Universal constraint on relaxation times for quantum dynamical semigroup

-- Physics of complete positivity

and my memory of Prof. Kossakowski

Based on joint works:

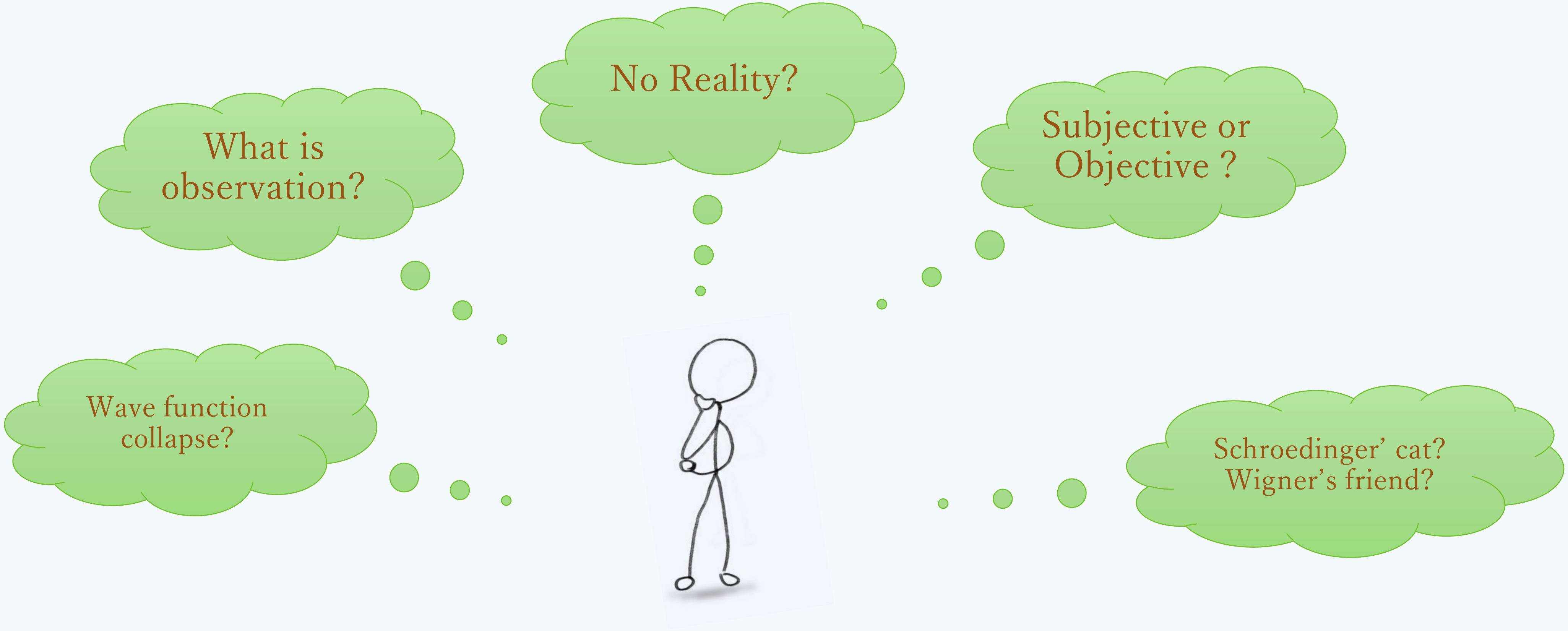
*G. K., Phys. Rev. A 66, 062113 (2002).

*G. K., S. Ajisaka, K. Watanabe, Open Syst. Inform. Dynam. 24(4): 1-8 (2017).

*D. Chruscinski, G. K., A. Kossakowski, Y. Shishido (arXiv:2011.10159).

*D. Chruscinski, R. Fujii, G. K., H. Ohno (arXiv:2106.08016).

When I was student, I have interested in measurement problem.



But many great people around me kindly advised “Not to do that”.

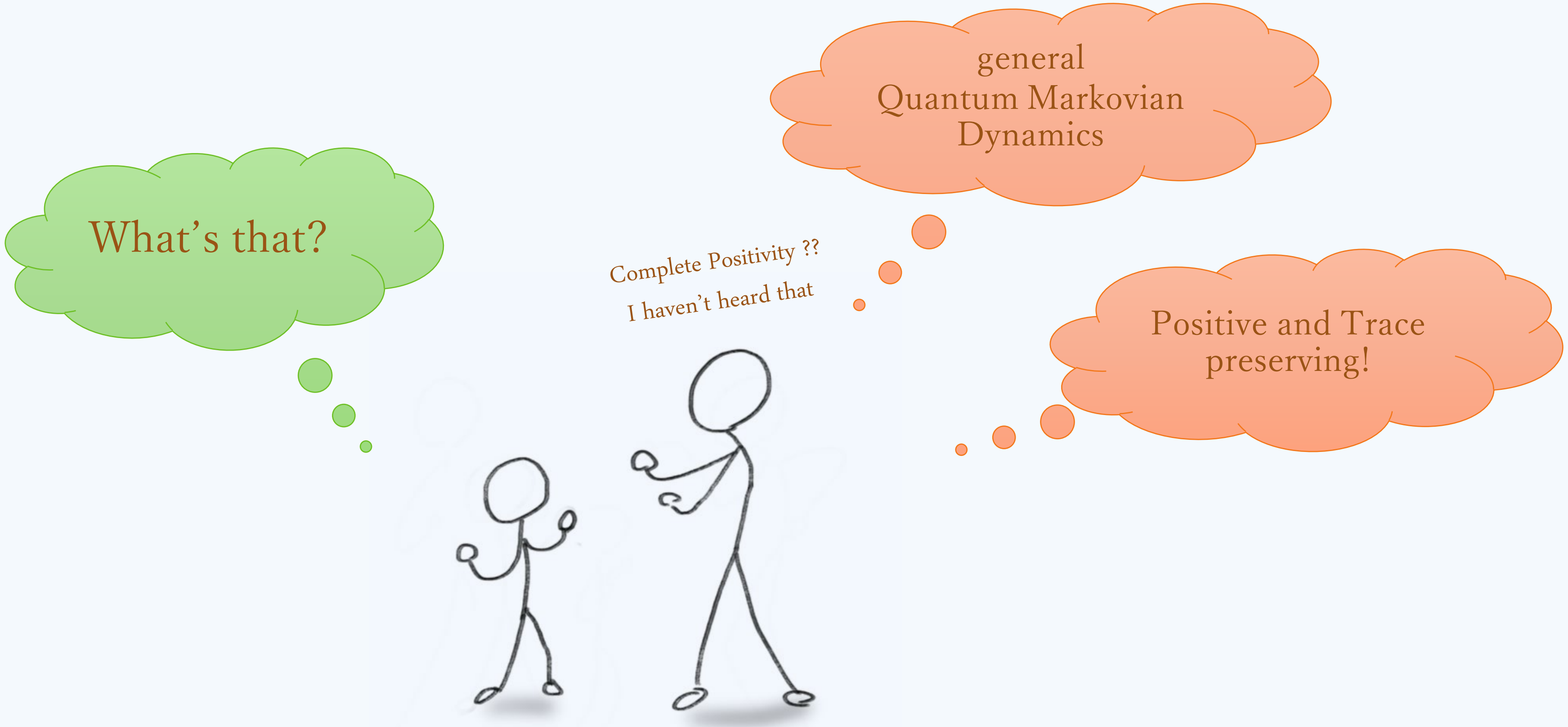
(Gently they told something like “SHUT UP AND CALCULATE!”)

So, I decided to study Open Quantum Systems

keeping the problem always in mind (even now).

Those times, many physicists used a beautiful master equation.

It was called the Lindblad master equation.



I love generality and universality.

I decided to study Lindblad's paper.

On the Generators of Quantum Dynamical Semigroups

G. Lindblad

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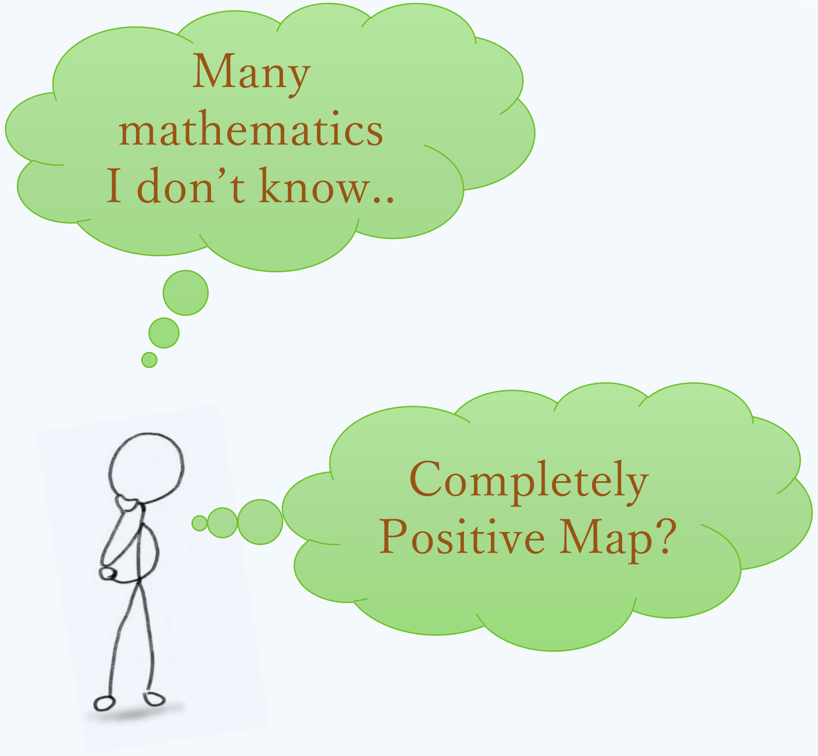
S-100 44 Stockholm 70, Sweden

Abstract. The notion of a quantum dynamical semigroup is defined using the concept of a completely positive map. An explicit form of a bounded generator of such a semigroup on $B(\mathscr{H})$ is derived. This is a quantum analogue of the Lévy-Khinchin formula. As a result the general form of a large class of Markovian quantum-mechanical master equations is obtained.

*Notice this paper is very clear and informative indeed !!



After the completion of this work we have recieved a preprint of a related work by V. Gorini et al. [34]. A result similar to Theorem 2 is derived for a system described by a finite-dimensional Hilbert space, using methods different from ours.

34. Gorini, V., Kossakowski, A., Sudarshan, E.C.G.: Preprint CPT 244, U. of Texas, Austin

That's how I encounter Prof. Kossakowski !!

Completely positive dynamical semigroups of *N*-level systems*

Vittorio Gorini† and Andrzej Kossakowski‡

Department of Physics, Center for Particle Theory, University of Texas at Austin, Austin, Texas 78712

E. C. G. Sudarshan

Department of Physics, Center for Particle Theory, U… …as 78712 and Centre for Theoretical Studies, Indian Institut…

(Received 19 March 1975)

We establish the general form of the generator o… …group of an *N*-level quantum system, and we apply the result to derive ex… …ysical parameters characterizing the Markovian evolution of a 2-level s…tem.

Proof. From (2.4) we have

$$LA = \frac{1}{N} c_{N^2N^2} A + \left(\frac{1}{N}\right)^{1/2} \sum_{i=1}^{N^2-1} (c_{iN^2} F_i A + c_{N^2 i} A F_i^*) + \sum_{i,j=1}^{N^2-1} c_{ij} F_i A F_j^*$$

$$= -i[H, A] + \{G, A\} + \sum_{i,j=1}^{N^2-1} c_{ij} F_i A F_j^*,$$

where $H = (1/2i)(F^* - F)$ and $G = (1/2N) C_{N^2N^2} \mathbb{1} + (1/2)(F^* + F)$,

[Quantum dynamical semigroup] … General Markovian CP quantum dynamics

1) Completely Positive Trace Preserving Map $\rho \mapsto \rho_t = \Lambda_t \rho$

2) One parameter (time) Dynamical Semigroup $\Lambda_{t+s} = \Lambda_t \Lambda_s \ (\forall t, s \geq 0)$

Markov property

Hille-Yoshida (1948)

➡ $$\frac{d\rho}{dt} = \mathcal{L}\rho$$ s.t. $\Lambda_t = \exp(t\mathcal{L})$

Master equation

* In this talk, we restrict to a d-level quantum system ($d < \infty$)

[Thm] (GKLS 1976) Generator of quantum dynamical semigroup is always written

$$\mathcal{L} = \mathcal{H} + \mathcal{D}$$

* Hamiltonian Part

$$\mathcal{H}(\rho) = -i[H, \rho]$$ where $H = H^\dagger$

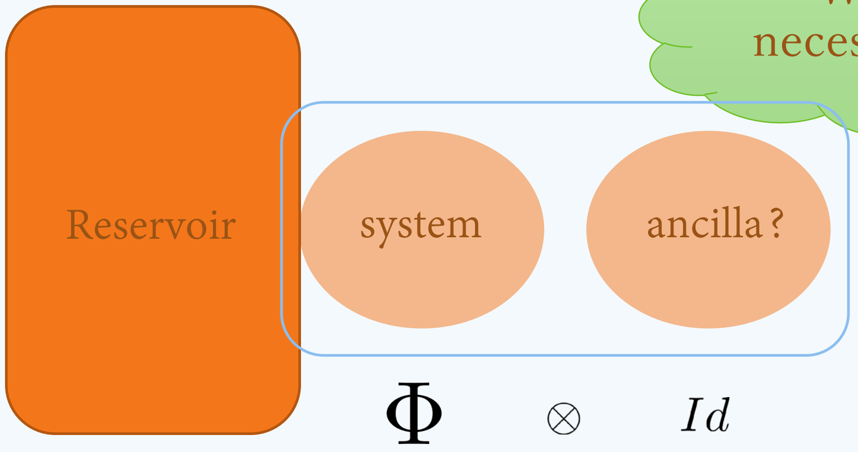
(effective) Hamiltonian

* Dissipative Part:

$$\mathcal{D}(\rho) = \frac{1}{2}\sum_k (2L_k \rho L_k^\dagger - L_k^\dagger L_k \rho - \rho L_k^\dagger L_k)$$

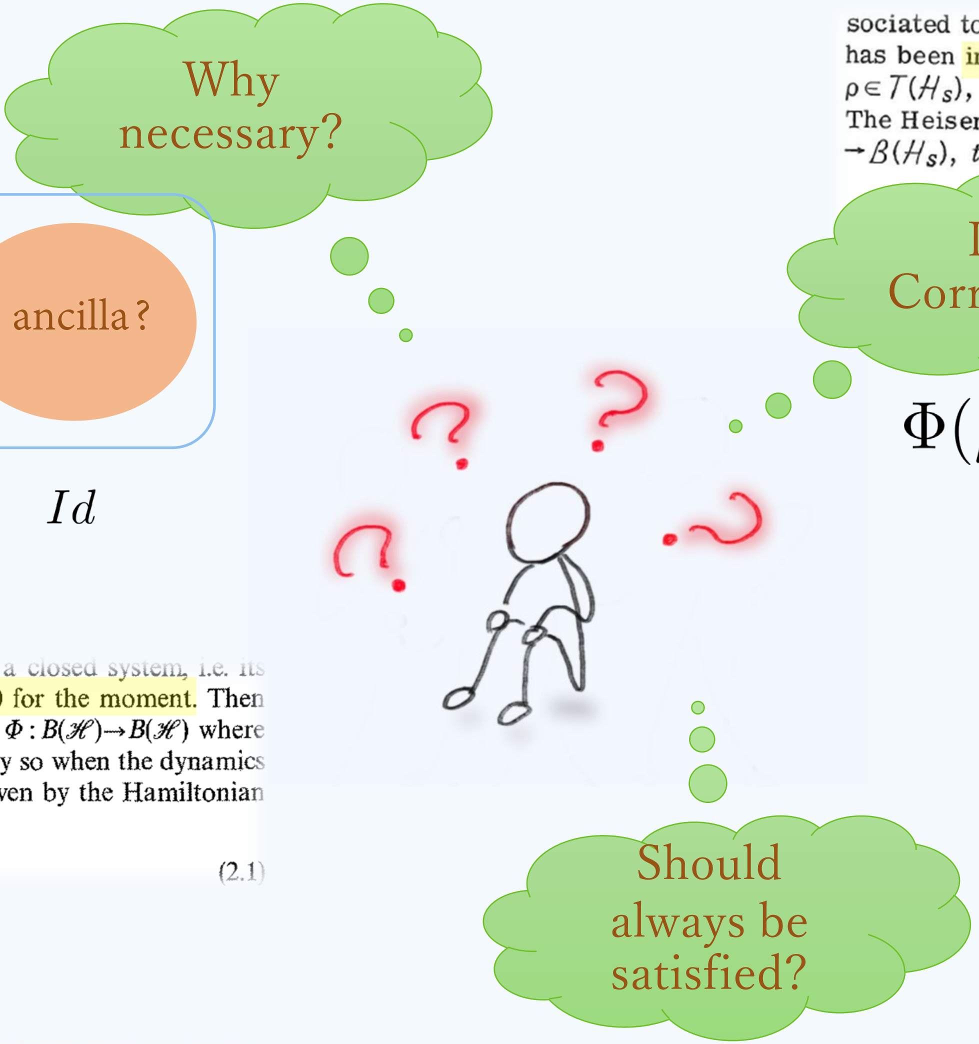
L_k: Jump/Noise Operator

But I still struggled with the completely positive condition.



drop the time parameter. We further assume that S_2 is a closed system, i.e. its dynamics is given by a Hamiltonian H_2. We put $H_2=0$ for the moment. Then we ask: can the map Φ_1 be extended to a positive map $\Phi: B(\mathscr{H}) \to B(\mathscr{H})$ where $\mathscr{H}=\mathscr{H}_1 \otimes \mathscr{H}_2$, such that S_2 is unaffected? This is obviously so when the dynamics of S_1 is Hamiltonian. Then the dynamics of S_1+S_2 is given by the Hamiltonian $H=H_1 \otimes I_2$. Φ is defined by

$$\Phi(X \otimes Y)=\Phi_1(X) \otimes Y \tag{2.1}$$

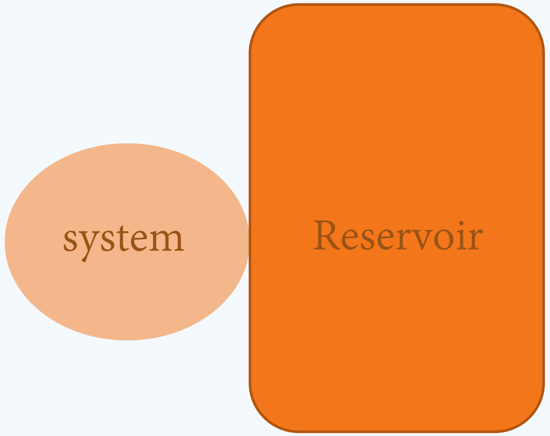


sociated to S and to R, respectively. Assume that $S+R$ has been initially prepared in a product state $\rho\otimes\sigma$, $\rho \in \mathcal{T}(\mathcal{H}_S)$, $\sigma \in \mathcal{T}(\mathcal{H}_R)$, in which S and R are uncorrelated. The Heisenberg reduced dynamics of S, $\Phi: t \to \Phi_t : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$, $t \in \mathbb{R}^+$, is defined by

$\Phi_t A)]$, (1.3)

). It is easy to see
proof can be found

$$\Phi(\rho_S) = \mathrm{Tr}_R U \rho_S \otimes \rho_R U^\dagger$$

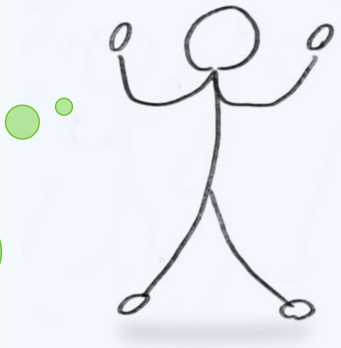


But according to discipline of natural science,

we must rely on **experiments** in order to determine something is right or wrong !!

Experiments

What is the
physics of CP
condition ??



Bell Theorem

Reality

Locality

"Free will"

→ Bell's Inequality

$$\langle AC \rangle + \langle AD \rangle + \langle BC \rangle - \langle BD \rangle \leq 2$$

Goal

CP

→ Experimentally testable condition.

Direct physical appearance of CP condition ??

Physical Manifestation of CP !

By Darek

Redundant !

Quantum Process Tomography $\Rightarrow$ Check CP condition

Complete Positive Witness

Redundant !

Quantum State Tomography $\Rightarrow$ Check Entanglement Criterion

Entanglement Witness

In terms of the γ_i's, (3.4) (a) can be written

$$\gamma_1 + \gamma_2 \geqslant \gamma_3, \quad \gamma_2 + \gamma_3 \geqslant \gamma_1, \quad \gamma_3 + \gamma_1 \geqslant \gamma_2, \tag{3.5}$$

showing that no two relaxation times can be much longer than the third.

$$\mathcal{L} u_\alpha = \lambda_\alpha u_\alpha \ (u_\alpha \neq 0)$$

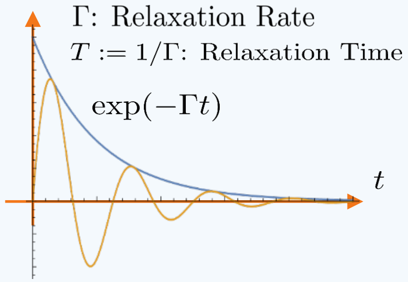
$\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$: Relaxation Rates

$T_\alpha := 1/\Gamma_\alpha$: Relaxation Times

$$L : \rho \to L\rho = -i[H, \rho] + \frac{1}{2}\sum_{i,j=1}^{3} c_{ij}\{[F_i, \rho F_j] + [F_i \rho, F_j]\}, \quad \rho \in M(2), \tag{3.1}$$

$$\text{(iii)} \ \{c_{ij}\} = \begin{pmatrix} \gamma - 2\gamma_1 & -ia_3 & i a_2 \\ ia_3 & \gamma - 2\gamma_2 & -ia_1 \\ -ia_2 & ia_1 & \gamma - 2\gamma_3 \end{pmatrix}, \quad \gamma = \gamma_1 + \gamma_2 + \gamma_3;$$

For d level system,
there are d^2-1 numbers
of decaying time scale !



they are in fact inverse relaxation times $\gamma_i = 1/T_i$, if L commutes with its Hamiltonian part $-i[H, \cdot\,]$.

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Bifurcation Phenomenon in Spin Relaxation

Gen Kimura,[1,*] Kazuya Yuasa,[1,†] and Kentaro Imafuku[2,‡]
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[2]*Centro Vito Volterra, Università di Roma Tor Vergata, 00133 Rome, Italy*
(Received 3 September 2001; published 16 September 2002; publisher error corrected 27 September 2002)

Spin relaxation in a strong-coupling regime (with respect to the spin system) is investigated in detail based on the spin-boson model in a stochastic limit. We find a bifurcation phenomenon in temperature dependence of relaxation constants, which is never observed in the weak-coupling regime. We also discuss inequalities among the relaxation constants in our model and show the well-known relation $2\Gamma_T \geq \Gamma_L$, for example, for a wider parameter region than before.

$$\Gamma_L^{(1)} + \Gamma_L^{(2)} \geq \Gamma_L^{(3)}, \qquad \Gamma_L^{(2)} + \Gamma_L^{(3)} \geq \Gamma_L^{(1)}, \qquad \Gamma_L^{(3)} + \Gamma_L^{(1)} \geq \Gamma_L^{(2)}. \tag{12}$$

even though $[\mathcal{L}, \mathcal{H}] \neq 0$

PHYSICAL REVIEW A **66**, 062113 (2002)

Restriction on relaxation times derived from the Lindblad-type master equations for two-level systems

Gen Kimura*
Department of Physics, Waseda University, Tokyo 169-8555, Japan
(Received 1 July 2002; published 31 December 2002)

We discuss a restriction on relaxation times derived from the Lindblad-type master equations for two-level systems and show that none of the inverse relaxation times can be greater than the sum of the others. The relation is experimentally proved or disproved and can be considered to be a measure for or against the applicability of the Lindblad-type master equations, and therefore, of the so-called completely positive condition.

DOI: 10.1103/PhysRevA.66.062113 PACS number(s): 03.65.Yz, 03.65.Ta, 76.20.+q

$$\Gamma_i + \Gamma_j \geqslant \Gamma_k, \quad (i,j,k) \text{ is a permutation of } (1,2,3), \tag{9}$$

Before I submitted this paper, I contacted Prof. Kossakowski for the first time by e-mail.

Dear Professor Kossakowski,

Please forgive me for sending you a sudden e-mail.
I am Gen Kimura, currently a master course student of Waseda University (Tokyo, Japan).
I am interested in the study of completely positive dynamical semigroup which was established by you and your collaborators in the 1970‘s. I had recently derived some relations of inverse relaxation times which hold universally in 2-level completely positive dynamical semigroup, and I have found that this result is a generalization of your result in J. Math. Phys. 17. 821 (1976).
You were introduced to me by Dr. Takashi Matsuoka (Tokyo University of Science) and he suggested me to ask for your comments and advice before submitting this to a journal.
Attached is a pdf file of my results and questions. [[[I‘m also sending a fax with the same contens.]]] I appreciate much if you can give me some commentary. I apologize for taking your precious time and I would like to thank you in advance for your consideration.
I have been much influenced by your work, I look forward to hearing from you, and hopefully, meeting you in the future.

Yours sincerely,

Gen Kimura

His reply was always simple

Please send a fax.

So, I sent him a fax:

Dr Andrzej Kossakowski
Institute of Physics
N. Copernicus University
Grudziadzka 5, 87-100 Torun, POLAND

Dear Professor Kossakowski

Please forgive me my boldness in writing to you by fax but I am doing so with the best of intensions. With reference to my e-mail about a 2-level completely positive dynamical semigroup, I'm sending my result.

As follows, I sum up my result with the same notations as your paper, "Completely positive dynamical semigroups of N-level systems" J. Math. Phys. **17**. 821 (1976): The generator $L : \rho \to L\rho, \rho \in M(2)$ of a dynamical semigroup of 2-level systems is

$$L : \rho \to L\rho = -i[H,\rho] + \frac{1}{2}\sum_{i,j=1}^{3} c_{ij}\{[F_i, \rho F_j] + [F_i\rho, F_j]\}, \text{ where}$$

(i) $H = \sum_{i=1}^{3} h_i F_i, h_i \in \mathbb{R}$; (ii) $F_i = F_i^*$ and $F_i F_j = \frac{1}{4}\delta_{ij}\mathbb{I} + \frac{i}{2}\sum_{k=1}^{3}\epsilon_{ijk}F_k$;

(iii) $\{c_{ij}\} = \begin{pmatrix} \gamma - 2\gamma_1 & -ia_3 & ia_2 \\ ia_3 & \gamma - 2\gamma_2 & -ia_1 \\ -ia_2 & ia_1 & \gamma - 2\gamma_3 \end{pmatrix}, \gamma = \gamma_1 + \gamma_2 + \gamma_3;$

His reply was again simple enough:

Congratulations !

It was the best message
I have ever got in my life !

In 2003 with Prof. Kossakowski and his wife !

During my visit, we have mainly discussed about Bloch vector representation and Positive maps.

Last meeting in 2018…

For d=2: 2-level system (qubit)

[Theorem 1] (Kimura 2002) For Arbitrary 2-level GKLS,

$$\Gamma_1 + \Gamma_2 \geq \Gamma_3,\ \Gamma_2 + \Gamma_3 \geq \Gamma_1,\ \Gamma_3 + \Gamma_1 \geq \Gamma_2$$

Experimentally Famous Relations*

[Case 1] Eigenvalues: $-\Gamma_L, -\Gamma_T \pm i\omega \quad \Rightarrow \Gamma_1 = \Gamma_L, \Gamma_2 = \Gamma_3 = \Gamma_T$

$T_L = 1/\Gamma_L$ Longitudinal Relaxation Time

$T_T = 1/\Gamma_T$ Transverse Relaxation Time

$$\Rightarrow 2T_L \geq T_T$$

* Flakowski et al. (2016)

[Case 2] Eigenvalues: $-\Gamma_1, -\Gamma_2, -\Gamma_3$ $\leftarrow$ Please let us know if you know of any such experiments!!

Three Damping Time Scales!!

For qubit system

CP

→

"Markovianity"

"Witness" of Complete Positivity !!

Experimentally testable condition.

$$\Gamma_1 + \Gamma_2 \geq \Gamma_3,\ \ \Gamma_2 + \Gamma_3 \geq \Gamma_1,\ \ \Gamma_3 + \Gamma_1 \geq \Gamma_2$$

$$\Leftrightarrow \quad 2\Gamma_\alpha \leq \sum_{\beta=1}^{3} \Gamma_\beta \ (\forall \alpha = 1, 2, 3)$$

Can we have similar results for general d-level system ?

Goal: to find the best constant c(d) s.t. $c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, d^2 - 1)$

is [illegible]ed for arbitrary quantum dynamical semigroup in d-level system

No single relaxation rate cannot be too large !!

[Theorem 2] (KAW 2017) For any d-level GKLS,

$$\frac{d}{\sqrt{2}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2-1)$$

[Proof] Omit. But we essentially used Böttcher-Wenzel Inequality (2008):

$$\|[A,B]\|^2 \leq 2\|A\|^2\|B\|^2 \ (\|A\|^2 := \mathrm{tr} A^\dagger A)$$

G. K., S. Ajisaka, K. Watanabe, Open Syst. Inform. Dynam. 24(4): 1-8 (2017).

[Conjecture] (CKKS 2020) For any d-level GKLS,

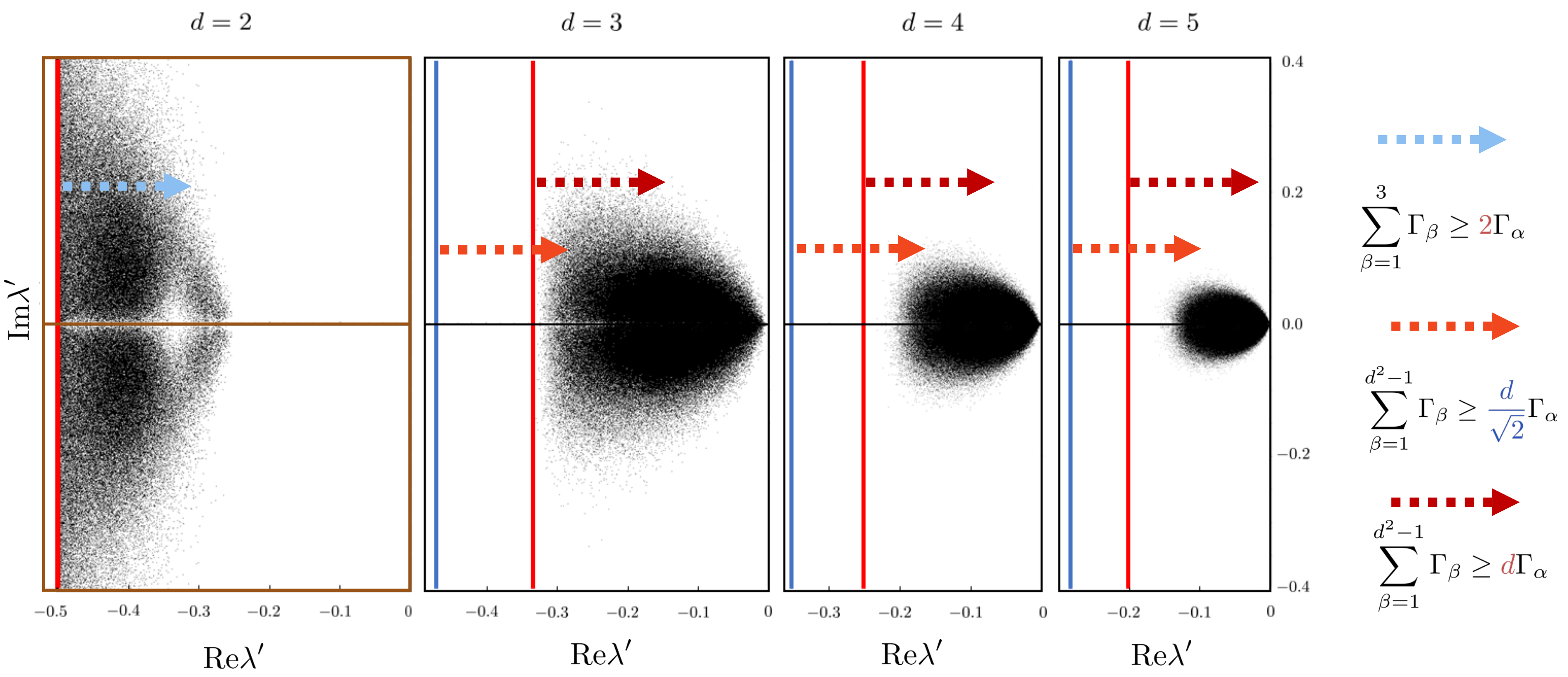
$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2-1)$$

* Numerically supported
* If this is true, c(d)=d is the best bound !!
* Satisfied in important classes of quantum dynamical semigroup

D. Chruscinski, G. K., A. Kossakowski, Y. Shishido (arXiv:2011.10159).

Numerical Simulation

(Rescaled) Eigenvalues of Randomly Generated GKLS generators $\lambda' := \lambda / (\sum_{\beta=1}^{d^2-1} \Gamma_\beta)$



* Kimura (2002)   * Kimura, Ajisaka, Watabe (2017)   * Chruściński, Kimura, Kossakowski, Shishido (2020)

A Tight Model

$$\mathcal{L}(\rho) = 2L\rho L - L^2\rho - \rho L^2 \quad (L^\dagger = L)$$

$$\mathcal{L}(|k\rangle\langle l|) = -(E_k - E_l)^2 |k\rangle\langle l| \qquad (E_i\text{: Eigenvalues of } L)$$

$$\sum_{\beta=1}^{d^2-1} \Gamma_\beta \geq d\Gamma_\alpha \iff 2\sum_{k<l}(E_k - E_l)^2 \geq d(E_i - E_j)^2$$

The best constant would be d

[M]oreover, the equality is attained iff

$$E_2 = \cdots = E_{d-1} = \frac{E_1 + E_d}{2}$$

Proof based on

[Lemma] For any $x_1, x_2, \ldots, x_n \in [a, b]$,

$$2\sum_{i=1}^{n}((x_i - a)^2 + (x_i - b)^2) + \sum_{i,j=1}^{n}(x_i - x_j)^2 \geq n(a-b)^2,$$

where the equality holds iff $x_1 = \cdots = x_n = \frac{a+b}{2}$.

* Chruściński, Kimura, Kossakowski, Shishido (2020)

Class of Covariant Generator

$$U_{\boldsymbol{x}}\mathcal{L}(X)U_{\boldsymbol{x}}^{\dagger} = \mathcal{L}(U_{\boldsymbol{x}}XU_{\boldsymbol{x}}^{\dagger})$$

where $U_{\boldsymbol{x}} = \sum_{k=1}^{d} e^{-ix_k}|k\rangle\langle k|$ and $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

[Theorem 3] For any covariant GKLS generator,

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall\alpha = 1, \ldots, d^2-1)$$

Including
* Pauli Master equation
* Weakly interacting model with non-degenerate invariant state (Davies 1974)

Class of entropy non-decreasing

$$\frac{d}{dt}S(\rho_t) \geq 0$$

$\Leftrightarrow$ Class of unital semigroup $\Lambda_t(\mathbb{I}) = \mathbb{I}$

Benatti (1988), Aniello, Chruscinski (2016)

[Theorem 4] For any unital generator,

$$d\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2-1)$$

[Proof] Omit. But through proving this,
we have found a nice characterization of relaxation rate.

Characterization of Relaxation Rate

* Invariant state $\mathcal{L}(\omega) = 0$
* Eigen equation $\mathcal{L}(u_\alpha) = \lambda_\alpha u_\alpha$

[Theorem 5] For any GKLS generator

$$\Gamma_\alpha = \frac{1}{2\|u_\alpha\|_\omega^2} \sum_k \|[L_k, u_\alpha]\|_\omega^2$$

where $\|X\|_\omega^2 := \mathrm{tr}(\omega X^\dagger X)$

[Proof] GKLS rep. reads: $\mathcal{L}^\ddagger(X^\dagger X) - \mathcal{L}^\ddagger(X^\dagger)X - X^\dagger \mathcal{L}^\ddagger(X) = \sum_k [L_k, X]^\dagger [L_k, X]$ $(\mathrm{Tr}(X\mathcal{L}(Y)) = \mathrm{Tr}(\mathcal{L}^\ddagger(X)Y))$

Taking $X = u_\alpha$ $\mathcal{L}^\ddagger(u_\alpha^\dagger u_\alpha) + 2\Gamma_\alpha u_\alpha^\dagger u_\alpha = \sum_k [L_k, u_\alpha]^\dagger [L_k, u_\alpha]$

which implies $\mathrm{tr}(\omega \mathcal{L}^\ddagger(u_\alpha^\dagger u_\alpha)) + 2\Gamma_\alpha \mathrm{tr}(\omega u_\alpha^\dagger u_\alpha) = \sum_k \mathrm{tr}(\omega [L_k, u_\alpha]^\dagger [L_k, u_\alpha]),$

Since $\mathcal{L}(\omega) = 0$, first term vanishes.

Approach based on r-function

[Definition] (r-function) For complex matrices $A, B \in M_d(\mathbb{C})$, we define

$$r(A,B) := \frac{1}{2}\operatorname{tr}(A^\dagger A B^\dagger B + A A^\dagger B^\dagger B - A^\dagger B A B^\dagger - B A^\dagger B^\dagger A)$$

$$\begin{aligned}
g(A,B) &= \frac{1}{2}\operatorname{tr}(\{A,A^\dagger\}B^\dagger B) - \Re\operatorname{tr}(A^\dagger B A B^\dagger), \\
&= \frac{1}{2}(\langle [B,A] | BA\rangle + \langle [B,A^\dagger] | BA^\dagger\rangle), \\
&= \frac{1}{2}(\|[A,B]\|^2 + \operatorname{tr} A^\dagger A[B^\dagger, B]) \\
&= \frac{1}{2}(\|[A^\dagger,B^\dagger]\|^2 + \operatorname{tr} A^\dagger A[B^\dagger, B]), \\
&= \frac{1}{2}(\|[A,B^\dagger]\|^2 + \operatorname{tr} AA^\dagger[B^\dagger, B]) \\
&= \frac{1}{2}(\|[A^\dagger,B]\|^2 + \operatorname{tr} AA^\dagger[B^\dagger, B]), \\
&= \frac{1}{4}(\|[A,B]\|^2 + \|[A^\dagger,B]\|^2 + \operatorname{tr}(\{A,A^\dagger\}[B^\dagger, B])).
\end{aligned}$$

Commutator

$$[A,B] := AB - BA$$

Anti-commutator

$$\{A,B\} := AB + BA$$

Hilbert-Schmidt Inner Prod.

$$\langle A,B\rangle := \operatorname{tr}A^\dagger B$$

Frobenius (Hilbert-Schmidt) Norm

$$\|A\| := \sqrt{\operatorname{tr}A^\dagger A}$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (arXiv:2106.08016).

Approach based on r-function

$$\mathcal{L}u_\alpha = \lambda_\alpha u_\alpha \quad (u_\alpha \neq 0)$$

$$\Gamma_\alpha = \frac{1}{\|u_\alpha\|^2}\sum_k r(u_\alpha, L_k)$$

$$\sum_{\alpha=1}^{d^2-1}\Gamma_\alpha = d\sum_k \|L_k\|^2$$

$$r(A,B) \leq c(d)\|A\|^2\|B\|^2$$

Universal Constraint !!

$$\frac{d}{c(d)}\Gamma_\alpha \leq \sum_\beta \Gamma_\beta$$

[Proof] $\Gamma_\alpha := -\mathrm{Re}\lambda_\alpha$ & $\|u_\alpha\|^2 = \mathrm{tr}u_\alpha^\dagger u_\alpha$

$$\mathrm{tr}\left[u_\alpha^\dagger \times \lambda_\alpha u_\alpha = \mathcal{L}(u_\alpha) = -i[H, u_\alpha] + \frac{1}{2}\sum_k (2L_k u_\alpha L_k^\dagger - L_k^\dagger L_k u_\alpha - u_\alpha L_k^\dagger L_k)\right]$$

- Re

$$\Gamma_\alpha = \frac{1}{2\|u_\alpha\|^2}\sum_k \mathrm{tr}(u_\alpha^\dagger u_\alpha L_k^\dagger L_k + u_\alpha u_\alpha^\dagger L_k^\dagger L_k - u_\alpha^\dagger L_k u_\alpha L_k^\dagger - L_k u_\alpha^\dagger L_k^\dagger u_\alpha)$$

$$= \frac{1}{\|u_\alpha\|^2}\sum_k r(u_\alpha, L_k)$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (arXiv:2106.08016).

Approach based on r-function

[Theorem 6] For any complex matrices $A, B \in M_d(\mathbb{C})$,

$$r(A,B) \leq \frac{1+\sqrt{2}}{2}\|A\|^2\|B\|^2,$$

where the equality can be achieved by a self-adjoint A.

[Proof] Omit. But we have first show this where A is normal first, and then used relation $r(A,B) = r(A_R,B)+i\, r(A_I,B)$

[Theorem 6'] (CFKO[1]) For any d-level GKLS,

$$\frac{\sqrt{2}d}{1+\sqrt{2}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1}\Gamma_\beta \ (\forall \alpha = 1,\ldots,d^2-1)$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (arXiv:2106.08016).

Approach based on r-function

Trace Preserving Property

**Eigenvector belonging to non-zero eigenvalue is traceless $\lambda_\alpha \neq 0 \Rightarrow \mathrm{tr}\, u_\alpha = 0$

[Theorem 7] For any complex matrices $A, B \in M_d(\mathbb{C})$ with $\mathrm{tr}A = 0$,

$$r(A,B) \leq \frac{1+\sqrt{2(1-\frac{1}{d})}}{2}\|A\|^2\|B\|^2$$

where the equality can be achieved by a self-adjoint A.

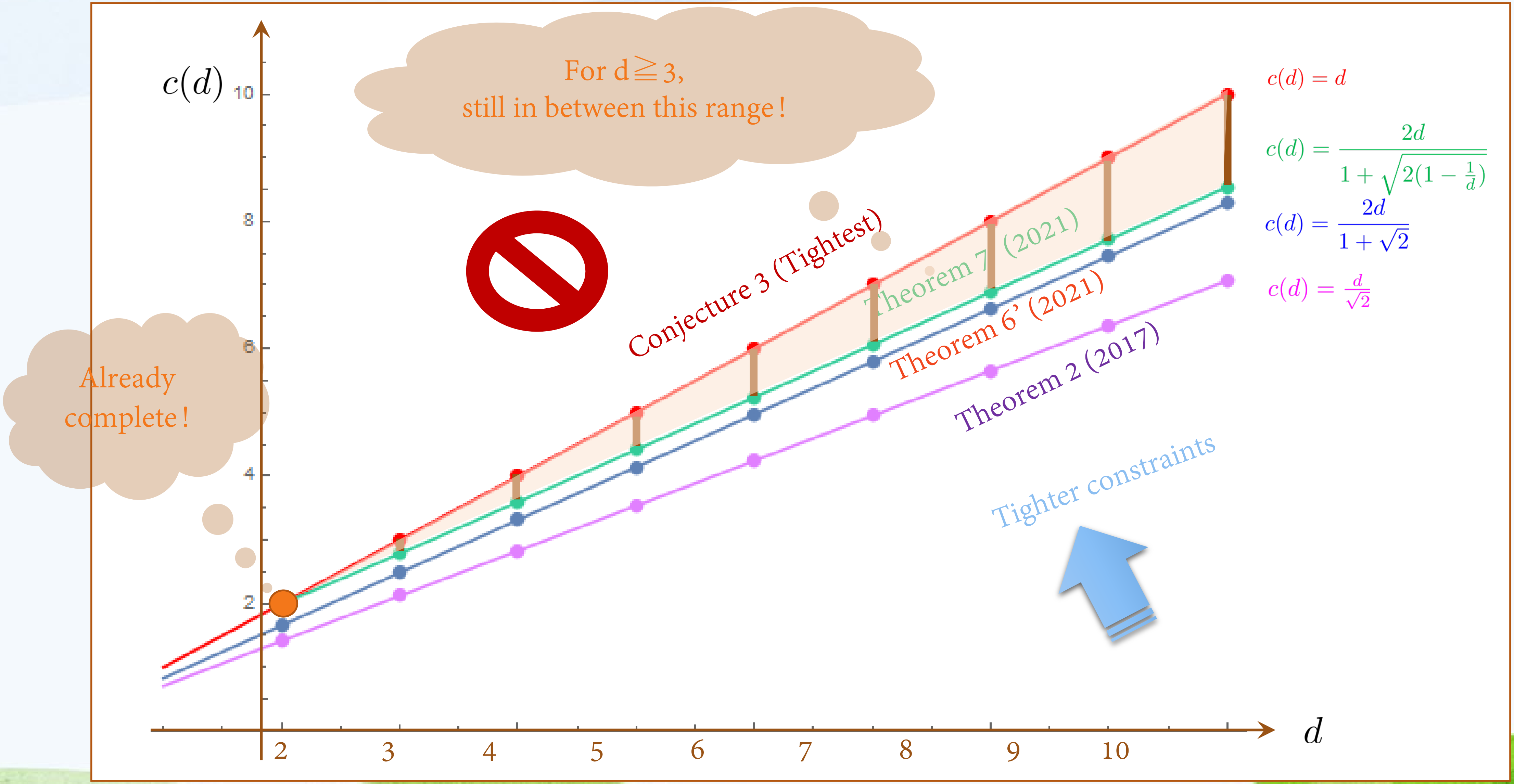
[Theorem 7'] (CFKO²) For any d-level GKLS,

$$\frac{2d}{1+\sqrt{2(1-\frac{1}{d})}}\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1}\Gamma_\beta \ (\forall \alpha = 1,\ldots,d^2-1)$$

D. Chruscinski, R. Fujii, G. K., H. Ohno (arXiv:2106.08016).

Universal Constraints for GKLS generator

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{d^2-1} \Gamma_\beta$$



Takeaway Message

1) Complete Positivity forces that

any single relaxation rate cannot be too large !!

Experimentally testable condition.

$$c(d)\Gamma_\alpha \leq \sum_{\beta=1}^{3} \Gamma_\beta \ (\forall \alpha = 1, \ldots, d^2 - 1)$$

2) This may serve as Completely Positive (at least QDS) witness !

3) Best bound is conjectured to be $c(d)=d$, but still open.

Takeaway Message

Thank you for your kind attention !!

1) Complete Positivity forces that
any single relaxation rate cannot be too large !!

Complete Positivity

"Markovianity"

Experimentally testable condition.

$c(d)\Gamma_\alpha \leq \sum^{3} \Gamma \ldots$

Thank Prof. Kossakowski for his kind, generous, passionate guidance of my life

2) This ma… …s !

3) Best … …ut still open.