Controllability of infinite dimensional quantum systems based on Quantum Graphs.

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Joint work with A. Balmaseda and D. Lonigro

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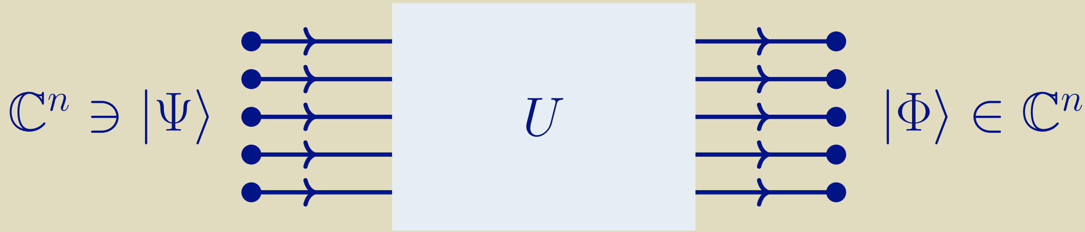
Outline

- The Control Problem in Quantum Mechanics
- Time dependent Boundary Conditions and the Schrödinger Equation
- Controllability of Magnetic Laplacians on Quantum Graphs

- A simplified scheme of a quantum computer



- $|\Psi\rangle$ input state; $|\Phi\rangle$ output state; U is a unitary operator in $\mathcal{U}(\mathbb{C}^n)$.
- Building a Quantum Computer $\Rightarrow$ Design a system capable of implementing any possible unitary operator
- This problem is equivalent to simultaneously control the evolution of n linearly independent states. Fixing n orthonormal states as input and other n as output.

- Given an evolution equation that depends on a family of parameters $\mathcal{C}$.
- Does it exist a curve $c : [0, T] \to \mathcal{C}$ such that the evolution can join any two given states?

$$i\frac{\partial}{\partial t}\Phi(t) = H(c(t))\Phi(t)$$

  - Initial State: Ψ_0 Target State: Ψ_T
  - Solution of the evolution equation $\Phi(t)$ is such that $\Phi(0) = \Psi_0$ and $\Phi(T) = \Psi_T$
- Typical case in Quantum Mechanics (Bilinear Control System):

$$i\frac{\partial}{\partial t}\Phi(t) = (H_0 + c(t)H_1)\,\Phi(t)$$

- $H(t)$ is the Hamiltonian Operator. A self-adjoint operator, possibly unbounded.
- Solutions of the *time dependent* Schrödinger equation are given in terms of a *unitary propagator* that maps the initial state at t_0 to the state at t_1
- Unitary propagator: strongly continuous two-parameter family of unitary operators.
  - $U(t,t) = \mathbb{I}_{\mathcal{H}}$
  - $U(t_2,t_1)U(t_1,t_0) = U(t_2,t_0)$
  - $U(t,t_0)\Psi_0$ is the solution with initial state Ψ_0 at $t = t_0$.
  - $\|U(t,t_0)\Psi_0\| = \|\Psi_0\|$

- The classic theory of control has been applied successfully to finite dimensional quantum systems.
- The success in the development of recent quantum technologies is a proof of this.
- This can be used even for infinite dimensional systems. What is the main idea?
  - Pick a suitable basis $\{\Phi_n\} \subset \mathcal{H}$
  - $\langle \Phi_n, H(c(t))\Phi_m \rangle = H_{nm}(c(t))$
  - Consider the truncated Schrödinger eq.:

$$i\dot{x}_n = \sum_{m=1}^{N} H_{nm}\left(c(t)\right) x_m$$

Quantum Control on Infinite Dimensions I

- Many important systems, including those appearing in the technological applications are infinite dimensional
- The truncation of the system to a finite dimensional subspace is an important source of errors.
- To study the appropriateness of the approximation one needs to address the problem in infinite dimensions directly.

- Results of control on finite dimensions cannot be applied directly to infinite dimension.
- The notion of controllability introduced in the previous slides is not appropriate for infinite dimensions
  - One can find examples where all the finite dimensional truncations are controllable but the infinite dimensional system is not, for instance the Harmonic oscillator.
  - This is reasonable. Suppose that the target state Ψ_T expressed in the basis $\{\Phi_n\}$ has countably many non-zero coefficients. Then $\|\Psi_T - \Psi_T^N\| > 0$ for any N.

Approximate Controllability:

A quantum control system is approximately controllable if for every Ψ_0, $\Psi_T \in \mathcal{S}$ and every $\epsilon > 0$ there exists $T > 0$ and $c : [0, T] \to \mathcal{C}$ such that the solution of the time dependent Schrödinger equation $\Phi(t)$ satsfies $\Phi(0) = \Psi_0$ and

$$\|\Psi_T - \Phi(T)\| < \epsilon$$

- One needs to study the existence of solutions of the Schrödinger equation.
  - Notice that the existence of solutions of the time-dependent Schrödinger equation is compromised. The Hamiltonians are in general unbounded operators (not continuous) even if they are linear.

If this is so difficult, why do we want to go this way?

- Even if in applications the information is codified in a finite dimensional subsystem, the extra dimensions can be used as a resource instead of as a drawback.
- Opens the possibility of a new type of control:

  Boundary Control

  Control the system by varying the boundary conditions.
- The Josephson junction, important for Superconducting Circuits, can be modelled as point like interaction.

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Prototypical Example

Hilbert space $\mathcal{H} = \mathcal{L}^2([0, 2\pi])$

Hamiltonian of a free particle (not driven by any force or external field) is the Laplacian $\Delta = -\frac{\partial^2}{\partial x^2}$

The time-independent Schrödinger equation governing the evolution of this system is:

$$i\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial^2 \Phi(t)}{\partial x^2}$$

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Quasi-periodic Boundary Conditions:

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{rl} \phi(0) & = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) & = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

The most general solutions of this problem date back to the 60's-70's and are due to T. Kato, J. Kisynski and B. Simon.

Theorem [J. Kisyński, Studia Mathematica 3 (23), 1964]:
Let $H(t)$ with domain $\mathcal{D}(t)$ be a time-dependent Hamiltonian with uniform lower bound and constant form domain $\mathcal{H}_+$. Let $h_t : \mathcal{H}_+ \times \mathcal{H}_+ \to \mathbb{C}$ be the quadratic form associated to $H(t)$. Suppose that for any $\Phi, \Psi \in \mathcal{H}_+$ one has that $t \mapsto h_t(\Phi, \Psi)$ is $C^2(\mathbb{R})$. Then there exists a strongly continuous unitary propagator $U(t, s)$ such that:

- $U(t, s)\mathcal{D}(s) = \mathcal{D}(t)$
- For $\Phi \in \mathcal{D}(s)$ one has that $\Phi(t) = U(t, s)\Phi$ solves the Schrödinger equation.

Stability

An important property that we were able to prove is the stability of the dynamics under perturbations/deformations of the Hamiltonian:

Theorem [Balmaseda, Lonigro, PP]:

Let $\{H_n(t)\}_{n=1,2}$ be two time-dependent Hamiltonians with constant form domain $\mathcal{H}_+$ that satisfy the conditions of Kisyński's Theorem and [a certain uniform bound on their derivatives]. Then the following inequality holds:

$$\|U_1(t,s) - U_2(t,s)\|_{+,-} \leq L\sqrt{\int_s^t \|H_1(\tau) - H_2(\tau)\|_{+,-}\mathrm{d}\tau},$$

where the constant L is independent of t and s.

The norm $\|\cdot\|_{+-}$ is the norm of linear operators $L : \mathcal{H}_+ \to \mathcal{H}_-$, where $\mathcal{H}_-$ is the canonical dual space of $\mathcal{H}_+$.

Improves previous results of B. Simon (1971) and A. Sloan (1981).

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Varying Quasiperiodic Boundary Conditions

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{H}^2 \middle| \begin{array}{ll} \phi(0) & = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) & = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

- This is a family of self-adjoint operators depending on α
- We want to consider $\alpha(t)$ the control parameter. These Hamiltonians do not have constant form domain.
- One can tackle with these systems by the unitary transformation $T(t) : \Phi(x) \mapsto \exp(-ix\alpha(t))\Phi(x)$
- Assuming that the parameter α depends smoothly with time this is equivalent to:

$$H(t) = \left[i\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(t) \right]^2 + \dot{\alpha}(t)x$$

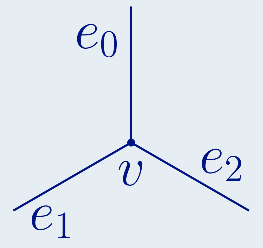
$\mathcal{D}_0 =$ "Periodic Boundary Conditions"

Laplacians on Quantum Graphs I

- Consider a planar Graph (V, E) and associate to each edge e a Hilbert space $\mathcal{H}_e = \mathcal{L}^2([0, l_e])$
- Take $\mathcal{H} = \bigoplus_{e \in E} \mathcal{H}_e$ and $\Delta = \bigoplus \Delta_e$ densely defined in it.
- The structure of the graph arises when one selects the boundary conditions.
- At each vertex we choose quasi-δ-boundary conditions:

$$\exp(-i\chi_{e_i,v})\Phi_e(v) = \Phi_{e_0}(v) \qquad i = 1, \ldots, n-1$$

$$\sum_e \exp(i\chi_{e_i,v})\dot{\varphi}_e = \delta_v \Phi_{e_0}(v)$$



There exist also time-dependent unitary maps that transform these Laplacians:

$$\Delta_e \rightsquigarrow \left[i\frac{\mathrm{d}}{\mathrm{d}x} - \alpha(t,x)\right]^2 + \Theta(t,x)$$

Boundary Conditions:

$$\Phi_e(v) = \Phi_{e_0}(v)$$

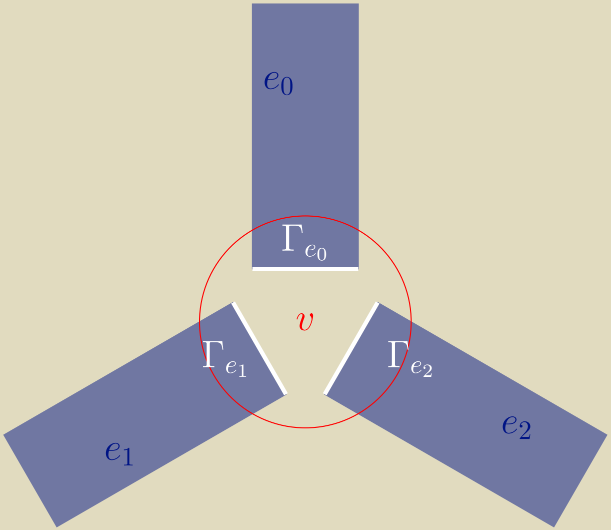
$$\sum_e \dot{\varphi}_{\alpha,e} = \delta_v \Phi_{e_0}(v)$$

Theorem [Balmaseda, Lonigro, PP]:

If $\sup_v \delta_v < \infty$ and $\sup_t \|\alpha\|_\infty < \infty$ the Magnetic Laplacians have a uniform lower bound. The form domain of the Magnetic Laplacian obtained this way does not depend on the parameter t (constant form domain).

Laplacians on Fattened Quantum Graphs

- Instead of associating an interval to each edge e one can associate a Riemannian manifold Ω_e.
- Magnetic Laplacians can be defined in an analogous way.
- There is a generalisation of the quasi-δ-type boundary conditions to the fattened graphs [Balmaseda, Lonigro, PP].



Γ_{e_i}: Boundary of the manifold Ω_{e_i}
Diffeomorphic to each other

Previous Results on Controllability

- To the best of our knowledge, the most general results on controllability that can be applied to this problem are results on bilinear control systems, for fixed domains obtained by [Boussaid, Caponigro, Chambrion, Mason, Sigalotti].

$$i\frac{\partial}{\partial t}\Phi(t) = (H_0 + c(t)H_1)\,\Phi(t)$$

Theorem [Chambrion, Mason, Sigalotti, Boscain. Ann. l'Inst. H. Poincare (C), 26 2009]

Consider a normal bilinear control system with $c : \mathbb{R} \to [0,\delta]$ for some $\delta > 0$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ denote the eigenvalues of H_0, each of them associated to the eigenfunction Φ_n. Then, if the elements of the sequence $\{\lambda_{n+1} - \lambda_n\}_{n\in\mathbb{N}}$ are $\mathbb{Q}$-linearly independent and if $\langle \Phi_{n+1}, H_1\Phi_n\rangle \neq 0$ for every $n \in \mathbb{N}$, the system is approximately controllable by piecewise constant controls.

Theorem [Balmaseda, Lonigro, PP]:

Let $u \in C^3(\mathbb{R})$ and consider $\chi_{v,e}(t) := u(t)\chi_{v,e}$. Let $H(t)$ be the time-dependent Hamiltonian defined by the Laplacian on a (Fattened) Quantum Graph (V, E) with quasi-δ boundary conditions. Then, the linear system defined by $H(t)$ is approximately controllable.

Ideas for the proof:

- Results by Chambrion et. al. imply that we have approximate controllability on an auxiliary system.
- Convergence of the evolution on the auxiliary system using the Stability Theorem.

Open Problems:

- Good approximation / stability results that allow to extend the results on finite dimensional controllability to the infinite dimensional case.
- Controllability by singular perturbations.

THANKS!

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