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Non-Markovianity and information backflow in terms of entropic quantities

Bassano Vacchini

Dipartimento di Fisica
"Aldo Pontremoli"
Università degli Studi di Milano

INFN
Sezione di Milano

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Outline ↔ Take home message

Information viewpoint on non-Markovianity

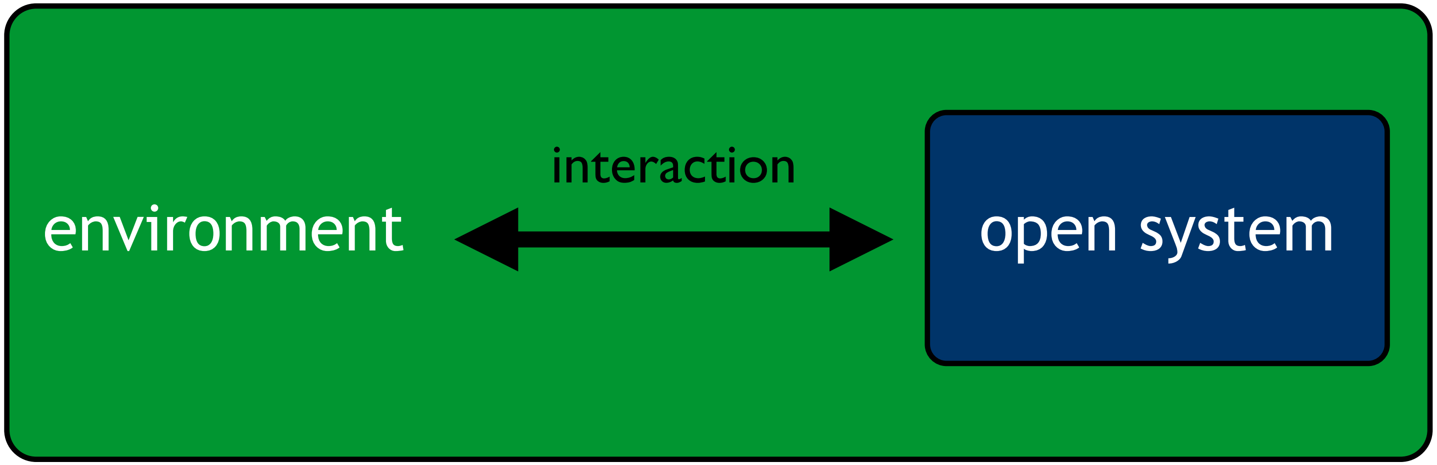
Entropic bounds on information flow

Reduced vs microscopic dynamics

N. Megier, A. Smirne and B. Vacchini
arXiv:2101.02720 to appear in PRL

A. Smirne, N. Megier and B. Vacchini
arXiv:2101.07282 Quantum, 5 439 (2021)

Open quantum systems



Bipartite setting

$$H = H_S + H_E + H_I$$

$$\rho_{SE} \in \mathscr{T}(\mathscr{H}_S \otimes \mathscr{H}_E) \qquad H \in \mathscr{B}(\mathscr{H}_S \otimes \mathscr{H}_E)$$

Reduced dynamics

$$\rho_S(0) \to \rho_S(t) = \Phi(t)\rho_S(0)$$

Correlations

$$\rho_{SE}(t) \neq \rho_S(t) \otimes \rho_E(t)$$

[Davies, 1976; Alicki & Lendi, 1987; Breuer & Petruccione, 2002; Rivas & Huelga, 2012]

Time dependent collection of evolution maps

$$\Phi(t)[\rho_S(0)] = \mathrm{Tr}_E(U(t)\rho_S(0)\otimes\rho_E U^\dagger(t)) = \sum_{\alpha,\beta} K_{\alpha,\beta}(t)\rho_S(0)K^\dagger_{\alpha,\beta}(t)$$

emergence of complete positivity

Quantum process

stochasticity of the dynamics
due to interaction with the environment

on top of

intrinsic probabilistic
quantum description

[Stinespring PAMS 1955; Hellwig & Kraus CMP 1969; Kraus LNP 1983]

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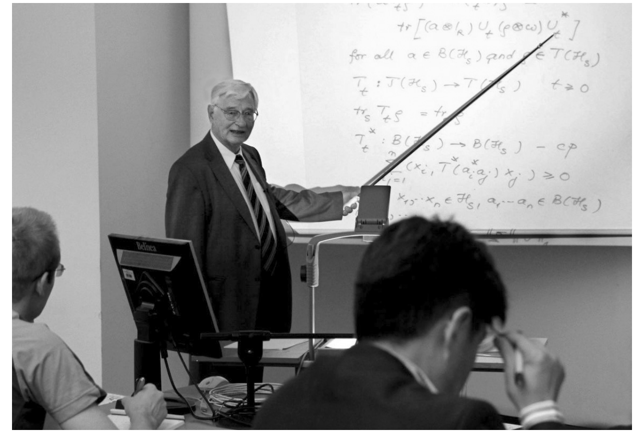
Markov process

Semigroup of (CPT) maps

$$\Lambda(t)\Lambda(s) = \Lambda(t+s) \qquad t, s \geq 0$$

$$\Lambda(t) = \exp(\mathscr{L}t)$$

$$\frac{d}{dt}\rho(t) = \mathscr{L}\rho(t)$$



Iff GKSL generator

$$\mathscr{L}\rho = -i[H,\rho] + \sum_k \gamma_k [A_k \rho A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho\}] \qquad \gamma_k \geq 0$$

[Kossakowski, RMP & Bull. Acad. Pol. Sci. 1972;
Gorini, Kossakowski & Sudarshan, JMP 1976; Lindblad, CMP 1976]

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Beyond Markovian dynamics*

Process viewpoint

$$P_n(t_n, x_n; t_{n-1}, x_{n-1}; \ldots t_1, x_1) \qquad t_n \geqslant t_{n-1} \geqslant \ldots \geqslant t_1 \geqslant 0$$

[Lindblad CMP 1979; B. V. & al. NJP 2011; Milz & Modi, arXiv 2020;
Giarmatzi & Costa, QUANTUM 2021]

Divisibility viewpoint

$$\Phi(t,\tau)\Phi(\tau,s) = \Phi(t,s) \qquad t \geq \tau \geq s \geq 0$$

[Rivas, Huelga & Plenio, PRL 2010; Rivas, Huelga & Plenio, RMP 2014]

Trajectory viewpoint

$$|\psi(t)\rangle \qquad t \geqslant 0$$

[Piilo & al., PRL 2008; Smirne & al., PRL 2020; Donvil & Muratore-Ginanneschi, arXiv 2021]

Distinguishability viewpoint

$$D(\rho_1(t), \rho_2(t)) \qquad t \geqslant 0$$

[Breuer, Laine & Piilo, PRL 2009; Breuer, Laine, Piilo & B.V., RMP 2016]

* Equations and references are a guide for the eye

Trace distance and distinguishability

Trace distance

Trace norm natural metric on the space of quantum states

$$D(\rho_1,\rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\| \qquad 0 \leqslant D \leqslant 1$$

$$D(\rho_1,\rho_2) = 0 \iff \rho_1 = \rho_2 \qquad D(\rho_1,\rho_2) = 1 \iff \rho_1 \perp \rho_2$$

(C)PT maps contractions for the trace distance

$$D(\Phi\rho_1, \Phi\rho_2) \leq D(\rho_1,\rho_2)$$

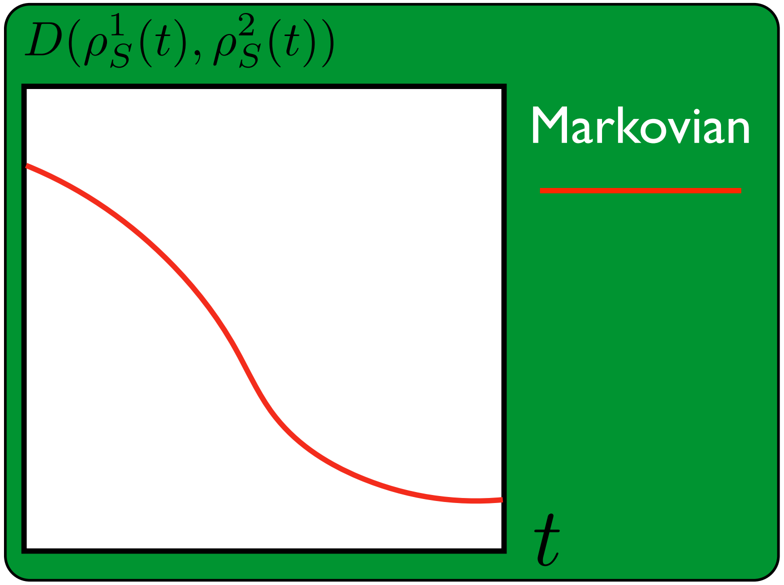
Triangle inequality

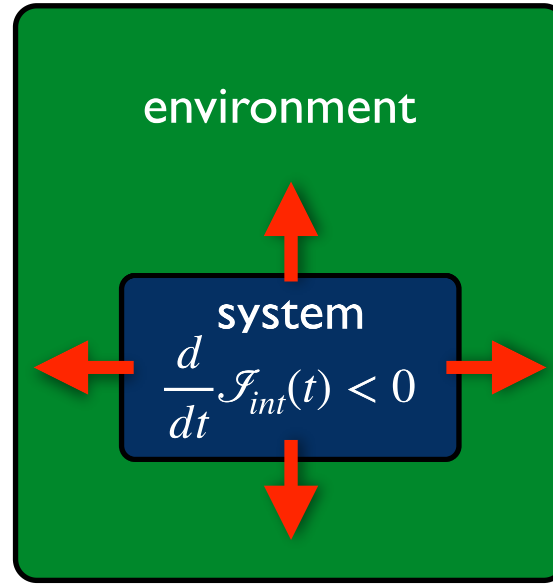
$$D(\varrho,\sigma) - D(\varrho,\tau) \leqslant D(\sigma,\tau)$$
$$D(\varrho,\sigma) - D(\eta,\sigma) \leqslant D(\varrho,\eta)$$

[Kossakowski Bull. Acad. Pol. Sci. 1972; Ruskai, RMP 1994]

Markovian versus non-Markovian dynamics

Monotonic loss of distinguishability





as in the presence of a well-defined composition law

$$D(\rho_1(t), \rho_2(t)) \leq D(\rho_1(s), \rho_2(s)) \quad \forall t \geq s \quad \forall \rho_1(0), \rho_2(0) \in \mathcal{S}(\mathcal{H})$$

Dynamics is said to be Markovian

[Breuer, Laine & Piilo, PRL 2009; Breuer, Laine, Piilo & B.V. RMP 2016]

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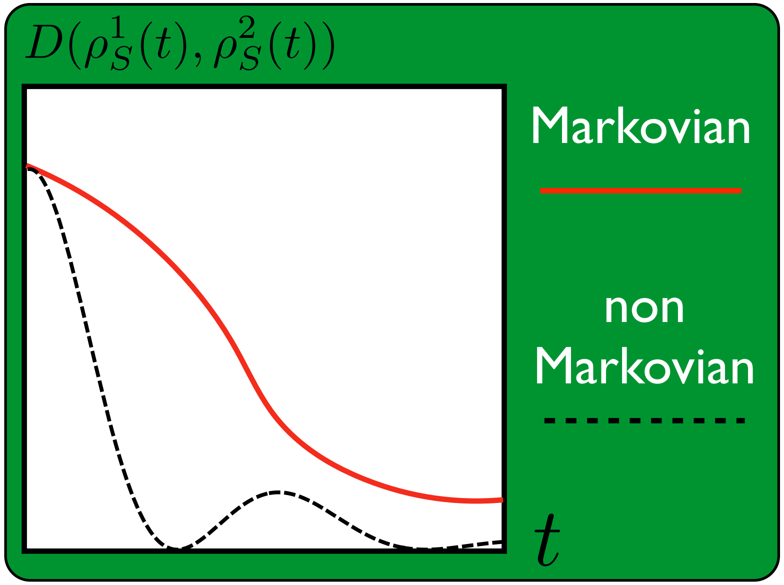
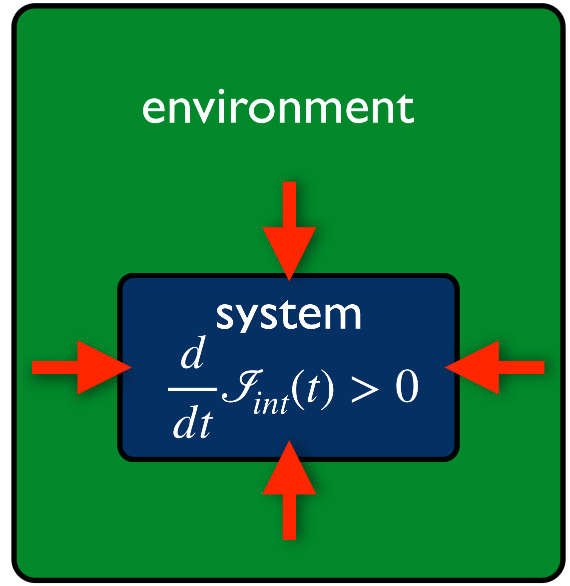
Markovian versus non-Markovian dynamics

Revival of distinguishability





e.g. due to revival in physical property

$$\exists \rho_1(0), \rho_2(0) \in \mathcal{S}(\mathcal{H}) \quad \exists t \geq s \quad D(\rho_1(t), \rho_2(t)) > D(\rho_1(s), \rho_2(s))$$

Dynamics is said to be non-Markovian

[Breuer, Laine & Piilo, PRL 2009; Breuer, Laine, Piilo & B.V. RMP 2016]

Information backflow

Internal vs external information

$$\mathcal{I}_{int}(t) = \mathrm{D}(\rho_S^1(t), \rho_S^2(t)) \quad \mathcal{I}_{ext}(t) = \mathrm{D}(\rho_{SE}^1(t), \rho_{SE}^2(t)) - \mathrm{D}(\rho_S^1(t), \rho_S^2(t))$$

so that

$$\mathcal{I}_{int}(t) + \mathcal{I}_{ext}(t) = \mathrm{cost}$$

$$\frac{d}{dt}\mathcal{I}_{int}(t) = -\frac{d}{dt}\mathcal{I}_{ext}(t)$$

Non-Markovian behaviour associated to

$$\mathcal{I}_{int}(t) - \mathcal{I}_{int}(s) \geqslant 0 \quad \text{for} \quad t \geqslant s$$

$$\mathcal{I}_{ext}(t) - \mathcal{I}_{ext}(s) \leqslant 0 \quad \text{for} \quad t \geqslant s$$

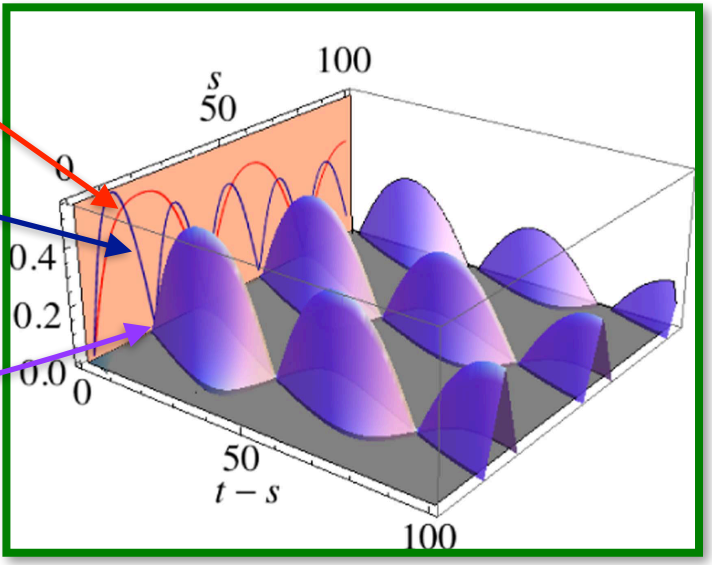
[Breuer, Laine, Piilo & B.V. RMP 2016; Laine & al. EPL 2010; Campbell & al. NJP 2019]

Internal vs external information

$$\mathcal{I}_{int}(t) - \mathcal{I}_{int}(s) = \mathcal{I}_{ext}(s) - \mathcal{I}_{ext}(t) \leqslant \mathcal{I}_{ext}(s)$$

leads to bound on revivals based on external information at previous times comparing states with product of their marginals and comparing environmental marginals

$$\mathcal{I}_{int}(t) - \mathcal{I}_{int}(s) \leq D(\rho_E^1(s), \rho_E^2(s)) + D(\rho_{SE}^1(s), \rho_S^1(s) \otimes \rho_E^1(s)) + D(\rho_{SE}^2(s), \rho_S^2(s) \otimes \rho_E^2(s))$$



[Breuer, Laine, Piilo & B.V. RMP 2016; Laine & al. EPL 2010; Campbell & al. NJP 2019]

Distinguishing states

Quantum divergence

Divergence as difference quantifier between classical or quantum probability distributions

Relaxing symmetry and triangle inequality

Relevant property of quantum divergence

Contractivity under (C)PT maps

$$\mathrm{f}(\Phi[\varrho], \Phi[\sigma]) \leqslant \mathrm{f}(\varrho, \sigma) \Longrightarrow \begin{cases} \mathrm{f}(\mathscr{U}[\varrho], \mathscr{U}[\sigma]) = \mathrm{f}(\varrho, \sigma) \\ \mathrm{f}(\varrho \otimes \eta, \sigma \otimes \eta) = \mathrm{f}(\varrho, \sigma) \end{cases}$$

Crucially unitary evolution, partial trace, assignment map are (C)PT transformation

Boundedness

$$0 \leqslant \mathrm{f}(\varrho, \sigma) \leqslant \text{cost}$$

Microscopic interpretation

Triangle-like inequality

Assume validity of inequalities

$$\mathsf{f}(\varrho,\sigma)-\mathsf{f}(\varrho,\tau)\leqslant\phi_R\left(\mathsf{f}(\sigma,\tau)\right)\leqslant\phi\left(\mathsf{f}(\sigma,\tau)\right)$$

$$\mathsf{f}(\varrho,\sigma)-\mathsf{f}(\eta,\sigma)\leqslant\phi_L\left(\mathsf{f}(\varrho,\eta)\right)\leqslant\phi\left(\mathsf{f}(\varrho,\eta)\right)$$

with monotonic subadditive function ϕ s.t. $\phi(0)=0$

Sufficient condition to derive the bound

$$\mathsf{f}(\varrho_S(t),\sigma_S(t))-\mathsf{f}(\varrho_S(s),\sigma_S(s))\leqslant\phi\circ\phi\left(\mathsf{f}(\varrho_E(s),\sigma_E(s))+\phi\left(\mathsf{f}(\varrho(s),\varrho_S(s)\otimes\varrho_E(s)\right)+\phi\left(\mathsf{f}(\sigma(s),\sigma_S(s)\otimes\sigma_E(s)\right)\right)$$

Connecting distinguishability revivals with correlations and environment changes

Quantum divergence → distance $\phi\to 1$

Outline

- Information viewpoint on non-Markovianity in open quantum systems
- **Entropic bounds on information flow**
- Reduced vs microscopic dynamics

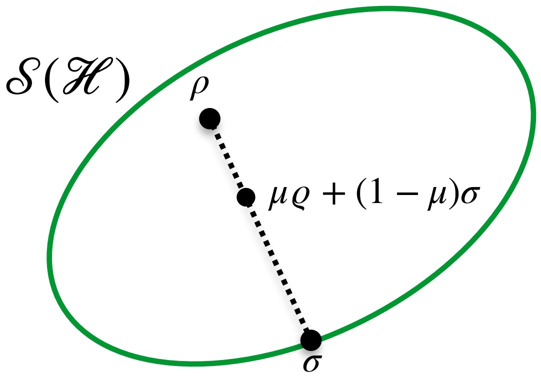
Telescopic relative entropy

Regularize quantum relative entropy

$$S(\varrho, \sigma) = \mathrm{Tr}\varrho \log \varrho - \mathrm{Tr}\varrho \log \sigma$$

$$\mathsf{S}_\mu(\varrho, \sigma) = \frac{1}{\log(1/\mu)} S(\varrho, \mu\varrho + (1-\mu)\sigma)$$

$$\mu \in (0,1)$$



Telescopic relative entropy or quantum skew divergence

Boundedness $\quad 0 \leqslant \mathsf{S}_\mu(\varrho, \sigma) \leqslant 1$

Contractivity under (C)PT maps $\quad \mathsf{S}_\mu(\Phi[\varrho], \Phi[\sigma]) \leqslant \mathsf{S}_\mu(\varrho, \sigma)$

[Audenaert arXiv 2011; Audenaert, JMP 2014]

Entropic divergences

Telescopic relative entropy

Joint convexity

$$\mathsf{S}_\mu(\lambda\rho_1 + (1-\lambda)\rho_2, \lambda\sigma_1 + (1-\lambda)\sigma_2) \leqslant \lambda\mathsf{S}_\mu(\rho_1,\sigma_1) + (1-\lambda)\mathsf{S}_\mu(\rho_2,\sigma_2)$$

Triangle-like inequalities

$$\mathsf{S}_\mu(\varrho,\sigma) - \mathsf{S}_\mu(\eta,\sigma) \leqslant \frac{\mathsf{D}(\varrho,\eta)}{\log(1/\mu)} \log\left(1 + \frac{1}{\mathsf{D}(\varrho,\eta)}\frac{1-\mu}{\mu}\right)$$

$$\mathsf{S}_\mu(\varrho,\sigma) - \mathsf{S}_\mu(\varrho,\tau) \leqslant \frac{1}{\log(1/\mu)} \log\left(1 + \mathsf{D}(\sigma,\tau)\frac{1-\mu}{\mu}\right)$$

[Audenaert arXiv 2011; Audenaert, JMP 2014]

Telescopic relative entropy

Telescopic Pinsker inequality

$$\mathrm{D}(\varrho,\sigma) \leqslant \frac{\sqrt{\log(1/\mu)/2}}{1-\mu}\sqrt{\mathrm{S}_\mu(\varrho,\sigma)}$$

together with estimate $\log(1+x) \leqslant \sqrt{x}$

leads to triangle-like inequalities

$$\mathrm{S}_\mu(\varrho,\sigma) - \mathrm{S}_\mu(\varrho,\tau) \leqslant \phi\left(\mathrm{S}_\mu(\sigma,\tau)\right)$$

$$\mathrm{S}_\mu(\varrho,\sigma) - \mathrm{S}_\mu(\eta,\sigma) \leqslant \phi\left(\mathrm{S}_\mu(\varrho,\eta)\right)$$

with

$$\phi(x) = \kappa(\mu)\sqrt[4]{x} \qquad \kappa(\mu) = 1\Big/\sqrt[4]{2\mu^2\log^3(1/\mu)}$$

[Megier, Smirne & B.V., PRL 2021]

Entropic bound on information flow

Telescopic relative entropy

Straightforward upper bound can be improved to

$$\mathsf{S}_\mu(\varrho_S(t),\sigma_S(t)) - \mathsf{S}_\mu(\varrho_S(s),\sigma_S(s)) \leqslant \kappa(\mu)\Big(\sqrt[4]{\mathsf{S}_\mu(\varrho_E(s),\sigma_E(s))}+ \sqrt[4]{\mathsf{S}_\mu(\varrho(s),\varrho_S(s)\otimes\varrho_E(s))}+\sqrt[4]{\mathsf{S}_\mu(\sigma(s),\sigma_S(s)\otimes\sigma_E(s))}\Big)$$

where

$$\kappa(\mu) = 1\Big/\sqrt[4]{2\mu^2\log^3(1/\mu)}$$

with minimum value

$$\kappa = (4\mathrm{e}^3/27)^{1/4} \approx 1.31 \text{ at } \mu = \mathrm{e}^{-3/2}$$

[Megier, Smirne & B.V., PRL 2021]

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Symmetrized telescopic relative entropy

Boundedness of telescopic relative entropy makes it natural to symmetrize

$$\bar{\mathsf{S}}_\mu(\varrho,\sigma) = \frac{1}{2}\Big(\mathsf{S}_\mu(\varrho,\sigma) + \mathsf{S}_\mu(\sigma,\varrho)\Big)$$

Special value $\mu = 1/2$ recovers quantum Jensen-Shannon divergence

$$\bar{\mathsf{S}}_\mu(\varrho,\sigma) \equiv \mathsf{J}(\varrho,\sigma) = \frac{1}{2}\left(S\left(\varrho, \frac{\varrho+\sigma}{2}\right) + S\left(\sigma, \frac{\varrho+\sigma}{2}\right)\right)$$

Square root of divergence recently proven to be distance

$$\sqrt{\mathsf{J}(\varrho,\sigma)} - \sqrt{\mathsf{J}(\varrho,\tau)} \leqslant \sqrt{\mathsf{J}(\sigma,\tau)}$$

$$\sqrt{\mathsf{J}(\varrho,\sigma)} - \sqrt{\mathsf{J}(\eta,\sigma)} \leqslant \sqrt{\mathsf{J}(\varrho,\eta)}$$

$$\phi(x) = x$$

[Virosztek, AdvMat 2021; Sra, LinAlgAppl 2021]

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Entropic bound on information flow

Quantum Jensen-Shannon divergence

For the symmetrised case and $\mu = 1/2$ taking the square root we can further improve using distance property

$$\sqrt{\mathrm{J}(\varrho_S(t), \sigma_S(t))} - \sqrt{\mathrm{J}(\varrho_S(s), \sigma_S(s))} \leqslant \sqrt{\mathrm{J}(\varrho_E(s), \sigma_E(s))} + \sqrt{\mathrm{J}(\varrho(s), \varrho_S(s) \otimes \varrho_E(s))} + \sqrt{\mathrm{J}(\sigma(s), \sigma_S(s) \otimes \sigma_E(s))}$$

Square root of quantum Jensen-Shannon divergence provides entropy based divergence sharing distance behavior of trace distance but sensitive to non-unital contribution

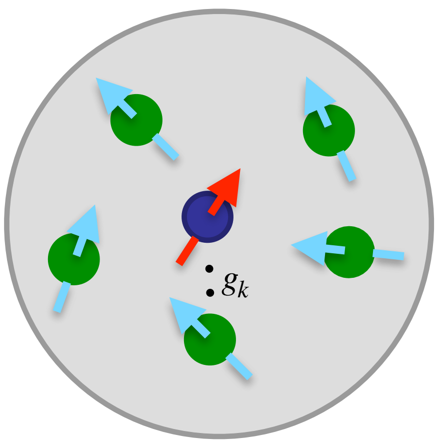
[Megier, Smirne & B.V., PRL 2021]

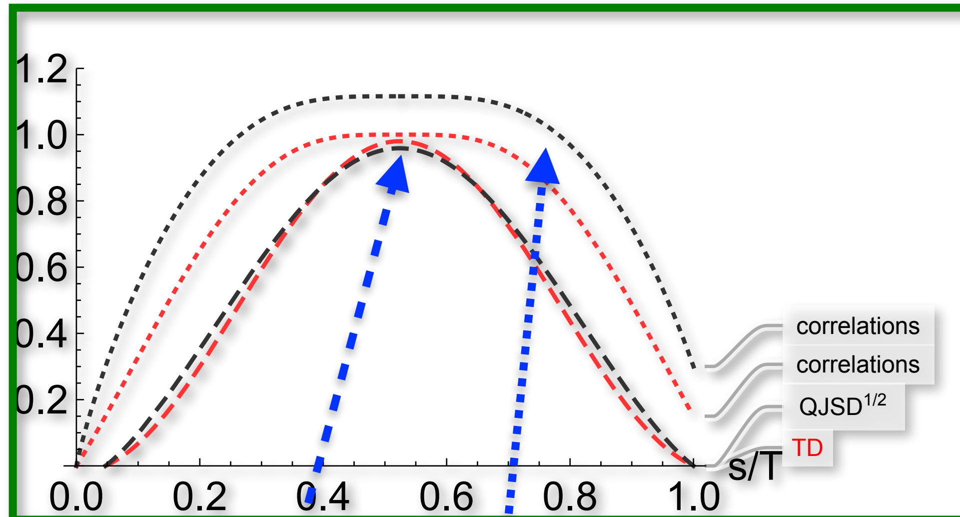
Dephasing spin star model

Revival due to correlations only

Spin star model

$$H_I = \sum_k g_k \sigma_z \otimes \sigma_z^k$$





$$\mathcal{J}_{int}(t) - \mathcal{J}_{int}(s) \leq \mathrm{f} \circ \mathrm{f}(\rho_E^1(s), \rho_E^2(s))$$

$$+\mathrm{f}(\rho_{SE}^1(s), \rho_S^1(s) \otimes \rho_E^1(s))$$

$$+\mathrm{f}(\rho_{SE}^2(s), \rho_S^2(s) \otimes \rho_E^2(s))$$

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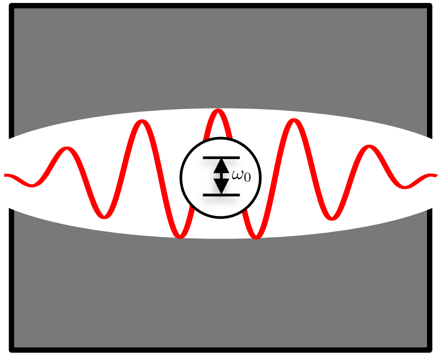
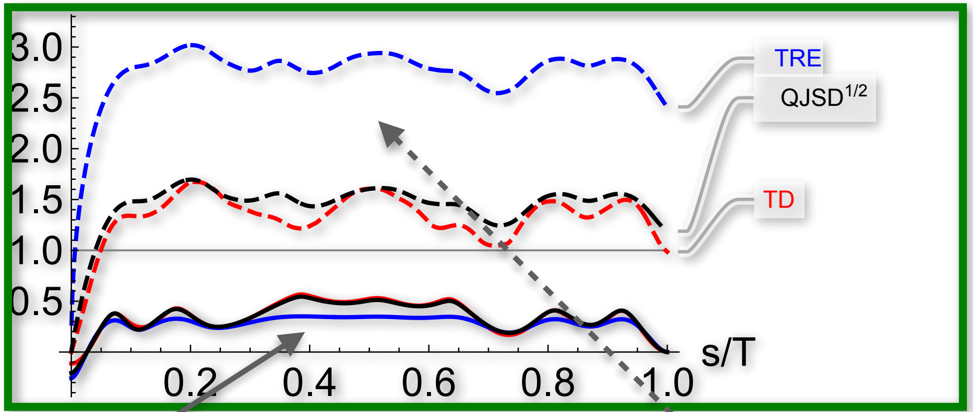
Jaynes-Cummings model

Divergences and bounds comparison

Two-level system interacting with a bosonic mode

$$H_I = g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger)$$





$$f(\varrho_S(T), \sigma_S(T)) - f(\varrho_S(s), \sigma_S(s)) \leqslant \text{Corr} + \text{Corr} + \text{Env}$$

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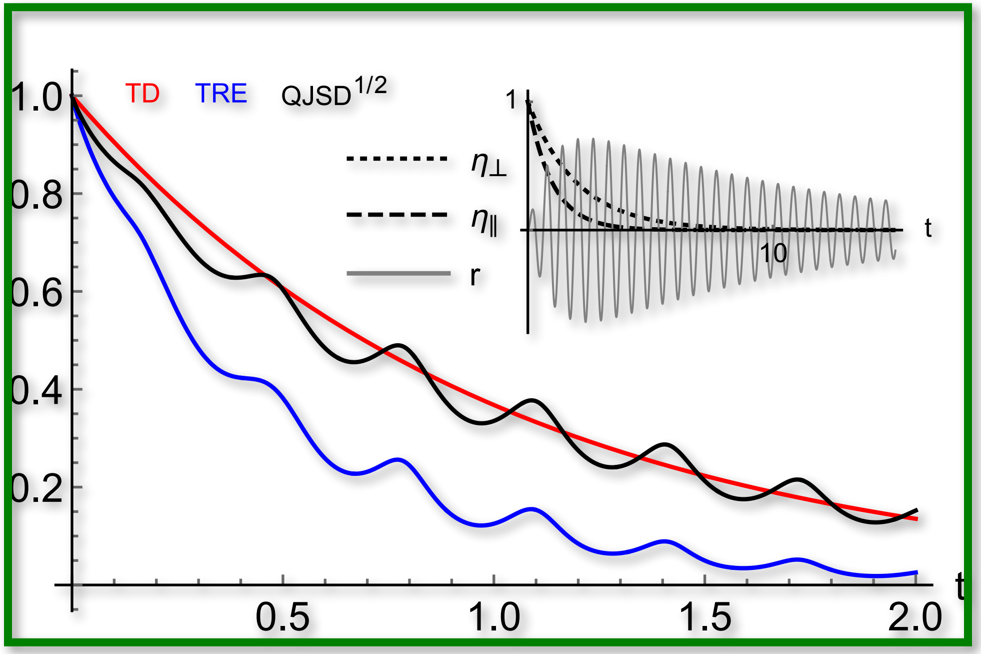
Phase-covariant dynamics

Behavior with respect to translations

Qubit phase covariant dynamics with long-lasting oscillations in non-unital component

$$\varrho(t) = \frac{1}{2}\left[1 + r(t)\sigma_z + \eta_\perp(t)(v_x\sigma_x + v_y\sigma_y) + \eta_\parallel(t)v_z\sigma_z\right] \qquad v_i = \mathrm{Tr}\{\sigma_i\varrho(0)\}$$



Avoid resorting to Generalized trace distance

$$|p_1 - p_2| \leqslant \|p_1\rho_1 - p_2\rho_2\| \leqslant 1$$

$$p_1, p_2 \geqslant 0, \quad p_1 + p_2 = 1$$

[Chruscinski, Kossakowski & Rivas, PRA 2011; Wißmann, Breuer & B.V., PRA 2015]

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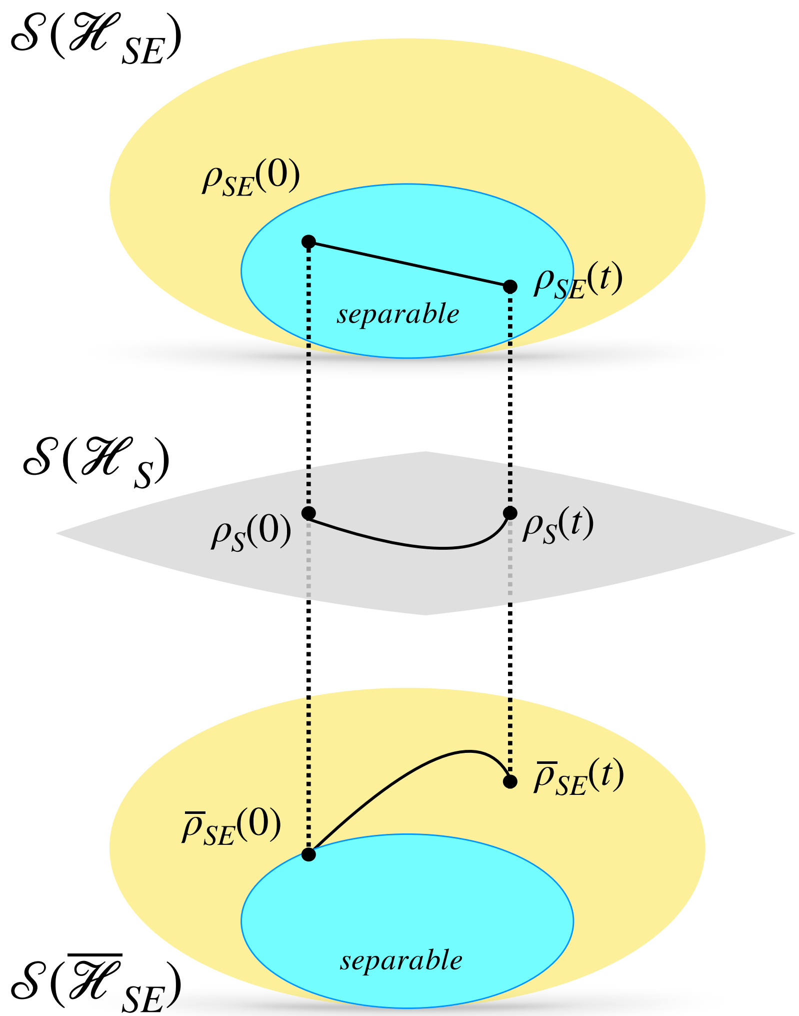
- Entropic bounds on information flow
- **Reduced vs microscopic dynamics**

Reduced vs microscopic description

Relevance and role of microscopic description



Quantum information viewpoint on quantum non-Markovianity based on local observations

Local signature of microscopic dynamics

[Tamascelli, Smirne, Huelga & Plenio, PRL 2018; Smirne, Megier & B.V., QUANTUM 2021]

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Reduced vs microscopic description

Different system-environment interaction

Consider system coupled to the same environment initialized in different states, with different coupling terms

$$H_{SE} = H_S + H_E + H_I \qquad \varrho_E(0)$$
$$H_{SE} = H_S + \bar{H}_E + \bar{H}_I \qquad \bar{\varrho}_E(0)$$

Same reduced dynamics

Constrain reduced dynamics to be the same for all initial system states

$$\mathrm{Tr}_E\{U_{SE}\varrho_S(0) \otimes \varrho_E(0) U_{SE}^\dagger\} = \mathrm{Tr}_E\{\bar{U}_{SE}\varrho_S(0) \otimes \bar{\varrho}_E(0) \bar{U}_{SE}^\dagger\}$$
$$\forall \varrho_S(0) \in \mathcal{S}(\mathscr{H})$$

[Smirne, Megier & B.V., QUANTUM 2021]

Generalized dephasing model

Simplify setting to obtain exact result

$$H_I = \sum_n |n\rangle\langle n| \otimes B_n \qquad \bar{H}_I = \sum_n |n\rangle\langle n| \otimes \bar{B}_n$$

Condition for same reduced dynamics becomes

$$\mathrm{Tr}_E \mathrm{e}^{-iBt} \varrho_E(0) = \mathrm{Tr}_E \mathrm{e}^{-i\bar{B}t} \bar{\varrho}_E(0) \iff \mathrm{Tr}_E B^k \varrho_E(0) = \bar{B}^k \bar{\varrho}_E(0)$$

Satisfied for the choice

$$\rho_E(0) = \frac{1}{2}(1 + \alpha \cdot \sigma) \qquad B = g\eta \cdot \sigma$$

$$\iff \quad \alpha \cdot \eta = \bar{\alpha} \cdot \bar{\eta}$$

$$\bar{\rho}_E(0) = \frac{1}{2}(1 + \bar{\alpha} \cdot \sigma) \qquad \bar{B} = g\bar{\eta} \cdot \sigma$$

[Smirne, Megier & B.V., QUANTUM 2021]

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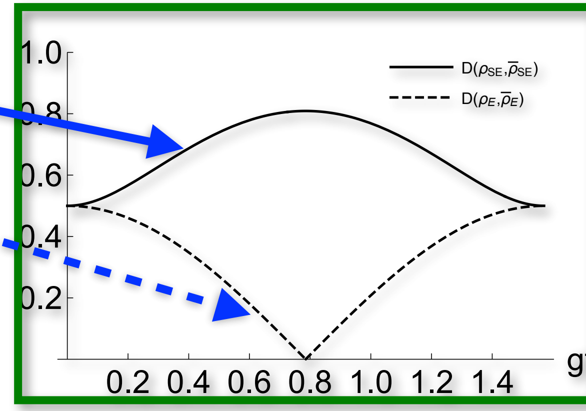
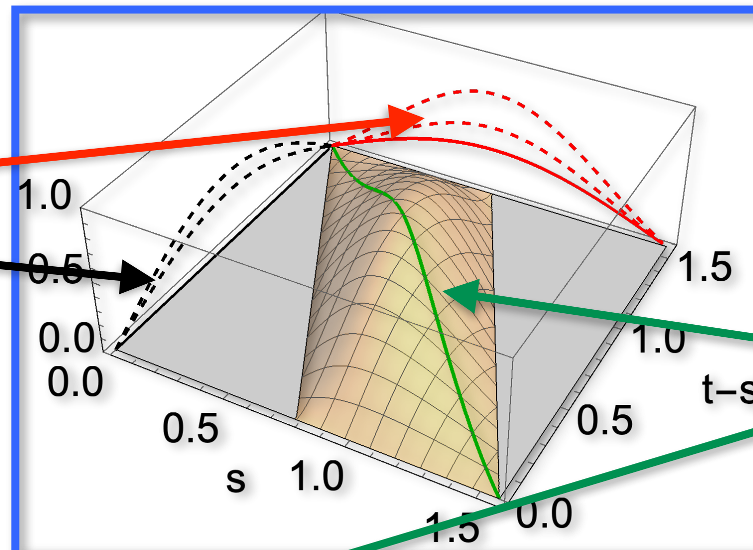
Generalized dephasing model

Different SE dynamics
different E dynamics
different correlations:

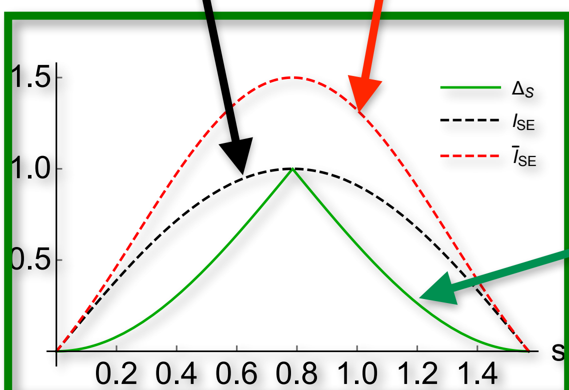
- $\rho_{SE}(t)$ remains zero-discord state
- $\bar{\rho}_{SE}(t)$ exhibits entanglement



Different external information



Identical reduced dynamics



$$\alpha = (0,0,c) \quad \eta = (0,0,1)$$

$$\bar{\alpha} = (0,0,1) \quad \bar{\eta} = (\sqrt{1-c^2},0,c)$$

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Recap

- Non-Markovianity and quantum info viewpoint
- Divergences of distance and entropic type
- Telescopic entropy and Jensen-Shannon
- Same reduced dynamics with different interactions and bath

N. Megier, A. Smirne and B. Vacchini
Entropic bounds on information backflow
arXiv:2101.02720 to appear in PRL

A. Smirne, N. Megier and B. Vacchini
On the connection between microscopic description and memory effects in open quantum system dynamics
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