

■ 4.2 Enter the u and d quarks

We noted in the previous lecture that the pions have a *negative* intrinsic parity with π^+ and π^- being a particle-antiparticle pair. Furthermore fermions and antifermions have *opposite parity* and hence it is reasonable to suggest that the mesons are made up of coupled pairs of fermionic and antifermionic partons. A minimal assumption for the baryons is that they involve the interaction of three fermions and the antibaryons three antifermions. We assume that the fermions are of spin 1/2, that allows us to construct mesons of spin 0 and 1 and baryons of spin 1/2 and 3/2. The pions form an isospin triplet (I = 1) so we shall assume a pair of basic entities that form an isospin doublet (I = 1/2). Let us call this pair of particles *quarks*[2] and designate the $I_3 = 1/2$ quark by the letter u (*the "up quark"*) and the $I_3 = -1/2$ quark by the letter d (*the "down quark"*) with electric charges q_u and q_d respectively. The corresponding antiquarks $\bar{u}$ and $\bar{d}$ will have opposite signs for their charge and isospin projection.

■ 4.3 Quark charges

The π^+ meson has $Q = 1$, $I = 1$, $J = 0$ and $I_3 = 1$ which would be compatible with the assignment

$$\pi^+ \sim u\bar{d}$$

if we take

$$Q = 1 = q_u - q_d \tag{4.1}$$

The proton has $Q = 1$, $I = 1/2$, $J = 1/2$ and $I_3 = 1/2$ which would be compatible with the assignment

$$p \sim uud$$

with

$$Q = 1 = 2q_u + q_d \tag{4.2}$$

Solving Eq.(4.1) and (4.2) gives the quark charges (in units of $+e$) as

$$q_u = 2/3 \qquad \text{and} \qquad q_d = -1/3 \tag{4.3}$$

It follows also that the quarks carry a baryonic charge $B = 1/3$ giving the baryonic charge of the nucleon as $B = 1$ and that of the pion as $B = 0$.

■ 4.4 Building π mesons

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[1] If magnetic monopoles exist, and to date there is no evidence for their existence, then it is possible to combine bosonic particles known as *dyons* (particles containing both electric and magnetic charge) to form fermions. The concept of the magnetic monopole was introduced by Dirac to explain electric charge quantisation.

[2] The name *quark* was introduced by M. Gell-Mann from James Joyce's book *Ulysses* "three quarks for Master Mark". It is *not* derived from the German soft quark cheese.

It is useful to regard the u and d quarks as forming a two component isospin spinor and likewise for $\bar{u}$ and $\bar{d}$. For reasons of phase consistency we shall write these two spinors as

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \tag{4.4}$$

We then have the isospin operations

$$I_-|u\rangle = |d\rangle \quad \text{and} \quad I_+|d\rangle = |u\rangle \tag{4.5a}$$

and

$$I_+|\bar{u}\rangle = -|\bar{d}\rangle \quad \text{and} \quad I_-|\bar{d}\rangle = -|\bar{u}\rangle \tag{4.5b}$$

The basic ansatz for constructing mesons out of quarks is:

Mesons are formed by coupling a quark to an antiquark

To construct the three pions let us take for π^+

$$|\pi^+\rangle = -|u\bar{d}\rangle \tag{4.6}$$

The wavefunction for the π^0 is found by acting with I_- on both sides of Eq.(4.6) and remembering Eq.(4.5) to give

$$\begin{aligned} I_-|\pi^+\rangle &= \sqrt{2}|\pi^0\rangle \\ (i_{1-} + i_{2-}) - |u\bar{d}\rangle &= -|d\bar{d}\rangle + |u\bar{u}\rangle \end{aligned} \tag{4.7}$$

leading to[3]

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \tag{4.8}$$

In a similar fashion the wavefunction of the π^- is found to be

$$\pi^-\rangle = |d\bar{u}\rangle \tag{4.9}$$

Note we could construct the wavefunction of a second neutral meson η^0 by demanding it be orthogonal to that of the π^0 to give

$$|\eta^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) \tag{4.10}$$

The wavefunctions for the two neutral mesons involve quark-antiquark pairs of the same type whereas the charged mesons involve different types. The charged π-mesons have a meanlife of $2.6 \times 10^{-8}s$ whereas the π^0 meson has a meanlife of $8.4 \times 10^{-17}s$. The charged pions decay via the weak interaction whereas the neutral pion decays by electromagnetic interaction giving rise to particle-antiparticle annihilation.

■ 4.5 Baryon wavefunctions

The corresponding ansatz for constructing baryons out of quarks is:

Baryons (Anti-baryons) are constructed out of a triplet of quarks (anti-quarks)

We have already suggested the proton is built out of the quark configuration uud and the neutron udd. Magnetic moment measurements are consistent with the quarks having zero orbital angular momentum in their groundstate. Let us temporarily suspend our belief in the Pauli exclusion principle and ask what particle would the quark configuration uuu correspond to? It would involve a charge of two units with a total isospin projection of 3/2 suggesting an isospin quartet multiplet completely symmetric in the spin space with J = 3/2. Could this be a member of the multiplet involving the four particles $(\Delta^-, \Delta^0, \Delta^+, \Delta^{++})$?

Let us consider a particle Δ^{++} in a spin state with $J = 3/2$ and maximal spin projection $J_3 = 3/2$. Ignoring the Pauli exclusion principle, the only possible way of constructing such a state out of u and d quarks is to make the assignment

$$|\Delta^{++}3/2, 3/2\rangle = |\overset{+}{u}\overset{+}{u}\overset{+}{u}\rangle \tag{4.11}$$

[3] As we shall see later, in the case of the neutral pion we are ignoring the possibility of other quarks being involved such as the *"strange quark" s*.

where the 3/2 refer to the respective isospin and spin projections while the + signs placed over the quarks indicate the quark spin projection $s_z = +1/2$ (We will later use $-$ for $s_z = -1/2$.). Remembering Eq. (4.5), we can apply the isospin operator I_- to both sides of Eq. (4.11) to give

$$I_-|\Delta^{++}3/2, 3/2\rangle = \sqrt{3}|\Delta^{+}1/2, 3/2\rangle \tag{4.12a}$$

and

$$\begin{aligned} &I_-|\overset{+}{u}\overset{+}{u}\overset{+}{u}\rangle + |\overset{+}{d}\overset{+}{u}\overset{+}{u}\rangle + |\overset{+}{u}\overset{+}{d}\overset{+}{u}\rangle + |\overset{+}{u}\overset{+}{u}\overset{+}{d}\rangle \\ &=|\overset{+}{u}\overset{+}{u}\overset{+}{d}\rangle_3 \end{aligned} \tag{4.12b}$$

where by the last line we understand we are to take all three distinct permutations of the orderings of the three quarks. Note that stepping down in isospin changes the *charge* of the particle by one unit. Comparison of Eq. (4.12a) with (4.12b) gives

$$|\Delta^{+}1/2, 3/2\rangle = \frac{1}{\sqrt{3}}|\overset{+}{u}\overset{+}{u}\overset{+}{d}\rangle_3 \tag{4.13}$$

How is this wavefunction related to that of the proton? Let us first apply a spin step down operator J_- to both sides of Eq. (4.13) to get

$$J_-|\Delta^{+}1/2, 3/2\rangle = \sqrt{3}|\Delta^{+}1/2, 1/2\rangle \tag{4.14a}$$

and

$$J_-\frac{1}{\sqrt{3}}|\overset{+}{u}\overset{+}{u}\overset{+}{d}\rangle_3 = \frac{1}{\sqrt{3}}(|\overset{+}{u}\overset{-}{u}\overset{+}{d}\rangle_6 + (|\overset{+}{u}\overset{+}{u}\overset{-}{d}\rangle_3) \tag{4.14b}$$

Comparison of Eq. (4.14a) with (4.14b) gives

$$|\Delta^{+}1/2, 1/2\rangle = \frac{1}{3}(|\overset{+}{u}\overset{-}{u}\overset{+}{d}\rangle_6 + |\overset{+}{u}\overset{+}{u}\overset{-}{d}\rangle_3) \tag{4.15}$$

■ 4.6 The nucleon wavefunctions

The proton has isospin $I = 1/2$ with isospin projection $I_3 = +1/2$ and spin $J = 1/2$ and hence the proton state $|p^{+}1/2, 1/2\rangle$ should be orthogonal to the state formed in Eq. (4.15) and hence of the form

$$|p^{+}1/2, 1/2\rangle = a|\overset{+}{u}\overset{-}{u}\overset{+}{d}\rangle_6 + b|\overset{+}{u}\overset{+}{u}\overset{-}{d}\rangle_3 \tag{4.16}$$

The coefficients a and b may be evaluated by noting that orthogonality requires $6a + 3b = 0$ and hence $b = -2a$ while normalisation requires $a^2 + b^2 = 1$ and hence choosing the phase of a as positive gives

$$|p^{+}1/2, 1/2\rangle = \frac{1}{\sqrt{18}}(|\overset{+}{u}\overset{-}{u}\overset{+}{d}\rangle_6 - 2|\overset{+}{u}\overset{+}{u}\overset{-}{d}\rangle_3 \tag{4.17}$$

The corresponding state for the neutron may be found by applying the isospin step-down operator I_- to both sides of Eq. (4.17) to give

$$|n^{0} - 1/2, 1/2\rangle = \frac{1}{\sqrt{18}}(-|\overset{+}{d}\overset{-}{d}\overset{+}{u}\rangle_6 + 2|\overset{+}{d}\overset{+}{d}\overset{-}{u}\rangle_3 \tag{4.18}$$

Eqs.(4.17) and (4.18) give us a quark description of the nucleon. In our next lecture we shall use this knowledge to compute the ratio of the magnetic moment of the proton to that of the neutron which will give us our first experimental test.

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Lecture Five

5.1 Introduction

In Lecture Four we introduced a quark model based on just two quarks (u, d) and their corresponding antiquarks $(\bar{u}, \bar{d})$. We then constructed wavefunctions for the proton and neutron in terms of their constituent quarks as

$$|p^{+}1/2, 1/2\rangle = \frac{1}{\sqrt{18}}(|\overset{+}{u}\overset{-}{u}\overset{+}{d}\rangle_6 - 2|\overset{+}{u}\overset{+}{u}\overset{-}{d}\rangle_3) \tag{4.17}$$

and

$$|n^{0} - 1/2, 1/2\rangle = \frac{1}{\sqrt{18}}(-|\overset{+}{d}\overset{-}{d}\overset{+}{u}\rangle_6 + 2|\overset{+}{d}\overset{+}{d}\overset{-}{u}\rangle_3) \tag{4.18}$$

In today's lecture I want to use this knowledge to calculate the ratio of the magnetic moment of the proton to that of the neutron.

5.2 Assumptions in the calculation of quark magnetic moments

The magnetic moment of an electron is proportional to its charge e and inversely proportional to its mass m. If isospin were exactly conserved we would have $m_u = m_d$. We shall assume isopspin conservation. Let us, by analogy, assume that the quarks carry a magnetic moment proportional to their charge. We wish to calculate a *ratio* and hence will ignore the mass term. We calculate the ratio for the same spin projection $J_z = 1/2$ for the proton and neutron. Let s_z denote the spin projection of a single quark, then the magnetic moments should be proportional to the matrix elements of the operator

$$\mu_z = \sum_{i=1}^{3} q_i s_{z_i} \tag{5.1}$$

We have the single quark matrix elements

$$\langle\overset{\pm}{u}|qs_z|\overset{\pm}{u}\rangle = \pm\frac{1}{3} \quad \text{and} \quad \langle\overset{\pm}{d}|qs_z|\overset{\pm}{d}\rangle = \mp\frac{1}{6} \tag{5.2}$$

5.3 Calculation of the proton-neutron magnetic moment ratio

Let us first consider the proton wavefunction of Eq. (4.17). Forming the matrix element of the operator given in Eq. (5.1) we have

$$\langle p^+ 1/2, 1/2|\mu_z|p^+ 1/2, 1/2\rangle = \frac{1}{18}\langle \overset{+}{u}\overset{-}{u}\overset{+}{d}_6 - 2\overset{+}{u}\overset{+}{u}\overset{-}{d}_3|\mu_z|\overset{+}{u}\overset{-}{u}\overset{+}{d}_6 - 2\overset{+}{u}\overset{+}{u}\overset{-}{d}_3\rangle \tag{5.3a}$$

$$= \frac{1}{18}(\langle \overset{+}{u}\overset{-}{u}\overset{+}{d}_6|\mu_z|\overset{+}{u}\overset{-}{u}\overset{+}{d}_6\rangle + 4\langle \overset{+}{u}\overset{+}{u}\overset{-}{d}_3|\mu_z|\overset{+}{u}\overset{+}{u}\overset{-}{d}_3\rangle) \tag{5.3b}$$

$$= \frac{1}{18}(6\langle \overset{+}{d}|qs_z|\overset{+}{d}\rangle + 4\times 3\times 2\langle \overset{+}{u}|qs_z|\overset{+}{u}\rangle + 4\times 3\langle \overset{-}{d}|qs_z|\overset{-}{d}\rangle) \tag{5.3c}$$

$$= \frac{1}{18}(24\langle \overset{+}{u}|qs_z|\overset{+}{u}\rangle - 6\langle \overset{+}{d}|qs_z|\overset{+}{d}\rangle) \tag{5.3d}$$

$$= \frac{1}{18}(24\times\frac{1}{3} - 6\times\frac{-1}{6}) \tag{5.3e}$$

$$= \frac{1}{2} \tag{5.3f}$$

Similarly for the neutron wavefunction given by Eq. (4.18) we have

$$\langle n^0 - 1/2, 1/2|\mu_z|n^0 - 1/2, 1/2\rangle == \frac{1}{18}\langle -\overset{+}{d}\overset{-}{d}\overset{+}{u}_6 + 2\overset{+}{d}\overset{+}{d}\overset{-}{u}_3|\mu_z| - \overset{+}{d}\overset{-}{d}\overset{+}{u}_6 + 2\overset{+}{d}\overset{+}{d}\overset{-}{u}_3\rangle \tag{5.4a}$$

$$= \frac{1}{18}(\langle \overset{+}{d}\overset{-}{d}\overset{+}{u}_6|\mu_z|\overset{+}{d}\overset{-}{d}\overset{+}{u}_6\rangle + 4\langle \overset{+}{d}\overset{+}{d}\overset{-}{u}_3|\mu_z|\overset{+}{d}\overset{+}{d}\overset{-}{u}_3\rangle) \tag{5.4b}$$

$$= \frac{1}{18}(6\langle \overset{+}{u}|qs_z|\overset{+}{u}\rangle + 4\times 3\times 2\langle \overset{+}{d}|qs_z|\overset{+}{d}\rangle + 4\times 3\langle \overset{-}{u}|qs_z|\overset{-}{u}\rangle) \tag{5.4c}$$

$$= \frac{1}{18}(24\langle \overset{+}{d}|qs_z|\overset{+}{d}\rangle - 6\langle \overset{+}{u}|qs_z|\overset{+}{u}\rangle) \tag{5.4d}$$

$$= \frac{1}{18}(24\times\frac{-1}{6} - 6\times\frac{1}{3}) \tag{5.3e}$$

$$= \frac{-1}{3} \tag{5.4f}$$

Comparison of Eq. (5.3f) with Eq. (5.4f) leads immediately to

$$\frac{\mu_p}{\mu_n}_{calc} = \frac{-3}{2} = -1.5 \tag{5.5a}$$

which may be compared with the experimental value of (see page 81 of the Particle Properties Data booklet)

$$\frac{\mu_p}{\mu_n}_{expt} = -1.46 \tag{5.5b}$$

Importantly, we have obtained a reasonable magnitude, with the correct sign.

The introduction of just two quarks has given a surprisingly good account of the nucleon and the pion. We now see how it is possible to have the neutron as an electrically neutral particle and yet have a sizable magnetic moment. We have also seen a reason for the difference in the lifetime of the charged and neutral π-mesons. However, two quarks are not sufficient to describe all of the observed baryons and mesons and we have not explained the observed groupings of mesons into octets and singlets and the baryons into octets and decuplets. Nor have we explained our apparent disregard of the Pauli exclusion principle. This is seen most strongly in the maximal spin state of the Δ^{++} baryon. In the latter case we need to introduce the *colour* quantum numbers and in the former the strange quark s. This we shall do in Lecture Six.

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Lecture Six

6.1 Introduction

In Lecture Four we introduced a quark model based on just two quarks (u, d) and their corresponding antiquarks $(\bar{u}, \bar{d})$. This led to a model for describing the pions, nucleons and Δ particles but was not rich enough to describe other members of the meson octet or baryon octet and decuplet. A richer scheme comes by introduction of the *strange* quark s and its antiparticle $\bar{s}$.

6.2 Strange Particles

The possibility of an additional quantum number capable of distinguishing different isospin multiplets pre-dates the quark model. In 1954 it became possible to bombard pions on protons and observe the reaction

$$\pi^- + p \to \Lambda + K^0 \tag{6-1}$$

The two neutral particles decayed as

$$\Lambda \to p + \pi^- \tag{6-2}$$
$$K^0 \to \pi^+ + \pi^- \tag{6-3}$$

The cross-sections measured for (6-1) were consistent with the process occurring via the strong interaction whereas the subsequent decays were of times normally associated with weak interactions. This behaviour seemed at the time strange with Λ and K being termed *strange particles.* It was suggested that there must be an additional quantum number $\mathcal{S}$ (*strangeness*) that is conserved in strong interactions but broken in weak interactions. The pions, nucleons and Δ's were assigned strangeness $\mathcal{S} = 0$ and the strange particles such as $\Lambda, \Xi, \Sigma, K, \ldots$ are assigned $\mathcal{S} \neq 0$.

If Eq.(6-1) is to conserve strangeness and the π^- and p have $\mathcal{S} = 0$ then the strangeness on the right-hand-side must sum to zero. This will be the case if we arbitrarily assign $\mathcal{S} = -1$ to the Λ and $\mathcal{S} = +1$ to the K^0. The decays given in Eq. (6-2) and (6-3) involve strangeness violation with

$$|\mathcal{S}| = 1 \tag{6-4}$$

States of a given isospin multiplet have the same strangeness quantum number $\mathcal{S}$ allowing us to assign $\mathcal{S} = +1$ to K^+ and $\mathcal{S} = -1$ to $\bar{K}^0$ and K^-. The assignment of strangeness to the Ξ^- particle follows by noting that its decay proceeds in two steps:-

$$\Xi^- \to \Lambda + \pi^-$$
$$\Lambda \to p + \pi^-$$

Each step involves a change of strangeness of -1 leading to the assignment of $\mathcal{S} = -2$ to the isospin doublet Ξ^-, Ξ^0

6.3 The Gell−Mann-Nishijima Charge Equation

The charges Q of the non-strange particles are observed to satisfy the equation

$$Q = I_3 + \frac{1}{2}B \tag{6-5}$$

The Λ particle has been assigned $\mathcal{S} = -1$ with $I_3 = 0$ and $B = 1$ suggesting that both strange and non-strange particles satisfy the so-called Gell−Mann-Nishijima relation

$$Q = I_3 + \frac{1}{2}(B + \mathcal{S}) \tag{6-6}$$
$$= I_3 + Y/2 \tag{6-7}$$

where Y is known as the *hypercharge*.

6.4 The Quark Triplet and Anti-Quark Anti-Triplet

The introduction of strangeness requires that our very simple two-quark model be extended by the introduction of a third quark, the *strange* quark s with strangeness $\mathcal{S} = -1$. The Gell−Mann-Nishijima relation requires $Q_s = -\frac{1}{3}$ and isospin $I = 0$. Note that the strange quark has the same charge as that of the down quark d. This suggests a grouping of particles into multiplets based upon their charge Q instead of isospin - the so-called $U-$spin multiplets. States belonging to a given $U-$spin would be expected to have similar electromagnetic properties. Thus the d, s quarks form a $U-$spin doublet with the $u-$quark being a $U-$spin singlet.

■ 6.5 The $U-$spin Algebra

Just as with isospin we may construct a $U-$spin algebra such that

$$U_3|U, M_U\rangle = M_U|U, M_U\rangle \qquad (6-8a)$$

$$U_\pm|U, M_U\rangle = \sqrt{U(U+1) - M_U(M_U \pm 1)}|U, M_U \pm 1\rangle \qquad (6-8b)$$

$$U^2|U, M_U\rangle = U(U+1)|U, M_U\rangle \qquad (6-8c)$$

Assigning the $d-$quark to $M_U = +\frac{1}{2}$ we then have

$$U_-|d\rangle = |s\rangle \qquad (6-9a)$$

$$U_+|s\rangle = 0 \qquad (6-9b)$$

$$U_\pm|u\rangle = 0 \qquad (6-9c)$$

$$U_+|\bar{d}\rangle = -|\bar{s}\rangle \qquad (6-9d)$$

■ 6.6 Strange Mesons

We earlier deduced that

$$|\pi^+\rangle = -|u\bar{d}\rangle \qquad (6-10a)$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \qquad (6-10b)$$

$$|\pi^-\rangle = |d\bar{u}\rangle \qquad (6-10c)$$

which are all non-strange mesons. Applying U_+ to Eq. (6-10a) and noting Eq. (6-9d) we have

$$U_+|\pi^+\rangle = |K^+\rangle$$
$$= |u\bar{s}\rangle$$

and hence we have the assignment

$$|K^+\rangle = |u\bar{s}\rangle \qquad (6-11)$$

We can now obtain the wavefunction for the K^0 meson by applying the isospin step-down operator I_- to Eq. (6-11) to give

$$I_-|K^+\rangle = |K^0\rangle$$
$$= |d\bar{s}\rangle \qquad (6-12)$$

With the three quarks u, d, s and three anti-quarks we may construct three orthogonal linear combinations of the pairs $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$. One linear combination has already been found and identified with that of the non-strange meson π^0. A complete scalar strange meson, η', with isospin $I = 0$ may be written as

$$|\eta'\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \qquad (6-13)$$

A state corresponding to the strange meson, η, arises by forming a linear combination that is orthogonal to the wavefunctions given by Eq. (6-10b) and (6-13) to give

$$|\eta\rangle = \frac{1}{\sqrt{6}}(2|s\bar{s}\rangle - |u\bar{u}\rangle - |d\bar{d}\rangle) \qquad (6-14)$$

Finally we have the two kaons

$$|\bar{K^0}\rangle = -|s\bar{d}\rangle \quad \text{and} \quad |K^-\rangle = |s\bar{u}\rangle \qquad (6-15)$$

Recall earlier the meson octet drawn earlier

$$\begin{array}{ccccc} & K^0 & & K^+ & \\ & \bullet & \cdots\cdots & \bullet & \\ \pi^- & & \pi^0 & & \pi^+ \\ \bullet & \cdots\cdots & \odot & \cdots\cdots & \bullet \\ & & \eta^0 & & \\ & \bullet & \cdots\cdots & \bullet & \\ & K^- & & \bar{K}^0 & \end{array}$$

Meson Octet $J^p = 0^-$

In terms of coupled quark-antiquark pairs we have the equivalent diagram

$$\begin{array}{ccccc} & d\bar{s} & & u\bar{s} & \\ & \bullet & \cdots\cdots & \bullet & \\ -u\bar{d} & & \pi^0 & & d\bar{u} \\ \bullet & \cdots\cdots & \odot & \cdots\cdots & \bullet \\ & & \eta^0 & & \\ & \bullet & \cdots\cdots & \bullet & \\ & s\bar{u} & & -s\bar{d} & \end{array}$$

Meson Octet $J^p = 0^-$

where the quark composition of the two neutral mesons, π^0 and η^0, are given in Eqs. (6-10b) and (6-14) respectively. The other neutral strange meson, η' occurs as a singlet. Altogether we have formed a nonet of mesons (an octet plus a singlet). The particles at the top of the octet have strangeness $\mathcal{S} = +1$ while the two at the bottom have $\mathcal{S} = -1$. All the particles on the central line have $\mathcal{S} = 0$. Thus the η and η' are said to have *hidden strangeness.*

■ 6.7 The Baryon Octets and Decuplets

With three quarks u, d, s we can form a total of 27 particles from triplets of these quarks as seen in the table below.

quarks	I_3	$\mathcal{S}$	$\mathcal{Q}$	Number of Baryons
uuu	$\frac{3}{2}$	0	2	1
uud	$\frac{1}{2}$	0	1	3
udd	$-\frac{1}{2}$	0	0	3
ddd	$-\frac{3}{2}$	0	-1	1
uus	1	-1	1	3
dds	-1	-1	-1	3
uds	0	-1	0	6
uss	$\frac{1}{2}$	-2	0	3
dss	$-\frac{1}{2}$	-2	-1	3
sss	0	-3	-1	1

There is one baryon for each independent quark wavefunction. Thus for uds we can form six independent orthonormal sets of quark wavefunctions. It is instructive to make a plot of isospin projection, I_3, versus strangeness , $\mathcal{S}$, as overleaf. The diagram can be resolved into a decuplet, two octets and a singlet corresponding to **3 x 3 x 3 = 1 + 8 + 8 + 10**

We have already identified the quark composition of the Δ^{++} as

$$|\Delta^{++} I = \frac{3}{2}, I_3 = \frac{3}{2}, J = \frac{3}{2}, J_3 = \frac{3}{2}\rangle \sim |\overset{+}{u}\overset{+}{u}\overset{+}{u}\rangle \qquad (6-16)$$

From that one state we may construct all the other states of the decuplet from a systematic application of the isospin ladder operators,$I_\pm$ to move along the states of a given isospin, I, and fixed strangeness, $\mathcal{S}$, and angular momentum, J, J_3. Application of the $U-$spin ladder operators $U_\pm$ allow us to change the strangeness quantum number $\mathcal{S}$ and hence move from one isospin multiplet to another. Note that changing the isospin projection I_3 changes the charge, Q, in steps of one unit while changing the $U-$spin projection, U_3, changes the strangeness, $\mathcal{S}$ in steps of unity.

The 27 baryons - a plot of isospin projection I_3 versus strangeness $\mathcal{S}$.

The quark composition for the baryon decuplet with $J = \frac{3}{2}^+$

Note that a change in strangeness involves the replacement of a non-strange quark, u or d, by a strange quark, s. With isospin conservation we would have $m_u = m_d$ and we would expect the mass of the strange quark to be greater than that of the up-down quarks. Hence we could predict that the mass intervals in the baryon decuplet will be approximately equal as is indeed the case.

$$m_\Delta = 1232 MeV, \quad m_{\Sigma^*} = 1385 MeV, \quad m_{\Xi^*} = 1530 MeV, \quad m_\Omega = 1672 MeV$$

The preceding plot may be compared with our earlier representation of the baryon decuplet given below

Δ^- $\quad$ Δ^0 $\quad$ Δ^+ $\quad$ Δ^{++}

• . . . . • . . . . • . . . . •

Σ^{*-} $\quad$ Σ^{*0} $\quad$ Σ^{*+}

• . . . . • . . . . •

Ξ^{*-} $\quad$ Ξ^{*0}

• . . . . •

•

Ω^-

The baryon decuplet with $J = \frac{3}{2}^+$

■ Concluding Remarks

In the preceding lectures I have endeavoured to introduce you to a small part of the role of symmetry in physics with mainly examples drawn from particle physics using chiefly your knowledge of angular momentum theory. Throughout, but without explicit statement, we have been using the theory of the group of rotations in two (SO_2) and three (SO_3) dimensions and their associated Lie algebras and in our discussion of spin the group of special unitary transformations (SU_2) and its Lie algebra A_1. In the background of our discussion of the singlets, octets and decuplets of the hadrons and the quark triplets and antitriplets has been the Lie algebras associated with two important group structures (SU_3) and ($SU_2 \times U_1$). Importantly, we note that methods developed in one area of physics can find similar applications in other seemingly very dissimilar areas of physics. While we stop here the never-ending story does not .....