Quasi-Inversion of Quantum Channels

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The lesson I learnt from him.

A lesson which transcends all kinds of borders.

For qubit channels

Vahid Karimipour, Fabio Benatti, Roberto Floreanini, Phys. Rev. A(2019)

Fabio Benatti

Roberto Floreanini

For arbitrary dimensions arXiv:2105.14581

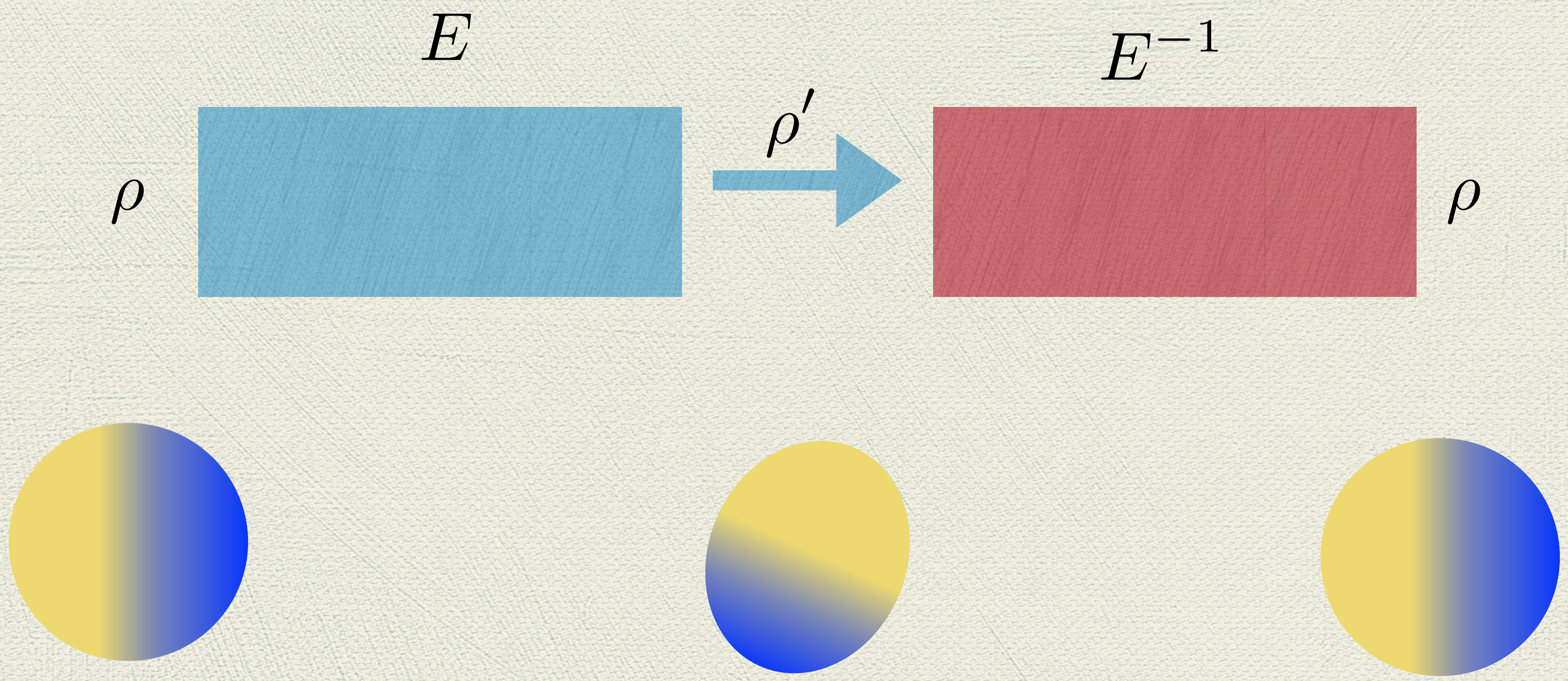
Fereshte Shahbeigi

Koorosh Sadri

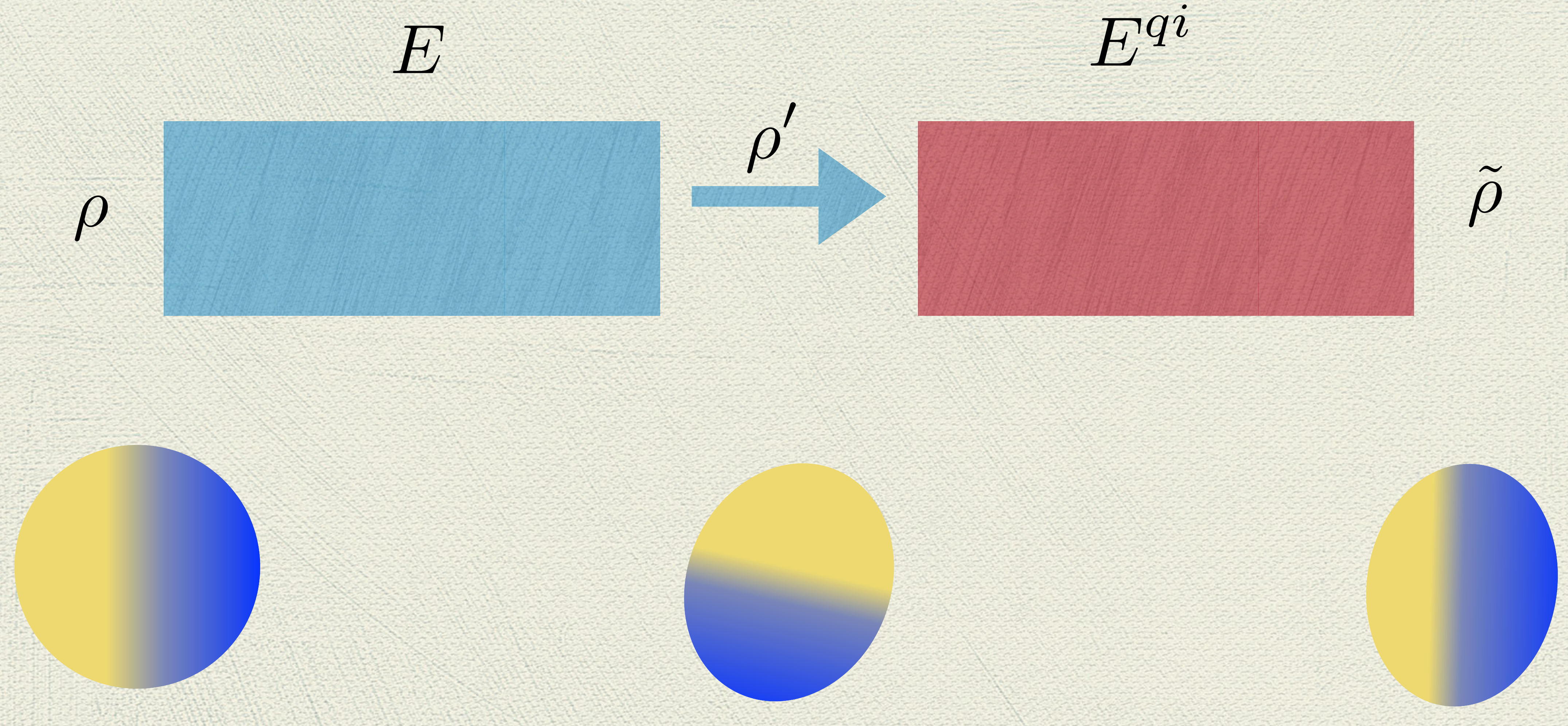
Morteza Moradi

Karol Życzkowski

A quantum channel cannot be inverted.



Can a quantum channel be quasi-inverted?



Formal Definition of the Quasi-Inverse

$\overline{F}(E^{qi} \circ E) > \overline{F}(E)$ 1-The average fidelity should increase

$\overline{F}(E^{qi} \circ E) \geq \overline{F}(E' \circ E) \quad \forall E'$ 2-Better than any other channel

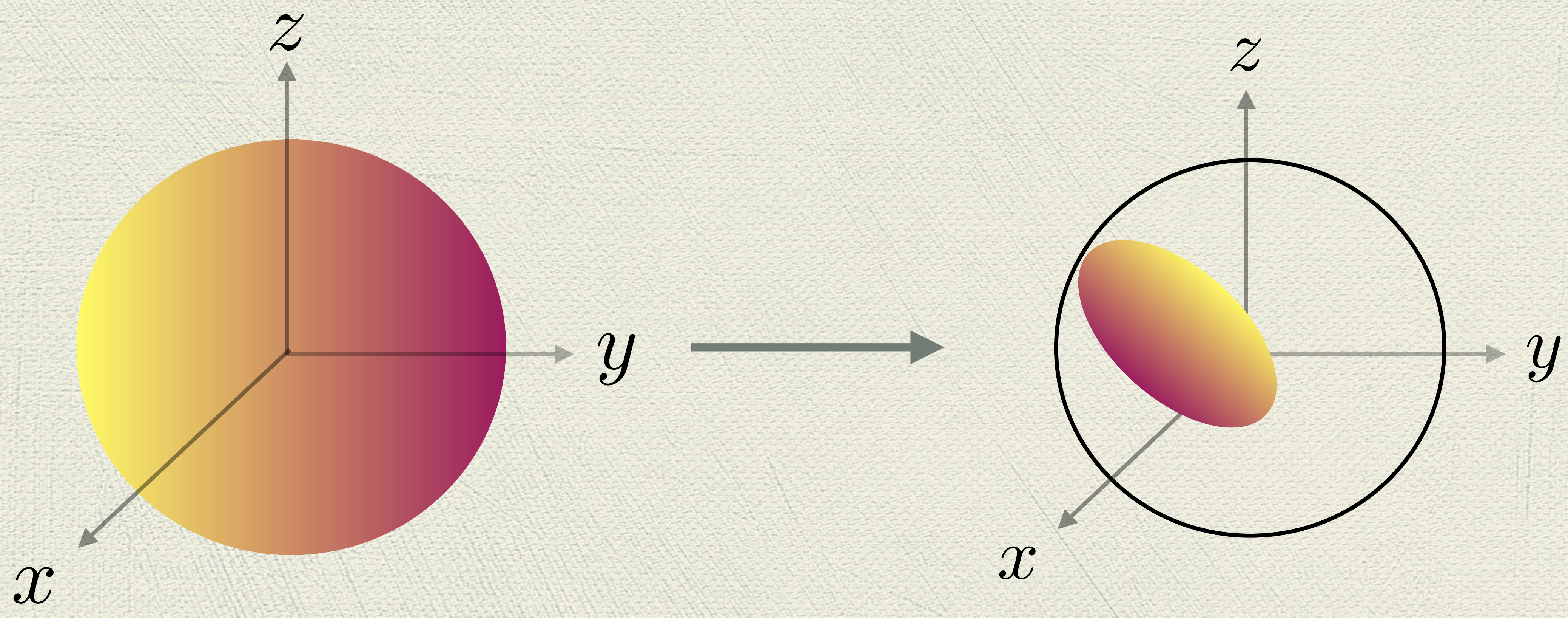
Qubit Channels

One of the joys of life is to listen to things you already know.

Structure of qubit channels

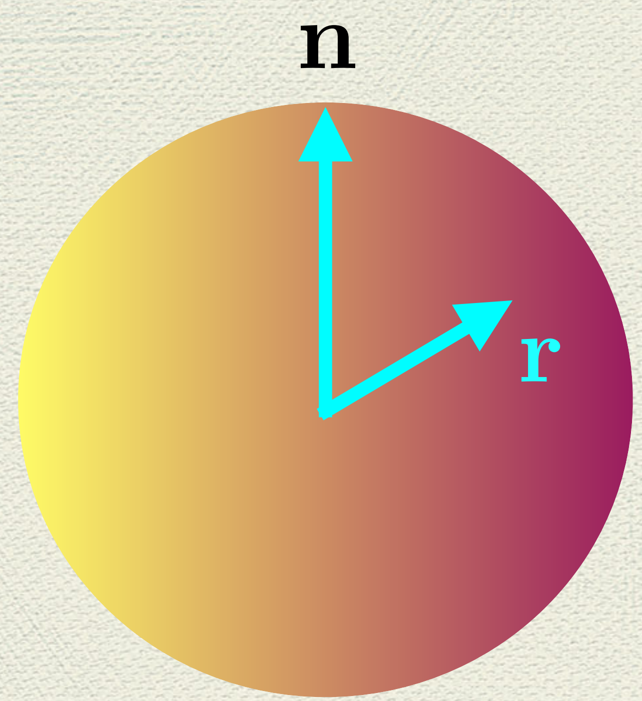
$$\rho = \frac{1}{2}(I + \mathbf{r}\cdot\sigma) \longrightarrow \rho' = \frac{1}{2}(I + \mathbf{r}'\cdot\sigma)$$

$$\mathbf{r}' = M\mathbf{r} + \mathbf{t}$$



Average Fidelity of a Channel

$$\overline{F}(\mathcal{E}) = \int \langle\psi|\; \mathcal{E}(|\psi\rangle\langle\psi|)\; |\psi\rangle$$



$$\mathbf{r}' = M\mathbf{r} + \mathbf{t}$$

$$\overline{F}(\mathcal{E}) = \frac{1}{2}(1 + \frac{1}{3} TrM)$$

$$(M, t) \longrightarrow (M^{qi}, t^{qi})$$

$$=$$

$$(M^{qi}M, M^{qi}t + t^{qi})$$

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) = \frac{1}{2}(1 + \frac{1}{3} Tr M^{qi} M)$$

Maximize this.

The average fidelity is a linear function on the convex set of all channels.

$$\overline{F}(E) = \int d\phi \langle \phi | E(\rho) | \phi \rangle$$

So it achieves its maximum on the extreme points of this set.

$$\mathcal{E}^{qi} = \sum_i \lambda_i \mathcal{E}_i$$

The quasi inverse of a qubit channel can be chosen to be unital.

if (M^{qi}, t^{qi}) is a channel $\rightarrow$ $(M^{qi}, 0)$ is also a channel

This is only true for qubit channels.

$\rightarrow$ A qubit channel which is both extreme and unital is a unitary map.

The explicit form of the quasi-inverse

$$\mathcal{E} = \mathcal{U} \circ \mathcal{E}_c \circ \mathcal{V}$$

$$M = S\Lambda T$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3$$

The explicit form of the quasi-inverse

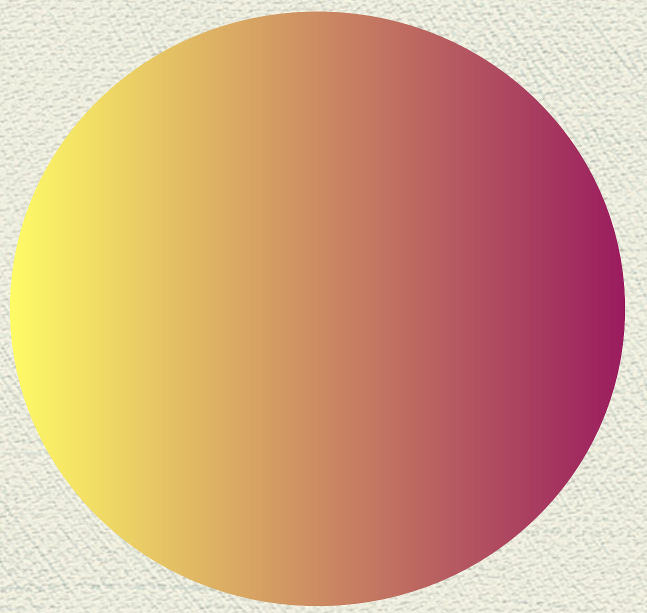
$$\lambda_2 + \lambda_3 \geq 0 \longrightarrow R^{qi} = T^{-1}S^{-1}$$

$$\lambda_2 + \lambda_3 < 0 \longrightarrow R^{qi} = T^{-1}XS^{-1}$$

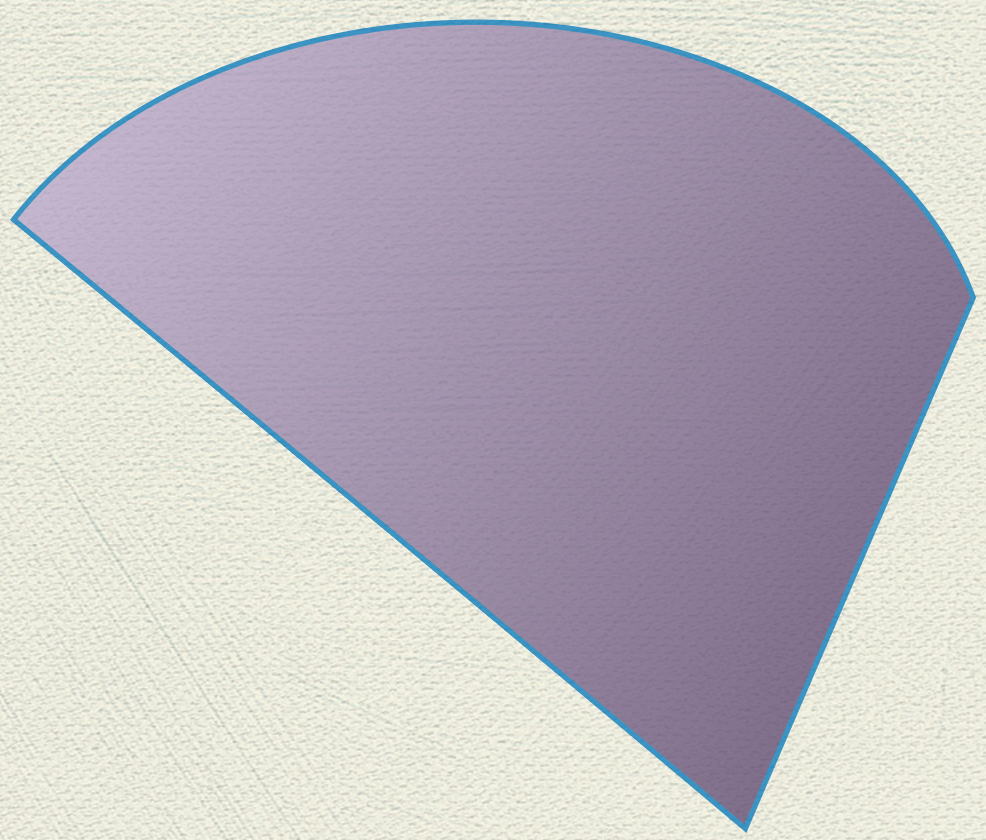
$$X = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

Higher dimensional Channels

1- We do not have the simple Bloch Sphere,



Qubits



D-dimensional states

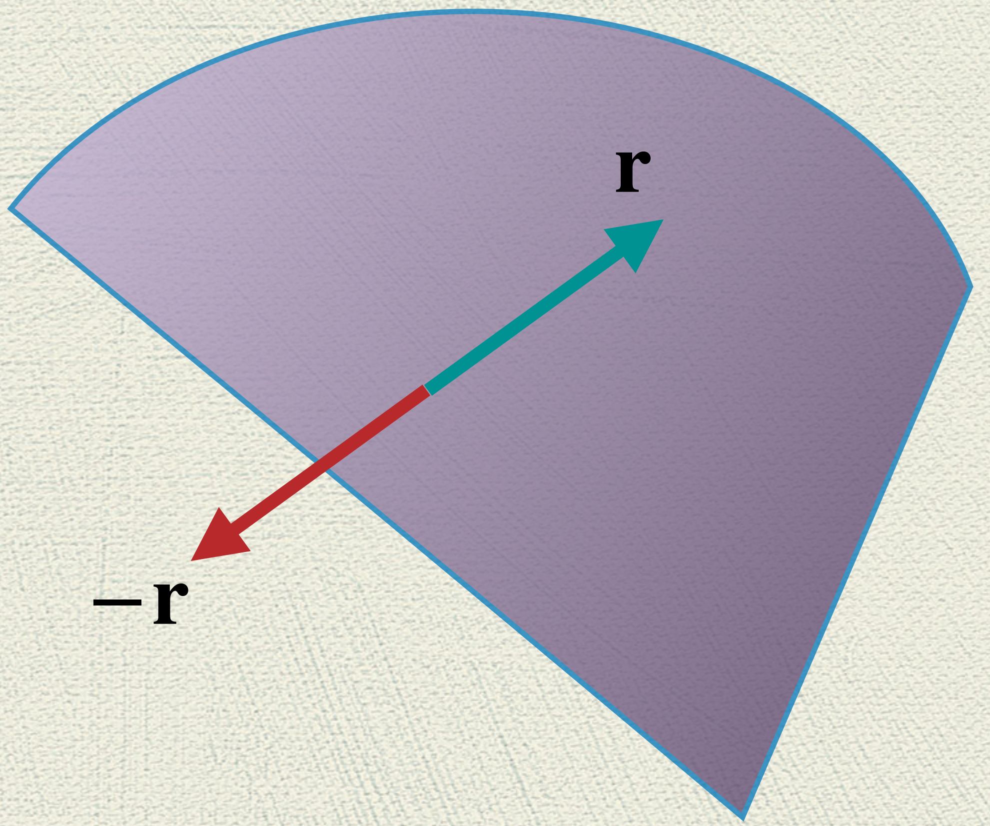
$$\rho = \frac{1}{d}(I + \mathbf{r} \cdot \mathbf{\Gamma})$$

$$Tr(\Gamma_i \Gamma_j) = d(d-1)\delta_{i,j}$$

2- The set is not symmetric anymore

$$\rho = \frac{1}{d}(I + \mathbf{r} \cdot \mathbf{\Gamma})$$

$$H = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

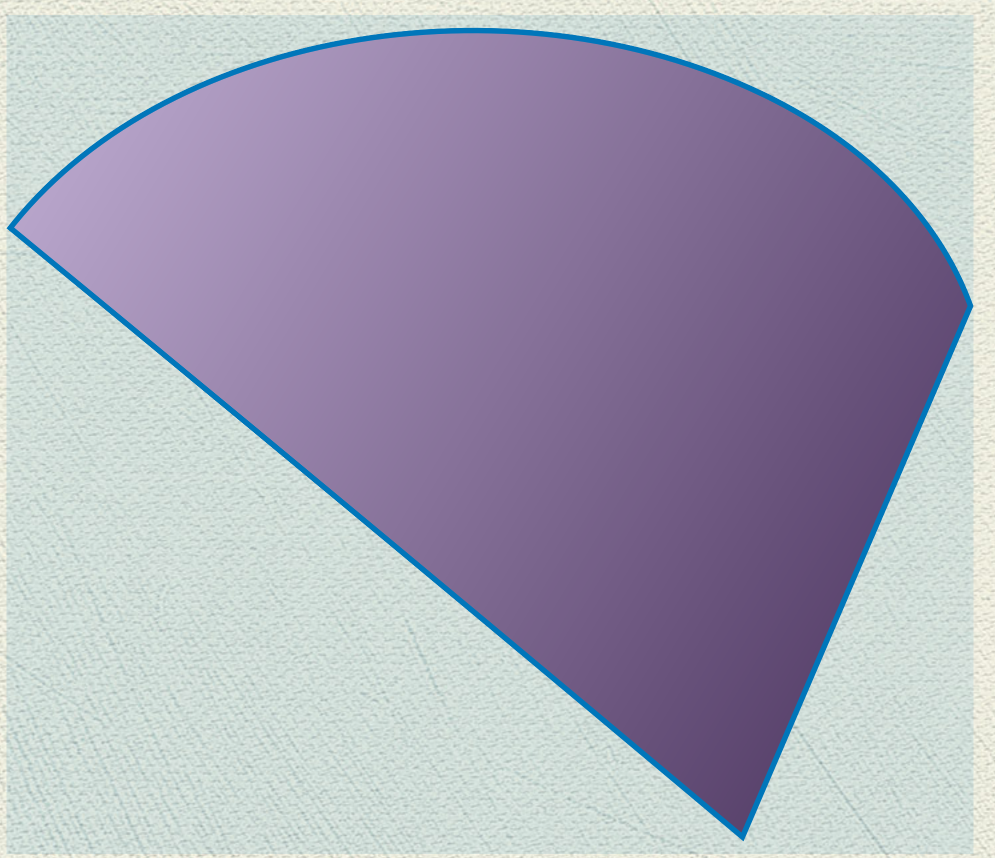


$$\frac{1}{3}(I - H) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$\frac{1}{3}(I + H) = \frac{1}{3}\begin{pmatrix} 2 & & \\ & 2 & \\ & & -1 \end{pmatrix}$$

3-Unitalization is not possible

$$(M^{qi}, t^{qi}) \not\longrightarrow (M^{qi}, 0)$$



1-The quasi-inverse is again an extreme channel.

The dynamical and the Choi-Jamiolkowski Matrix

$$\mathcal{E}(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$$

The Choi Matrix

$$C_\mathcal{E} = \sum_\alpha |K_\alpha\rangle\langle K_\alpha|$$

$$|K_\alpha\rangle = \sqrt{d}(K_a \otimes I)|\phi^+\rangle$$

The Dynamical Matrix

$$\Phi_\mathcal{E} = \sum_\alpha K_\alpha \otimes K_\alpha^*$$

The average fidelity

$$\overline{F}(\mathcal{E}) = \int_{S_{2d-2}} \langle\psi|\ \mathcal{E}(|\psi\rangle\langle\psi|)\ |\psi\rangle$$

$$\overline{F}(\mathcal{E}) = \frac{1}{d(d+1)}\left[d + Tr\ \Phi_{\mathcal{E}}\right]$$

$$\overline{F}(\mathcal{E}^{qi}\mathcal{E}) = \frac{1}{d(d+1)}\left[d + Tr\ \Phi_{\mathcal{E}^{qi}}\Phi_{\mathcal{E}}\right]$$

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) = \frac{1}{d(d+1)}\left[d + Tr\ \Phi_{\mathcal{E}^{qi}} \Phi_{\mathcal{E}}\right]$$

Goal: Find quantum channels

$$\mathcal{E}'$$

which maximize

$$Tr(\Phi'_{\mathcal{E}} \Phi_{\mathcal{E}})$$

A basic theorem

$$\boxed{Tr(\Phi_{\mathcal{E}'}\Phi_{\mathcal{E}}) \leq d\lambda_{max}}$$

$$\lambda_{max} = \lambda_{max}(C_{\mathcal{E}})$$

$$Tr(\Phi_{\mathcal{E}}^{qi}\Phi_{\mathcal{E}}) = d\lambda_{max}$$

Example one: Mixture of orthogonal Unitaries

$$\mathcal{E}(\rho) = \sum_{\alpha=1}^{r} p_\alpha \; V_\alpha \rho V_\alpha^\dagger \qquad p_m \geq p_i \quad \forall \; i \, .$$

$$C_{\mathcal{E}} = \sum_\alpha p_\alpha |V_\alpha\rangle\langle V_\alpha|$$

$$\boxed{\mathcal{E}^{qi}(\rho) = V_m^\dagger \rho V_m \qquad \lambda_{max} = p_m}$$

Example Two: Uniform mixture of orthogonal conjugations

$$\mathcal{E}(\rho) = \sum_{\alpha=1}^{q} X_\alpha \rho X_\alpha^\dagger$$

$$\mathcal{E}^{qi}(\rho) = \sum_{\alpha=1}^{q} X_\alpha^\dagger \rho X_\alpha$$

$$\boxed{\overline{F}(\mathcal{E}) = \frac{1}{d+1} \qquad \overline{F}(\mathcal{E} \circ \mathcal{E}) = \frac{1+\frac{d}{q}}{1+d}}$$

Channels which are their own inverse: Landau-Streater channel

$$\mathcal{E}(\rho) = \frac{1}{j(j+1)}\big(J_x\rho J_x + J_y\rho J_y + J_z\rho J_z\big)$$

$$\mathcal{E}_\mu(\rho) = \frac{1}{C_\mu}\sum_{\alpha=1}^{n} D_\mu(T_\alpha)\rho D_\mu^\dagger(T_\alpha)$$

Example Three: Transpose-depolarizing channel

$$\mathcal{E}_p^{td}(\rho) = (1-p)\rho^T + \frac{p}{d} Tr(\rho) I$$

$$\frac{d}{d+1} \leq p \leq \frac{d}{d-1}$$

$p = \frac{d}{d+1}$  $p = \frac{d}{d-1}$

$$\mathcal{E}_p^{td}(\rho) = (1-p)\rho^T + \frac{p}{d} Tr(\rho) I$$

$$\mathscr{E}_+(\rho) = \frac{1}{d+1}(Tr(\rho)I + \rho^T)$$

$$\mathscr{E}_-(\rho) = \frac{1}{d-1}(Tr(\rho)I - \rho^T)$$

$$p = \frac{d}{d+1}$$

$$p = \frac{d}{d-1}$$

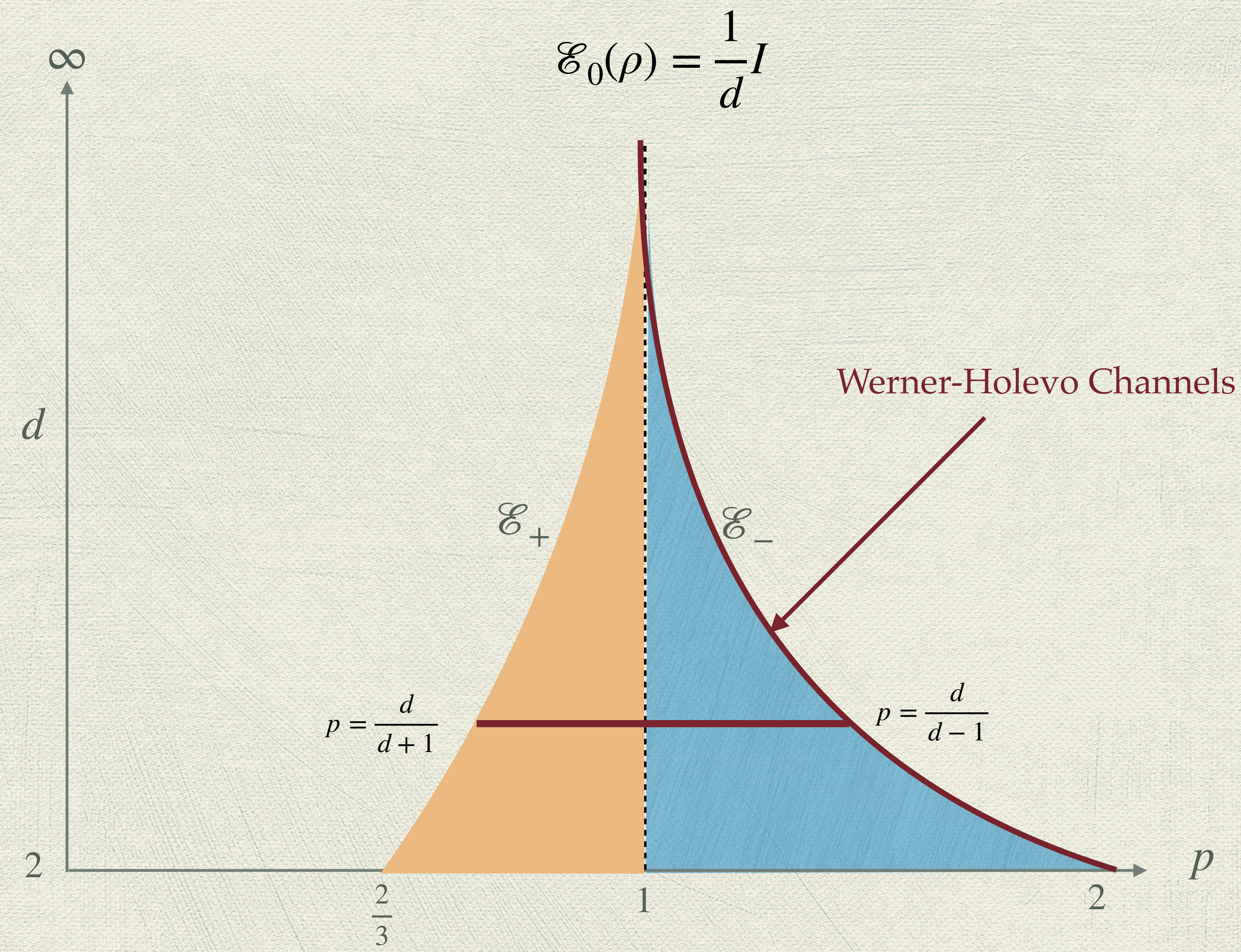
Werner-Holevo Channel

$$\mathcal{E}_p^{td}(\rho) = (1-p)\rho^T + \frac{p}{d}Tr(\rho)I$$

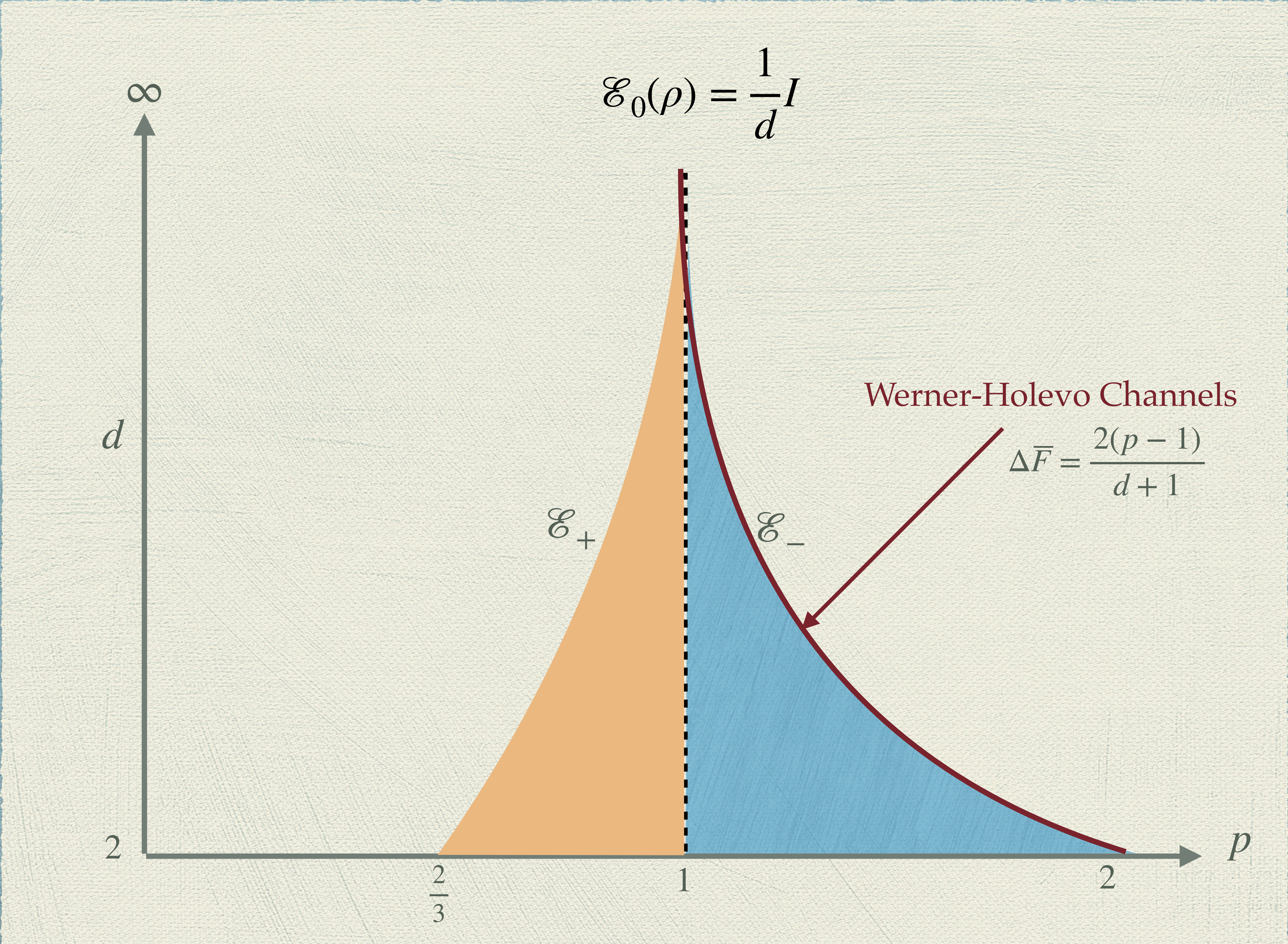
$$\Phi_p^{td} = (1-p)S + p|\phi^+\rangle\langle\phi^+|$$

$$C_p^{td} = (1-p)S + \frac{p}{d}I \otimes I$$

$$\lambda_{max} = (1-p) + \frac{p}{d}$$



$\mathcal{E}_0(\rho) = \frac{1}{d} I$
∞
d
Werner-Holevo Channels
$\mathcal{E}_+$
$\mathcal{E}_-$
$p = \frac{d}{d+1}$
$p = \frac{d}{d-1}$
2
p
$\frac{2}{3}$
1
2

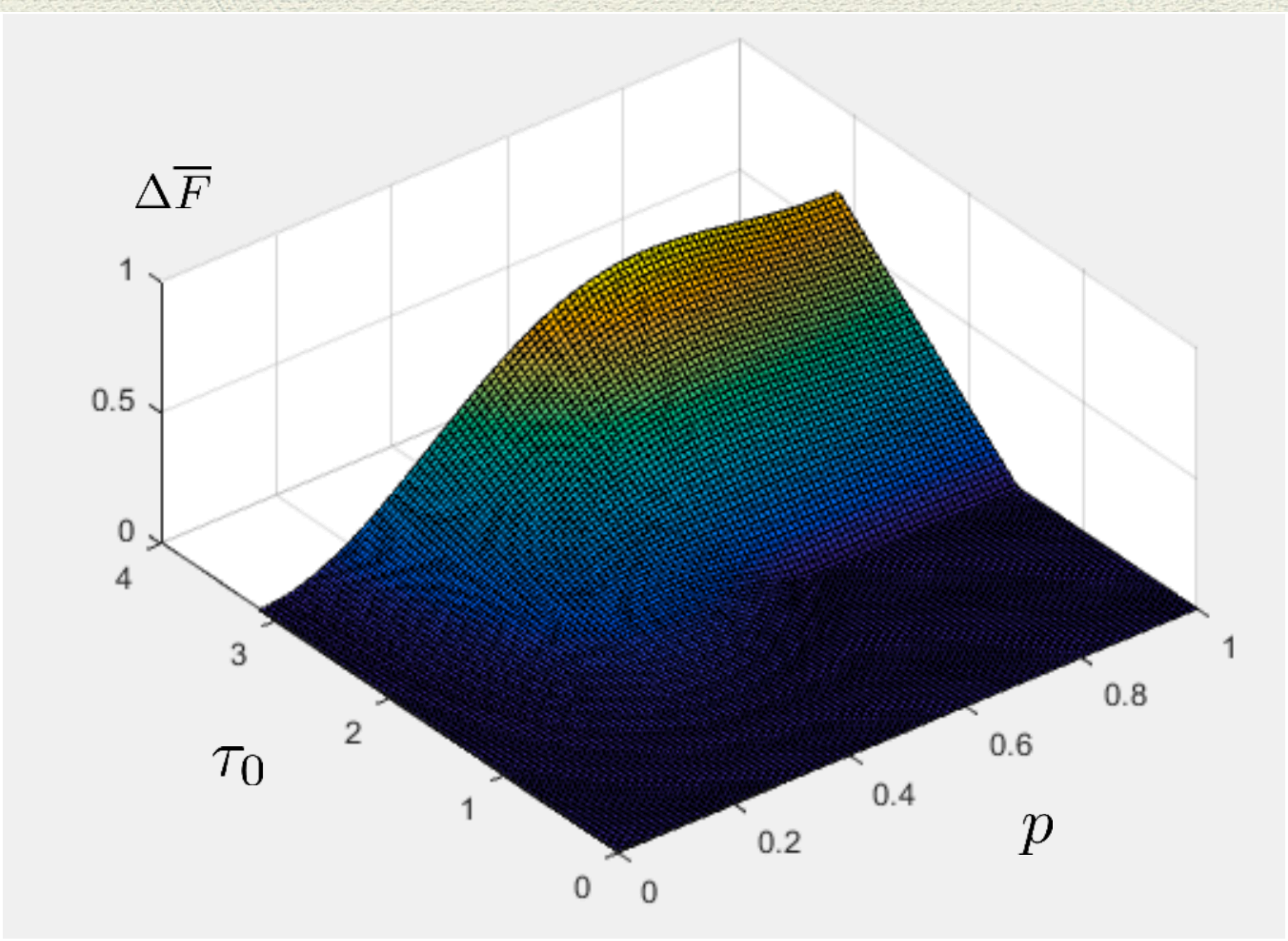


$\mathscr{E}_0(\rho) = \frac{1}{d}I$
∞
d
2
Werner-Holevo Channels
$\Delta\overline{F} = \frac{2(p-1)}{d+1}$
$\mathscr{E}_+$
$\mathscr{E}_-$
p
$\frac{2}{3}$
1
2

Example four: Mixture of commuting unitaries

$$\mathcal{E}(\rho) = \int d\tau f(\tau) U(\tau) \rho U^\dagger(\tau)$$

$$U(\tau) = \begin{pmatrix} e^{i\tau} & & \\ & 1 & \\ & & e^{-i\tau} \end{pmatrix}$$



Classical Channels

Stochastic Matrices

Classical Channels

$$|P\rangle \longrightarrow |P'\rangle = T|P\rangle$$

Probability vector

Stochastic Matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdot & \cdot & T_{13} \\ T_{21} & T_{22} & \cdot & \cdot & T_{24} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T_{n1} & T_{n2} & \cdot & \cdot & T_{nn} \end{pmatrix}$$

=1

=1

The inverse of a stochastic matrix!

$$T = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}$$

$$T = \begin{pmatrix} 1-x & y \\ x & 1-y \end{pmatrix} \qquad T^{qi} = ?$$

Average fidelity

$$F(|P\rangle, |P'\rangle) := (\sum_{i=1}^{d} \sqrt{p_i p_i'})^2$$

$$\overline{F}(T) := \frac{1}{d} \sum_{i=1}^{d} F(|i\rangle, T|i\rangle)$$

$$\overline{F}(T) = \frac{1}{d} Tr(T)$$

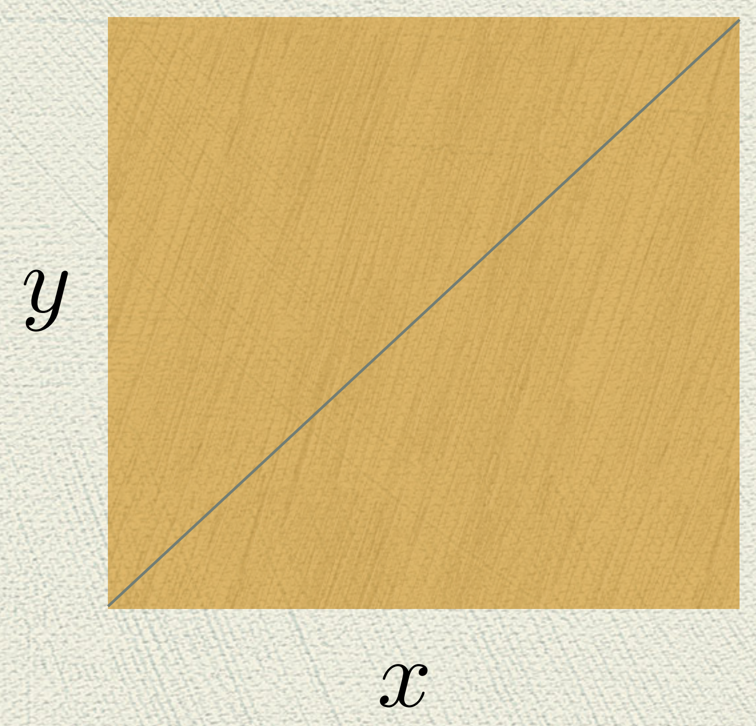
The quasi-inverse

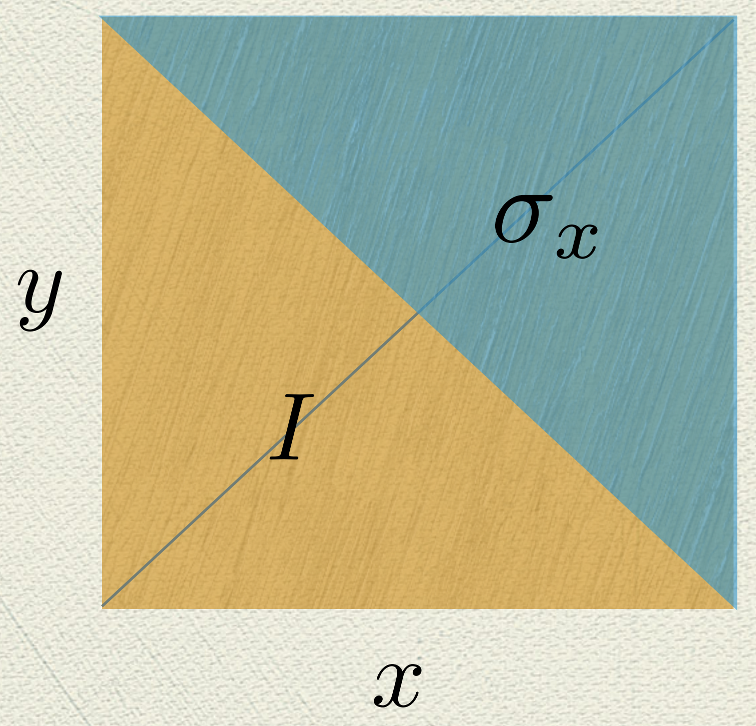
Maximize $Tr(T'T)$

Example 1:

$$T = \begin{pmatrix} 1-x & y \\ x & 1-y \end{pmatrix}$$

$$T^{qi}$$





Example 2:

$$T = \begin{pmatrix} 1/4 & 3/4 & 1/4 \\ 1/2 & 1/8 & 1/8 \\ 1/4 & 1/8 & 5/8 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A permutation

$$T^{qi} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3:

$$T = \begin{pmatrix} \boxed{1/4} & 1/8 & 1/16 \\ 1/12 & 3/4 & \boxed{7/8} \\ \boxed{2/3} & 1/8 & 1/16 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^{qi} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Not a permutation

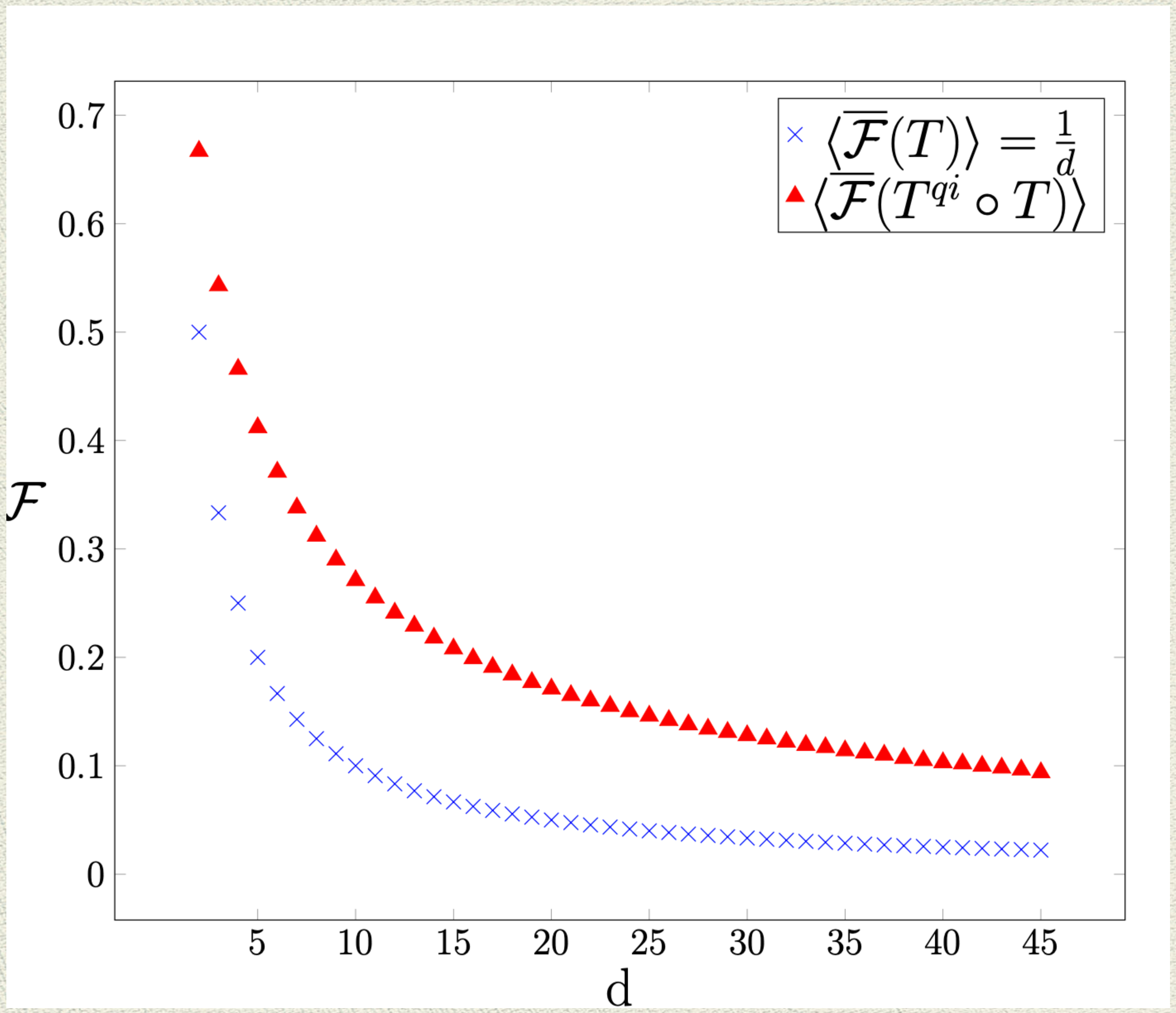
Example 4:

$$T = \begin{pmatrix} 1/3 & 1/8 & 1/3 \\ 1/2 & 1/4 & 1/4 \\ 1/6 & 5/8 & 5/12 \end{pmatrix}$$

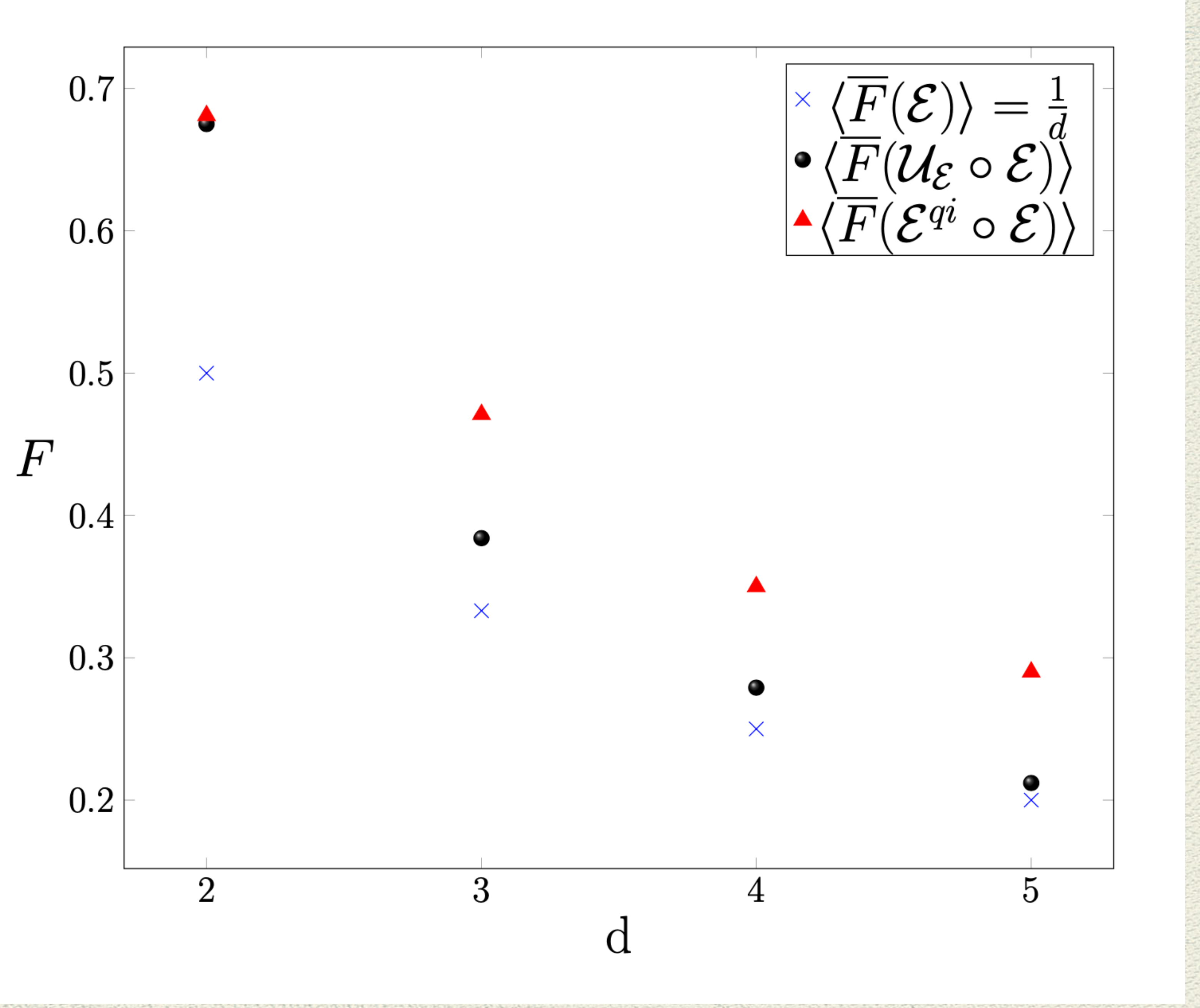
$$T^{qi} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & 1 \\ 1-\lambda & 0 & 0 \end{pmatrix}$$

Not an extreme point

Statistical Properties of Classical Channels



Statistical Properties of Quantum Channels



Thank you for your attention