General results.— We are now in the position to define the the central mathematical object of this paper: *geometric algebra anti-correlator* (GAAC) by

$$G_{\mathcal{A}}(U) := 1 - \frac{\langle \mathbb{P}_{\mathcal{A}'}, \mathbb{P}_{\mathcal{U}(\mathcal{A}')} \rangle}{\|\mathbb{P}_{\mathcal{A}'}\|_{HS}^2}. \tag{2}$$

The geometrical meaning of GAAC should be evident from Eq. (2): the larger $G_{\mathcal{A}}(U)$ the smaller is the intersection between $\mathcal{A}'$ and its unitarily evolved image $\mathcal{U}(\mathcal{A}')$ [20]. Algebraically, (2) measures how much the symmetries of the generalized quantum subsystem associated to $\mathcal{A}$ are dynamically broken by the channel $\mathcal{U}$. From Eq. (2 it easily follows

Proposition 1. *i)* $G_{\mathcal{A}}(U) = 0 \Leftrightarrow \mathcal{U}(\mathcal{A}') = \mathcal{A}' \Leftrightarrow \mathcal{U}(\mathcal{A}) = \mathcal{A}$. *In words: the GAAC Eq. (2) vanishes if and only if both algebras* $\mathcal{A}$ *and* $\mathcal{A}'$ *are invariant under* $\mathcal{U}$ *i.e., there is no algebra scrambling. ii) If the pair* $(\mathcal{A}, \mathcal{A}')$ *is collinear then* $G_{\mathcal{A}}(U) = G_{\mathcal{A}'}(U)$, $(\forall U)$.

Proposition 2.

$$G_{\mathcal{A}}(U) = \frac{1}{2} \frac{D^2\left(\mathcal{A}', \mathcal{U}(\mathcal{A}')\right)}{d(\mathcal{A}')} \tag{3}$$

In words: the GAAC measures the (squared and normalized) distance between the algebra $\mathcal{A}'$ *and its image* $\mathcal{U}(\mathcal{A}')$.

tions. In fact, a short manipulation of Eq. (4) shows that

$$1 - G_{\mathcal{A}}(U) = \frac{1}{d(\mathcal{A}')} \sum_{\alpha,\beta=1}^{d(\mathcal{A})} |\langle e_\alpha, \mathcal{U}(e_\beta)\rangle|^2, \qquad (6)$$

1) Now we consider a bipartite quantum system with $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{A} = L(\mathcal{H}_A) \otimes \mathbb{1}_B$ and, therefore, $\mathcal{A}' = \mathbb{1}_A \otimes L(\mathcal{H}_B)$. In this case one finds that $\mathbb{P}_{\mathcal{A}'}(X) = \frac{\mathbb{1}}{d_A} \otimes \mathrm{Tr}_A(X)$, $\Omega_{\mathcal{A}} = \frac{S_{AA'}}{d_A}$, where $S_{AA'}$ is the swap operator between the A factors in $\mathcal{H}^{\otimes 2}$ and $d_X = \dim \mathcal{H}_X$ $(X = A, B)$. Using Eq. (4) one gets

$$G_{\mathcal{A}}(U) = 1 - \frac{1}{d^2} \langle S_{AA'}, \mathcal{U}^{\otimes 2}(S_{AA'})\rangle, \qquad (7)$$

where $d = d_B d_A = \dim \mathcal{H}$. The same relation is true with $S_{AA'} \to S_{BB'} = S S_{AA'} = d_A \tilde{\Omega}_{\mathcal{A}}$.

Eq. (7) *coincides exactly* with he averaged OTOC discussed in [9] i.e., $d^{-1}\mathbb{E}_{X\in\mathcal{A}, Y\in\mathcal{A}'}\left[\|[X, \mathcal{U}(Y)]\|_2^2\right]$ (here $\mathbb{E}$ denotes an Haar average over the unitary groups of $\mathcal{A}$ and $\mathcal{A}'$.). Remarkably, this quantity was shown to be equal to the *operator entanglement* [6, 22] of the unitary U. The latter concept (and variations thereof) has found important applications to a variety of quantum information-theoretic problems [7, 23–26]

2) Let $\mathcal{A}_B$ the algebra of operators which are diagonal with respect to an orthonormal basis $B := \{|i\rangle\}_{i=1}^d$ i.e., $\mathcal{A} = \mathbf{C}\{\Pi_i := |i\rangle\langle i|\}_{i=1}^d$. This is a d-dimensional maximal abelian subalgebra of $L(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}'$. In this case $\mathbb{P}_{\mathcal{A}'}(X) = \sum_{i=1}^d \Pi_i X \Pi_i$, $\Omega_{\mathcal{A}} = \sum_{i=1}^d \Pi_i^{\otimes 2}$, and, using Eq. (4) again, one finds

$$G_{\mathcal{A}_B}(U) = 1 - \frac{1}{d} \sum_{i,j=1}^{d} |\langle i|U|j\rangle|^4, \qquad (8)$$

This relation shows that in this case the GAAC is nothing but the *coherence generating power* (CGP) of U defined as the average coherence (measured by the the sum of the square of off-diagonal elements, with respect B,) generated by U starting from any of the pure incoherent states Π i.e., $G_{\mathcal{A}_B}(U) =$

$\frac{1}{d}\sum_{i=1}^{d}\|\mathbb{Q}_B\mathcal{U}(\Pi_i)\|_2^2$, where $\mathbb{Q} = 1 - \mathbb{P}_{\mathcal{A}_B}$ is the projector onto the orthogonal complement of $\mathcal{A}_B$. [10, 21].

5) Let $|\psi\rangle \in \mathcal{H}$ and $\Pi = |\psi\rangle\langle\psi|$. We define $\mathcal{A}_{LE} = \mathbf{C}\{\mathbb{1}, \Pi\}$ The commutant $\mathcal{A}'_{LE}$ is the algebra of operators leaving the subspace $\mathbf{C}|\psi\rangle$ and its orthogonal complement invariant, $d(\mathcal{A}'_{LE}) = (d-1)^2+1$. One has, $\Omega_{\mathcal{A}_{LE}} = \Pi^{\otimes 2} + (\mathbb{1}-\Pi)^{\otimes 2}$ and, using (4), one finds

$$G_{\mathcal{A}_{LE}}(U) = \frac{2(1-\mathcal{L}^2)[d-2(1-\mathcal{L}^2)]}{(d-1)^2+1}, \qquad (11)$$

where $\mathcal{L} := |\langle\psi|U|\psi\rangle|$ is the Loschmidt echo. Notice, $G_{\mathcal{A}_{LE}}(U) = \frac{2}{d}(1-\mathcal{L}^2) + O(1/d^2)$ and that $2(1-\mathcal{L}^2) = \|\Pi - \mathcal{U}(\Pi)\|_2^2$. That is to say : the distance between the algebras $\mathcal{A}'_{LE}$ and its image $\mathcal{U}(\mathcal{A}'_{LE})$, as captured by the GAAC [see Eq. (3)], in high dimension is directly related to the Hilbert-Schmidt distance between the states Π and $\mathcal{U}(\Pi)$. From Eq. (11) one can see that the GAAC is a monotonic decreasing function of $\mathcal{L}$ for $d > 4$ and that $\mathcal{L} = 1 \Rightarrow G_{\mathcal{A}_{LE}}(U) = 0$ [29]

Proposition 4. *i)*

$$G_{\mathcal{A}}(U) \leq \min\{1 - \frac{1}{d(\mathcal{A})}, 1 - \frac{1}{d(\mathcal{A}')}\} =: G_{UB}(\mathcal{A}) \quad (12)$$

ii) if $d(\mathcal{A}') \leq d(\mathcal{A})$ then the bound above is achieved iff $\mathbb{P}_{\mathcal{A}'}\mathcal{U}\mathbb{P}_{\mathcal{A}'} = \mathcal{T}$ where $\mathcal{T}: X \mapsto \mathrm{Tr}(X)\frac{\mathbb{1}}{d}$. iii) If $\mathcal{A}'$ is Abelian the bound $1 - \frac{1}{d(\mathcal{A}')}$ is always achieved. iv) In the collinear case ii) and iii) above hold true with $\mathcal{A} \leftrightarrow \mathcal{A}'$.

Proposition 5.

$$i) \qquad \overline{G_{\mathcal{A}}(U)}^{U} = \frac{(d^2 - d(\mathcal{A}'))(d(\mathcal{A}') - 1)}{d(\mathcal{A}')(d^2 - 1)} \tag{13}$$

ii) $\mathbf{Prob}_U \left[|G_{\mathcal{A}}(U) - \overline{G_{\mathcal{A}}(U)}^{U}| \geq \epsilon \right] \leq \exp[-\frac{d\epsilon^2}{4K^2}]$.

iii) In the collinear case $G_{UB}(\mathcal{A}) - \overline{G_{\mathcal{A}}(U)}^{U} = O(1/d)$ *and* $\mathbf{Prob}_U \left[G_{UB}(\mathcal{A}) - G_{\mathcal{A}}(U) \geq d^{-1/3} \right] \leq \exp[-\frac{d^{1/3}}{16K^2}]$.
In ii) and iii) one can choose $K \geq 40$.

Proposition 6. $\overline{G_{\mathcal{A}}(U_t)}^{t} \leq \overline{G_{\mathcal{A}}(U_t)}^{NRC} \leq \overline{G_{\mathcal{A}}(U)}^{U}$ *where*

$$1 - \overline{G_{\mathcal{A}}(U_t)}^{NRC} = \frac{1}{d(\mathcal{A}')} \sum_{\alpha=0,1} \left[\|R^{(\alpha)}\|_2^2 - \frac{1}{2} \|R_D^{(\alpha)}\|_2^2 \right] \tag{14}$$

$R_{lk}^{(0)} := \|\mathbb{P}_{\mathcal{A}'}(|\Psi_l\rangle\langle\Psi_k|)\|_2^2$, $R_{lk}^{(1)} := \langle \mathbb{P}_{\mathcal{A}'}(\Pi_l), \mathbb{P}_{\mathcal{A}'}(\Pi_k) \rangle$, *and* $(R_D^{(\alpha)})_{lk} := \delta_{lk} R_{lk}^{(\alpha)}$, $(l, k = 1, \ldots, d)$. *Moreover, the first inequality in i) becomes an equality if H fulfills the so-called Non Resonance Condition (NRC)[30]*

In fact, it is rather tempting to *define* $\mathcal{A}$-chaotic the dynamics generated by U_t's such the (relative) difference between its infinite-time average and the Haar-average of the GAAC is approaching zero sufficiently fast as the system dimension grows. More formally,

$$1 - \overline{G_{\mathcal{A}}(U_t)}^{t} / \overline{G_{\mathcal{A}}(U)}^{U} =: \epsilon = O(d^{-\gamma}) \quad (\gamma \geq 1). \tag{17}$$

In particular, in the collinear case, this condition would allow one to prove the “equilibration" result for the GAAC (16). The intuition behind this definition is quite simple: if Eq. (17) holds the long time behavior of the GAAC gets, as the system dimension grows, quickly indistinguishable from the one of a typical Haar random unitary.

Before concluding, we would like to illustrate $\mathcal{A}$-chaos with the simple Loschmidt case **5)**. Here one has $\epsilon = \overline{\mathcal{L}_t^2}^t + O(1/d)$ where $\mathcal{L}_t = |\langle\psi|U_t|\psi\rangle|$. The infinite-time average is given by the purity of the Hamiltonian dephased state $\overline{\mathcal{L}_t^2}^t = \|\overline{\mathcal{U}_t(|\psi\rangle\langle\psi|)}^t\|_2^2$ [32] Whence the "chaoticity" condition is achieved if this purity is $O(1/d)$ which in turn implies that the dephased state is $O(1/d)$ away from the the maximally mixed state. Interestingly, this condition is known to be a sufficient one for temporal equilibration of many observables with initial state $|\psi\rangle$ [32, 33]

Conclusions.— In this paper we have proposed a novel approach to quantum scrambling based on algebras of observables. A quantitative measure of scrambling for is introduced in terms of anti-correlation between the whole algebra and its (unitarily) evolved image.

This quantity, which we named the Geometric Algebra Anti Correlator (GAAC) has a clear geometrical meaning as it described the distance between the algebras or, equivalenty, the degree of self-orthogonalization induce by the dynamics.

We explicitly computed the GAAC for several physically motivated cases and characterized its behavior in terms of typical values, upper bounds and temporal fluctuations.

One of the main result is to show that the the GAAC formalism generalizes concepts like operator entanglement, averaged bipartite OTOC, coherence power and Loschmdt echo. Finally, we suggested a GAAC based approach to quantum chaos in terms of the asymptotic behavior of GAAC for large system dimension. To prove the effectiveness of such and approach is one of the challenges of future investigations.