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Markovian approximation for correlated initial states

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The question

Open system - reservoir (SR) Hamiltonian

$$H_\lambda = H_S + H_R + \lambda I$$

coupling parameter

acting on Hilbert space

$$\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$$

field of oscillators

finite-dimensional

Schrödinger dynamics

$$\rho_{SR}^{t} = e^{-itH_\lambda} \rho_{SR} e^{itH_\lambda}$$

initial state

Reduced system state

$$\rho_S^t = \mathrm{tr}_R \, \rho_{SR}^t$$

partial trace

Two kinds of initial states:

$\rho_{SR} = \rho_S \otimes \rho_R$ <u>uncorrelated</u>

ρ_{SR} not of product form <u>correlated</u>

Reduced dynamics of uncorrelated initial states $\rho_S \otimes \rho_R$

Born approximation $\qquad \rho_{SR}^t \approx \rho_S^t \otimes \rho_R$

(ρ_R stationary, $S \lll R$ & λ small)

Markov approximation $\qquad \rho_S^t \approx e^{t\mathcal{L}} \rho_S$

(R loses memory quickly enough)

Master equation $\qquad \rho_S \mapsto V_t \rho_S = e^{t\mathcal{L}} \rho_S$

(Dynamical map for system)

Davies generator $\mathcal{L}$ from weak coupling (van Hove) limit

$\tau = \lambda^2 t$ fixed, $\lambda \to 0$, $t \to \infty$:

$$e^{\tau \mathcal{L}} \rho_S = \lim_{\lambda \to 0} \operatorname{tr}_R \left[e^{-i \frac{\tau}{\lambda^2} H_\lambda} e^{i \frac{\tau}{\lambda^2} H_0} (\rho_S \otimes \rho_R) e^{-i \frac{\tau}{\lambda^2} H_0} e^{i \frac{\tau}{\lambda^2} H_\lambda} \right]$$

interaction picture dynamics, rescaled

$$\mathcal{L} = \mathcal{L}(\lambda) = -i [H_S, \cdot] + \lambda^2 K$$

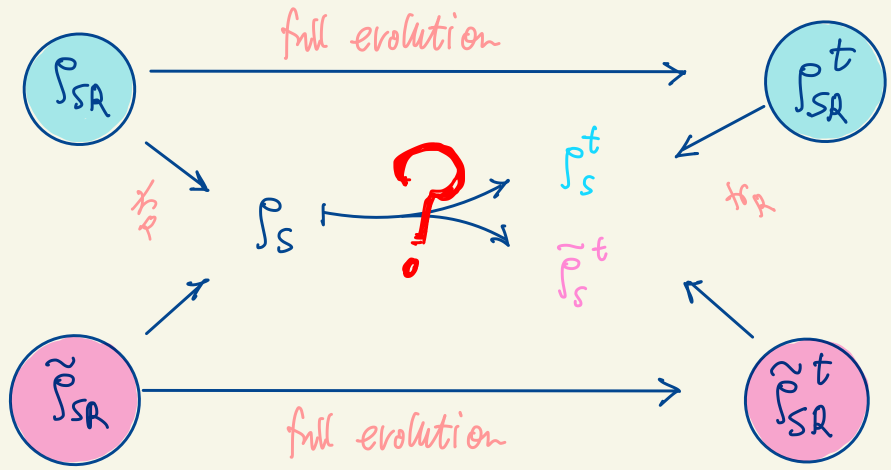
generates CPTP semigroup

Reduced dynamics of correlated initial states ??

$$\rho_S^t = \mathrm{tr}_R \, e^{-itH_\lambda} \rho_{SR} \, e^{itH_\lambda}$$

- Born approximation X $\qquad \rho_{SR}^t \cancel{\approx} \rho_S \otimes \rho_R$

- Dynamical map X $\qquad \rho_S \overset{V_t}{\cancel{\longmapsto}} \rho_S^t$



Cannot define dynamical map V_t but can still analyze structure of reduced S dynamics

PHYSICAL REVIEW A **67**, 062109 (2003)

Kraus representation in the presence of initial correlations

Hiroyuki Hayashi,* Gen Kimura,† and Yukihiro Ota‡

PHYSICAL REVIEW A **100**, 042120 (2019)

Dynamics of initially correlated open quantum systems: Theory and applications

Gerardo A. Paz-Silva,[1] Michael J. W. Hall,[1,2] and Howard M. Wiseman[1]

Works similar in spirit to ours

Available online at www.sciencedirect.com
ScienceDirect
Annals of Physics 322 (2007) 631–656
ANNALS of PHYSICS
www.elsevier.com/locate/aop

On the assumption of initial factorization in the master equation for weakly coupled systems I: General framework

S. Tasaki [a], K. Yuasa [b,*], P. Facchi [c,d], G. Kimura [b,1], H. Nakazato [b], I. Ohba [b], S. Pascazio [e,d]

Available online at www.sciencedirect.com
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Annals of Physics 322 (2007) 657–676
ANNALS of PHYSICS
www.elsevier.com/locate/aop

On the assumption of initial factorization in the master equation for weakly coupled systems II: Solvable models

K. Yuasa [a,*], S. Tasaki [b], P. Facchi [c,d], G. Kimura [a,1], H. Nakazato [a], I. Ohba [a], S. Pascazio [e,d]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_{\mathrm{I}}(\tau) = \mathcal{K}\rho_{\mathrm{I}}(\tau), \qquad \rho_{\mathrm{I}}(0) = \mathcal{P}\rho_0 = \mathrm{tr}_{\mathrm{B}}\{\rho_0\} \otimes \Omega_{\mathrm{B}}. \tag{4.4}$$

That is, even if the initial state ρ_0 is not in a factorized form, but rather there is entanglement, or simply a classical correlation, between system S and reservoir B, all correlations disappear in van Hove's limit and system S behaves as if the total system started from the factorized initial state in (4.4) with a reservoir state Ω_{B} specified below.

PHYSICAL REVIEW X **10,** 041024 (2020)

Correlation-Picture Approach to Open-Quantum-System Dynamics

S. Alipour,[1,*] A. T. Rezakhani,[2,†] A. P. Babu,[1] K. Mølmer,[3] M. Möttönen,[4] and T. Ala-Nissila[1,5,‡]

Lindblad-type ME with jump operators depending on initial correlation

Our main results: Perturbation theory small λ, all times $t \geq 0$.

- Initial correlations decay and then Markov approximation becomes valid for SR dynamics.

- Markovian approximation for reduced S dynamics is valid for all times.

SCIENTIFIC REPORTS
Aug 15, 2012

OPEN

Operational approach to open dynamics and quantifying initial correlations

SUBJECT AREAS:
QUANTUM PHYSICS
THEORETICAL PHYSICS
CHEMICAL PHYSICS
QUANTUM CHEMISTRY

Kavan Modi[1,2]

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Quantum process tomography

epl A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS
December 2010
EPL, **92** (2010) 60010
www.epljournal.org
doi: 10.1209/0295-5075/92/60010

Witness for initial system-environment correlations in open-system dynamics

E.-M. LAINE[1(a)], J. PIILO[1(b)] and H.-P. BREUER[2(c)]

Initial SE correlation can increase distinguishability of system states.

Model

SR open quantum system : N-level S coupled to thermal R

$$\mathcal{H} = \mathbb{C}^N \otimes \mathcal{F}$$

$\bigoplus_{n \geq 0} L^2_s(\mathbb{R}^{3n}, d^{3n}k)$

Bosonic Fock space

$$H_\lambda = H_S + H_R + \lambda I$$

coupling constant

Uncoupled Hamiltonians

$$H_S = \sum_{j=1}^{N} E_j \, |\phi_j\rangle\langle\phi_j| \quad , \qquad H_R = \int_{\mathbb{R}^3} \omega(k) \; a^*(k) a(k) \, d^3k$$

$|k|$

Interaction operator

$$I = G \otimes \varphi(g)$$

any hermitian matrix

$$\varphi(g) = \frac{1}{\sqrt{2}} \left[a^*(g) + a(g) \right]$$

$$a^*(g) = \int_{\mathbb{R}^3} \overline{g(k)} \; a^*(k) \, d^3k$$

$g(k) \in L^2(\mathbb{R}^3, d^3k)$

'form factor'

Equilibrium states

S : Gibbs density matrix $\qquad \rho_{S,\beta} = \dfrac{e^{-\beta H_S}}{\operatorname{tr} e^{-\beta H_S}}$

R : Infinitely extended (continuous modes) KMS state $\omega_{R,\beta}$

Two-point function

$$\omega_{R,\beta}\left(a^*(k)a(\ell)\right) = \frac{\delta(k-\ell)}{e^{\beta\omega(k)}-1}$$

Initial states

quantum channel

$$\rho_{SR}(\cdot) = \Lambda\left(\rho_S \otimes \omega_{R,\beta}\right) \equiv \sum_\alpha \rho_S \otimes \omega_{R,\beta}\left(K_\alpha^* \cdot K_\alpha\right)$$

$$\sum_\alpha K_\alpha^* K_\alpha = \mathbb{1}$$ Kraus ops

$K_\alpha \in \mathcal{C}$ 'correlator algebra'

Observable and Correlator algebras

$$\mathcal{O} = \mathcal{B}(\mathbb{C}^N) \otimes \mathcal{W}(L^2_{obs})$$ Weyl algebra

$$\mathcal{C} = \text{alg span}\{\mathcal{X}, \mathcal{F}\}$$ sums and products

$\mathcal{F}$: linear comb. of $\quad B \otimes a^*(f_1) \cdots a^*(f_k)$, $f_j \in L^2_{cor}$

$\mathcal{X}$: linear comb. of $\quad \exp\left(\sum_{j=1}^{k} B_j \otimes a^\#(f_j)\right)$, $f_j \in L^2_{cor}$

Test function spaces

L^2_{obs} : $f \in L^2(\mathbb{R}^3, d^3k)$ 3 times differentiable w.r.t. $|k|>0$ &

- $|k| \sim 0$: $f(k) \sim |k|^p$ $p = -\frac{1}{2}, \frac{1}{2}$ or $p>1$
- $|k| \to \infty$: $f(k) \sim |k|^{-q}$ $q>3$

L^2_{cor} : $f \in L^2_{obs}$ and also $e^{\beta|k|} f(k) \in L^2(\mathbb{R}^3, d^3k)$

Example of initial state:

$$\rho_{SR} = \frac{1}{Z}\, e^{\mathcal{E}} \left(\rho_S \otimes \omega_{R,\beta}\right) e^{\mathcal{E}^*}$$

$$\mathcal{E} = \sum_j B_j \otimes a^*(f_j) + \sum_\ell D_\ell \otimes a(h_\ell)$$

Assumption on effectiveness of SR coupling

Fermi Golden Rule Condition

spectral density of reservoir

$$\langle \phi_m , G \phi_n \rangle \, J(|E_m - E_n|) \neq 0 \qquad \forall \quad E_m \neq E_n$$

Second order transition processes are not supressed

Main result

Theorem 3.4 (SR dynamics for correlated initial states) *There is a constant* $\lambda_0 > 0$ *such that if* $|\lambda| < \lambda_0$*, then the following holds true. Let* $\rho_{\mathrm{SR}} = \Lambda(\rho_{\mathrm{S}} \otimes \omega_{\mathrm{R},\beta})$ *and let* ρ_{S} *be its reduction to S. Then for all* $t \geq 0$, $O \in \mathcal{O}$,

$$\rho_{\mathrm{SR}}\big(e^{\mathrm{i}tH_\lambda} O e^{-\mathrm{i}tH_\lambda}\big) = \big(e^{t\mathcal{L}_{\mathrm{S}}(\lambda)}\rho_{\mathrm{S}} \otimes \omega_{\mathrm{R},\beta}\big)(O) + \chi(\lambda, t, O) + R(\lambda, t, O),$$

where $\mathcal{L}_{\mathrm{S}}(\lambda)$ *is the Davies generator and*

$$|R(\lambda, t, O)| \leq C(O)|\lambda|^{1/4}.$$

The correlation term χ *satisfies*

$$\begin{aligned} \chi(\lambda, t, O) &= 0 \qquad \text{if} \quad \rho_{\mathrm{SR}} = \rho_{\mathrm{S}} \otimes \omega_{\mathrm{R},\beta} \\ \chi(\lambda, t, O_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}) &= 0 \\ |\chi(\lambda, t, O)| &\leq C(O)\,(1 + t^2)^{-3/2}. \end{aligned}$$

The constants $C(O)$ *depend on* O *as well as the initial state* ρ_{SR} *(but are independent of* t, λ*). For* $O = O_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}$ *we have* $C(O_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}) \leq c\|O_{\mathrm{S}}\|$ *with* c *independent of* O_{S}*.*

Consequences

1. The born approximation holds $\rho_{SR}^t \approx \rho_S^t \otimes \omega_{R\beta}$ & the Markov approximation $\rho_S^t \approx e^{t\mathcal{L}}$ holds, as soon as the initial correlation has decayed $(\sim 1/t^3)$

2. The system dynamics is Markovian at all times:

$$\operatorname{tr}_R(\rho_{SR}^t) = e^{t\mathcal{L}}(\operatorname{tr} \rho_{SR}^0) + O(|\lambda|^{1/4})$$

Outline of proof

$\omega_0 \equiv \rho_{S,\beta} \otimes \omega_{R,\beta}$ uncoupled equilibrium state

$\alpha_\lambda^t(\cdot) \equiv e^{itH_\lambda} \cdot e^{-itH_\lambda}$ Heisenberg dynamics

KMS condition $\omega_0(A\,\alpha_0^{i\beta}(B)) = \omega_0(BA)$

$$\omega_0(K^* \alpha_\lambda^t(0) K)$$

$$= \omega_0(\underline{\alpha_0^{-i\beta}(K) K^*} \alpha_\lambda^t(0))$$

$$\equiv \omega_0(\mathcal{C}\, \alpha_\lambda^t(0))$$

GNS representation

$$= \langle \Omega_0, \pi(\mathcal{C})\, \pi(\alpha_\lambda^t(0))\, \Omega_0 \rangle$$

Liouville operator

$$= \langle \Omega_0, \pi(\mathcal{C})\, e^{itL_\lambda}\, \pi(0)\, e^{-itL_\lambda}\, \Omega_0 \rangle$$

Perturbation theory of KMS states: $\| \Omega_0 - \Omega_\lambda \| \leq C|\lambda|$

Invariance $L_\lambda \Omega_\lambda = 0$

$$= \langle \Omega_0, \pi(\mathcal{C})\, e^{itL_\lambda}\, \pi(0)\, \Omega_0 \rangle + O(|\lambda|)$$

indep. of t

Use <u>resonance expansion</u> [KM 2017, M 2021]

$$e^{itL_\lambda} = e^{itM(\lambda)} \otimes P_\Omega + P_\Omega^\perp e^{itL_\lambda^\perp} P_\Omega^\perp + O(|\lambda|^{1/4})$$

weakly on dense set of vectors

$P_\Omega = |\Omega_0\rangle\langle\Omega_0|$

proj. onto uncoupled equilibrium

$\Omega_0 = \Omega_S \otimes \Omega_R$

$$\omega_0 (K^* \alpha_\lambda^t (0) K)$$

$$= \langle \Omega_0, \pi(e) \left(e^{itM(\lambda)} \otimes P_\Omega \right) \pi(0) \Omega_0 \rangle + \chi(\lambda, t, 0)$$

$$+ O(|\lambda|^{1/4})$$

Term along P_Ω:

$$\left(e^{itM(\lambda)} \otimes P_\Omega\right) \pi\left(O_S \otimes O_R\right) \Omega_0$$

$$= \omega_{R,\beta}(O_R) \cdot \left(e^{itM(\lambda)} \otimes \mathbb{1}_R\right) \pi\left(O_S \otimes \mathbb{1}_R\right) \Omega_0$$

$$= \omega_{R,\beta}(O_R) \cdot \pi\left(\left(e^{t\mathcal{L}^*} O_S\right) \otimes \mathbb{1}_R\right) \Omega_0 \qquad \text{Defines } \mathcal{L}^*$$

$$\langle \Omega_0, \pi(e) \left(e^{itM(a)} \otimes P_\Omega\right) \pi(O_S \otimes O_R) \Omega_0 \rangle$$

$$= \omega_{R,\beta}(O_R) \langle \Omega_0, \pi\left(\alpha^{-i\beta}(K) K^*\right) \pi\left(e^{t\mathcal{L}^*} O_S \otimes \mathbb{1}_R\right) \Omega_0 \rangle$$

KMS cond.

$$= \omega_{R,\beta}(O_R) \langle \Omega_0, \pi(K^*) \pi\left(e^{t\mathcal{L}^*} O_S \otimes \mathbb{1}_R\right) \pi(K) \Omega_0 \rangle$$

$$= \omega_{R,\beta}(O_R) \underbrace{\rho_{SR}\left(e^{t\mathcal{L}^*} O_S \otimes \mathbb{1}_R\right)}_{\rho_S\left(e^{t\mathcal{L}^*} O_S\right)}$$

reduced initial state

$$= \rho_S \otimes \omega_{R,\beta}\left(e^{t\mathcal{L}^*} O_S \otimes O_R\right)$$

$$= \left(e^{t\mathcal{L}} \rho_S \otimes \omega_{R,\beta}\right)(O_S \otimes O_R) \qquad \blacksquare$$

Thank you 🙏