DISCRETE PHASE SPACE TECHNIQUES IN QUANTUM SPIN CHAINS

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"Channels, Maps and All That"

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In memory of Prof. Andrzej Kossakowski

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DISCRETE PHASE SPACE TECHNIQUES IN QUANTUM SPIN CHAINS

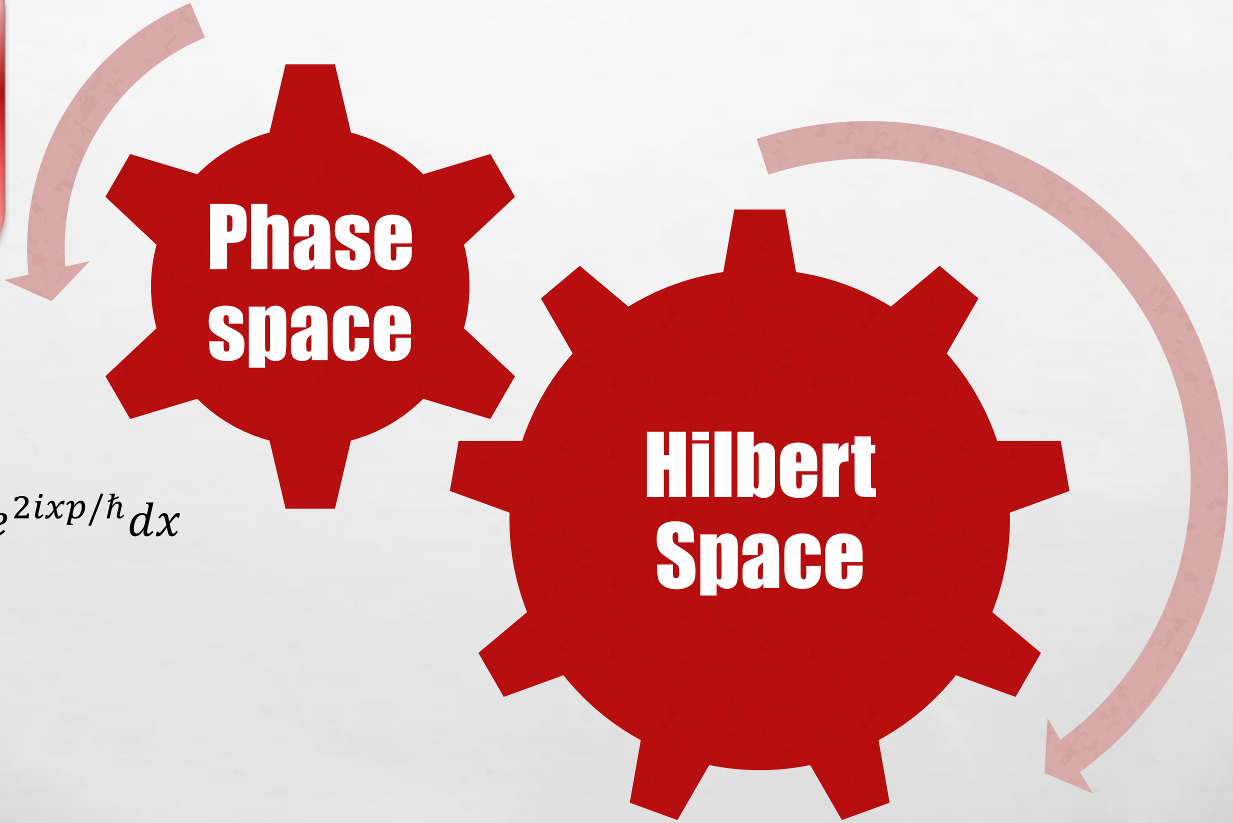
Zakaria Mzaouali, Steve Campbell and MEB,
"Discrete and generalized phase space techniques in critical quantum spin chains"
Phys. Lett. A 383, 125932 (2019)

- **INTRODUCTION**
- **DISCRETE WIGNER FUNCTION AND DISCRETE PHASE SPACE**
- **XY MODEL**
- **XXZ MODEL**
- **CONCLUSION**

INTRODUCTION

Wigner quasi-probability distribution

$$W(q,p) = \frac{1}{2\pi\hbar} Tr[\rho A(q,p)]$$



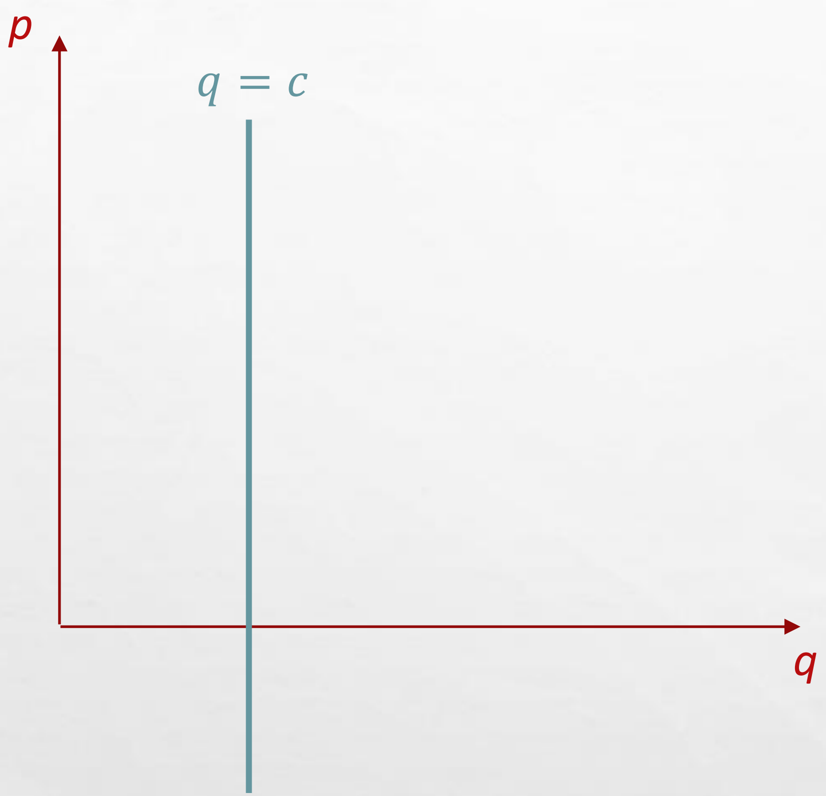
Density operator

$$\rho = \sum p_k |\psi_k\rangle\langle\psi_k|$$

$$W(q,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle q - x|\rho|q + x\rangle e^{2ixp/\hbar} dx$$

Phase space

Quasi-probability? No problem!

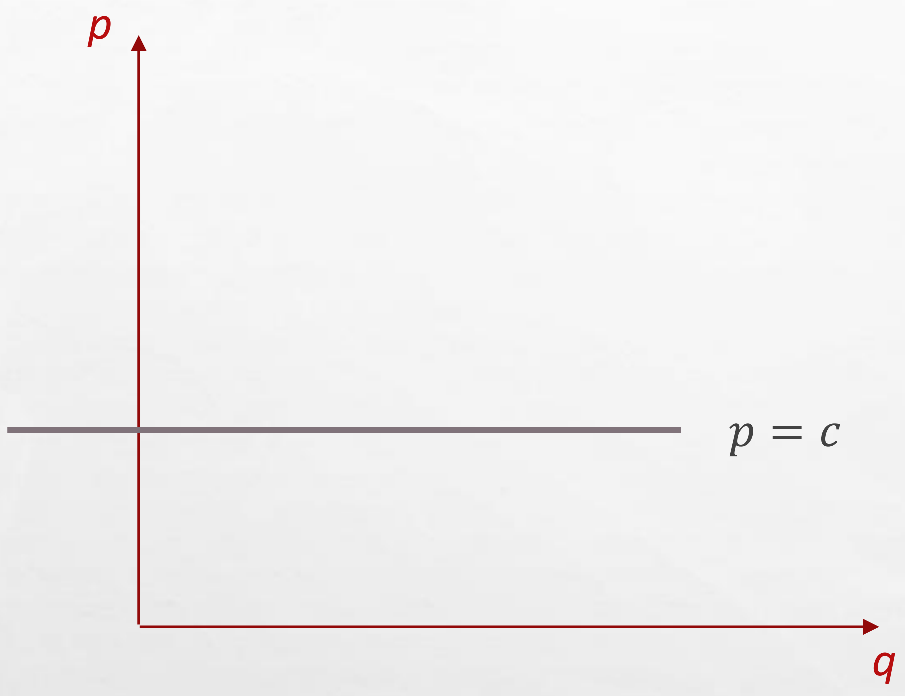


$$\int_{-\infty}^{+\infty} W(q = c, p)\,dp$$

probability that a measure of the observable $\hat{q}$ yields the eigenvalue c.

Phase space

Quasi-probability? No problem!

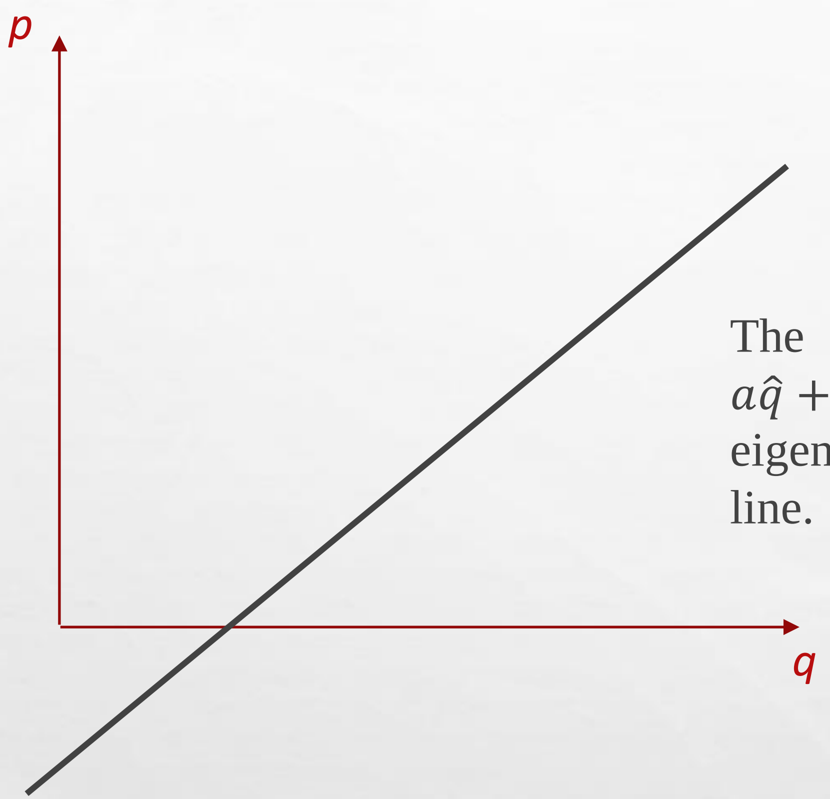


$$\int_{-\infty}^{+\infty} W(q, p = c)\,dq$$

probability that a measure of the observable $\hat{p}$ yields the eigenvalue c.

Phase space

Quasi-probability? No problem!



$$aq + bp = c$$

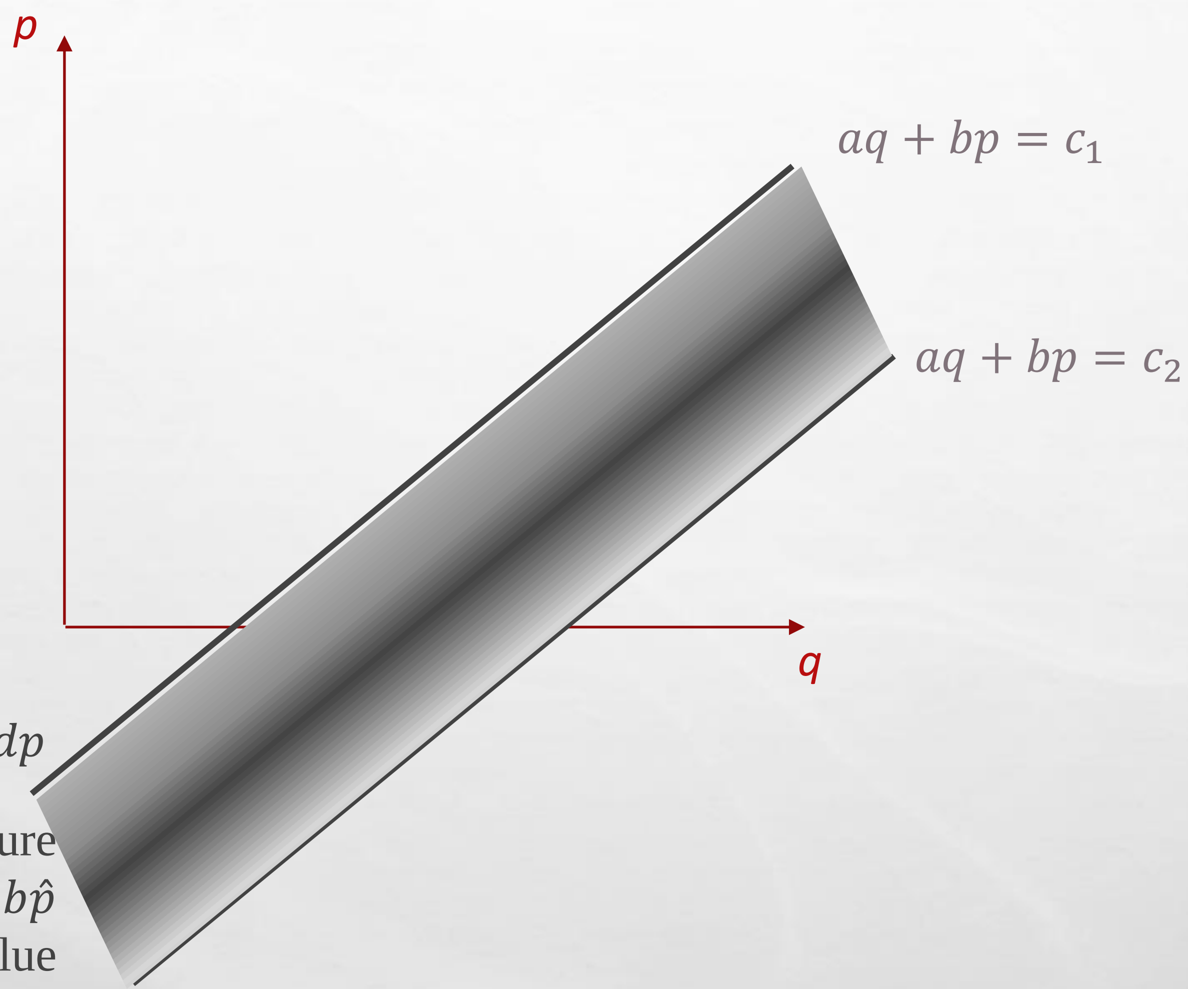
The eigenstate of the operator $a\hat{q} + b\hat{p}$ corresponding to the eigenvalue c is assigned to this line.

$$\int_{aq+bp=c} W(q,p)\,dq\,dp$$

probability that a measure of the observable $a\hat{q} + b\hat{p}$ yields the eigenvalue c.

Phase space

Quasi-probability? No problem!

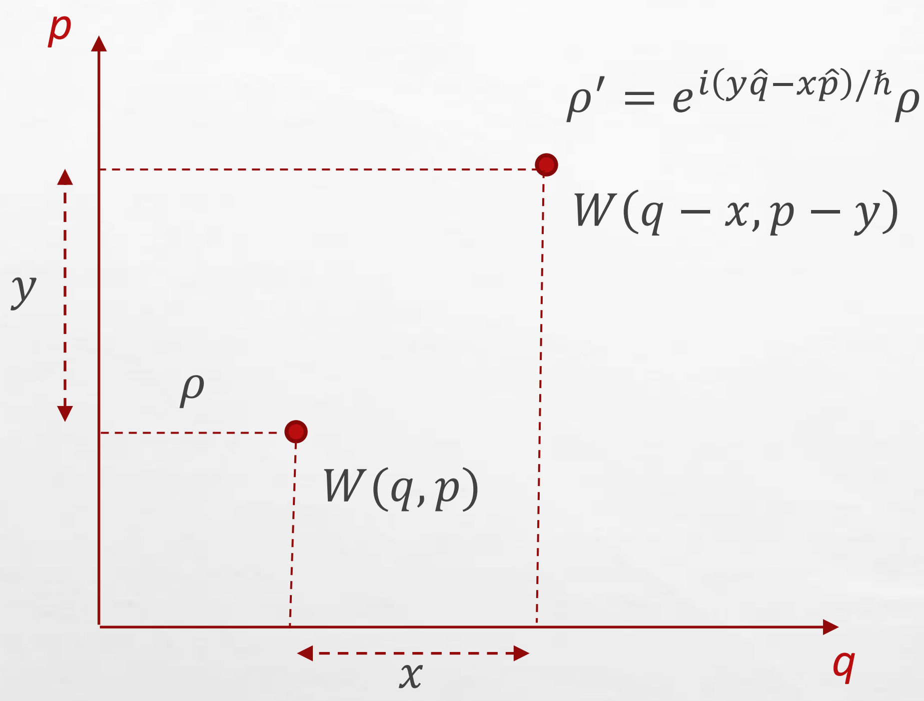


$\int_{c_2<aq+bp<c_1} W(q,p)dqdp$ probability that a measure of the observable $a\hat{q} + b\hat{p}$ yields an eigenvalue between c_1 and c_2.

Phase space

Translational covariance:

$$\rho' = e^{i(y\hat{q}-x\hat{p})/\hbar}\rho e^{-i(y\hat{q}-x\hat{p})/\hbar}$$



DISCRETE PHASE SPACE

- **F. A. Buot,** "Method for calculating TrHn in solid-state theory" **Phys. Rev. B 10, 3700 (1974)**
- **R. Feynman,** "Negative Probabilities" **in Quantum Implications: Essays in Honour of David Bohm, edited by B. Hiley and D. Peat (Routledge, London, 1987)**
- **W. K. Wootters,** "A Wigner-function formulation of finite-state quantum mechanics" **Annals of Physics 176, 1 (1987)**
- **K. S. Gibbons, M. J. Hoffman, and W. K. Wootters,** "Discrete phase space based on finite fields" **Phys. Rev. A 70, 062101 (2004)**

DISCRETE PHASE SPACE

- **K. S. Gibbons, M. J. Hoffman, and W. K. Wootters,** "Discrete phase space based on finite fields" **Phys. Rev. A 70, 062101 (2004)**

The phase space is a vector space over a finite field.

(q, p) are elements of a Galois field $\mathbb{F}_n$.

n, the number of elements in the field is the dimension of the state space of the system.

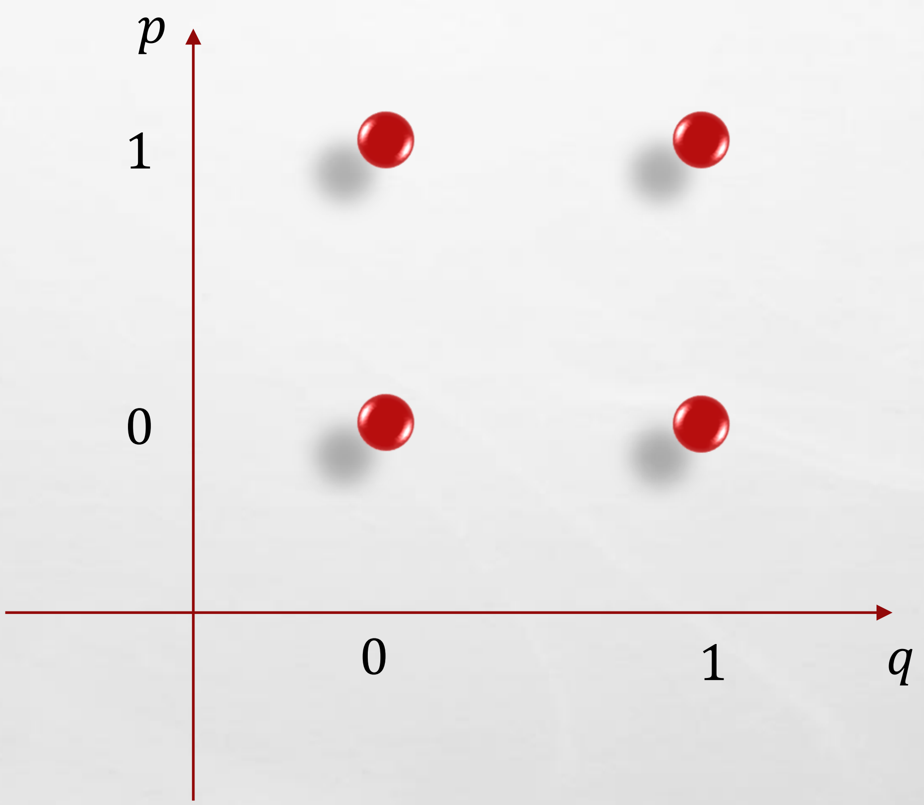
$\mathbb{F}_n$ exists *if and only if* n is a prime power.

If n is a prime number, then the field is $\{0,1,\dots,n-1\}$ with addition and multiplication $modulo\ n$.

If $\mathrm{n} = r^n$ with r being a prime number, then the field $\mathbb{F}_n$ is an extension of the filed $\mathbb{F}_r$.

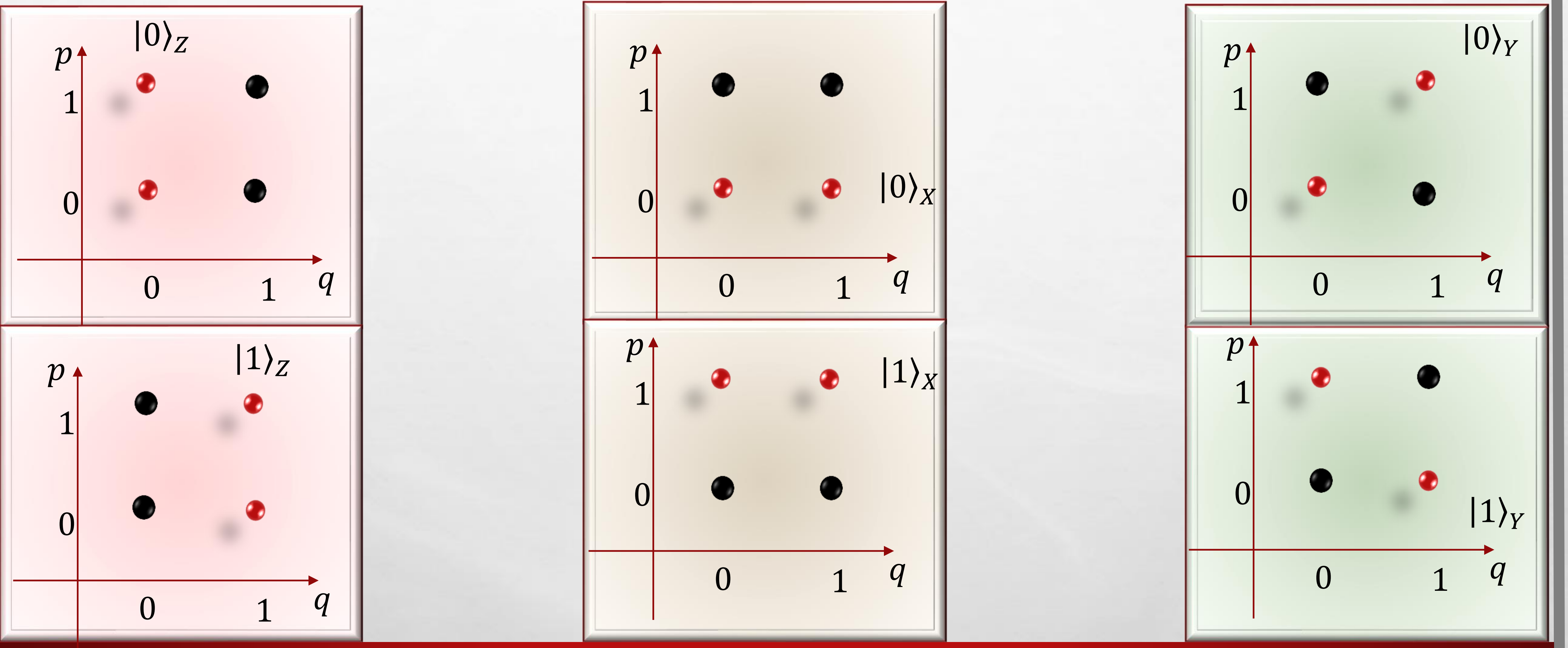
One q-bit discrete phase space

(1 qubits $\Rightarrow n = 2$) $\mathbb{F}_2 = (0,1)$



One q-bit discrete phase space

Line equation: $aq + bp = c$ $\quad a, b, c \in \mathbb{F}_2$

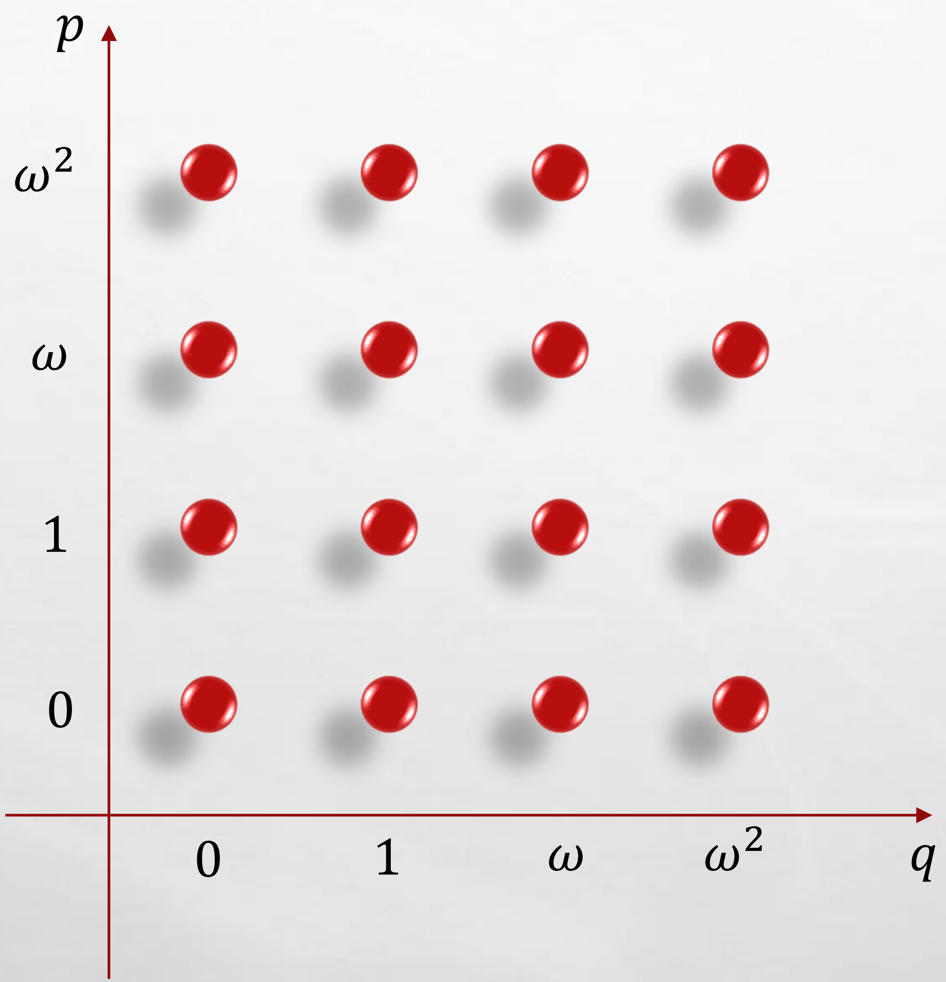


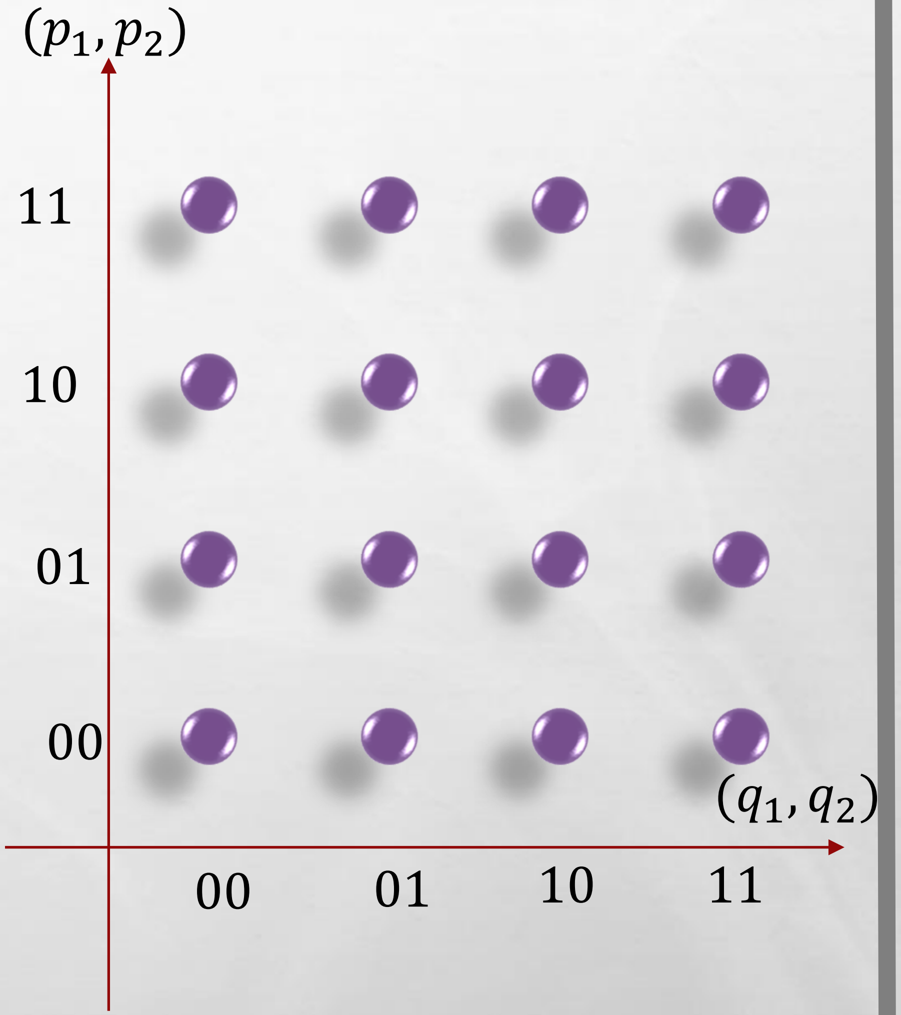
Two q-bit discrete phase space

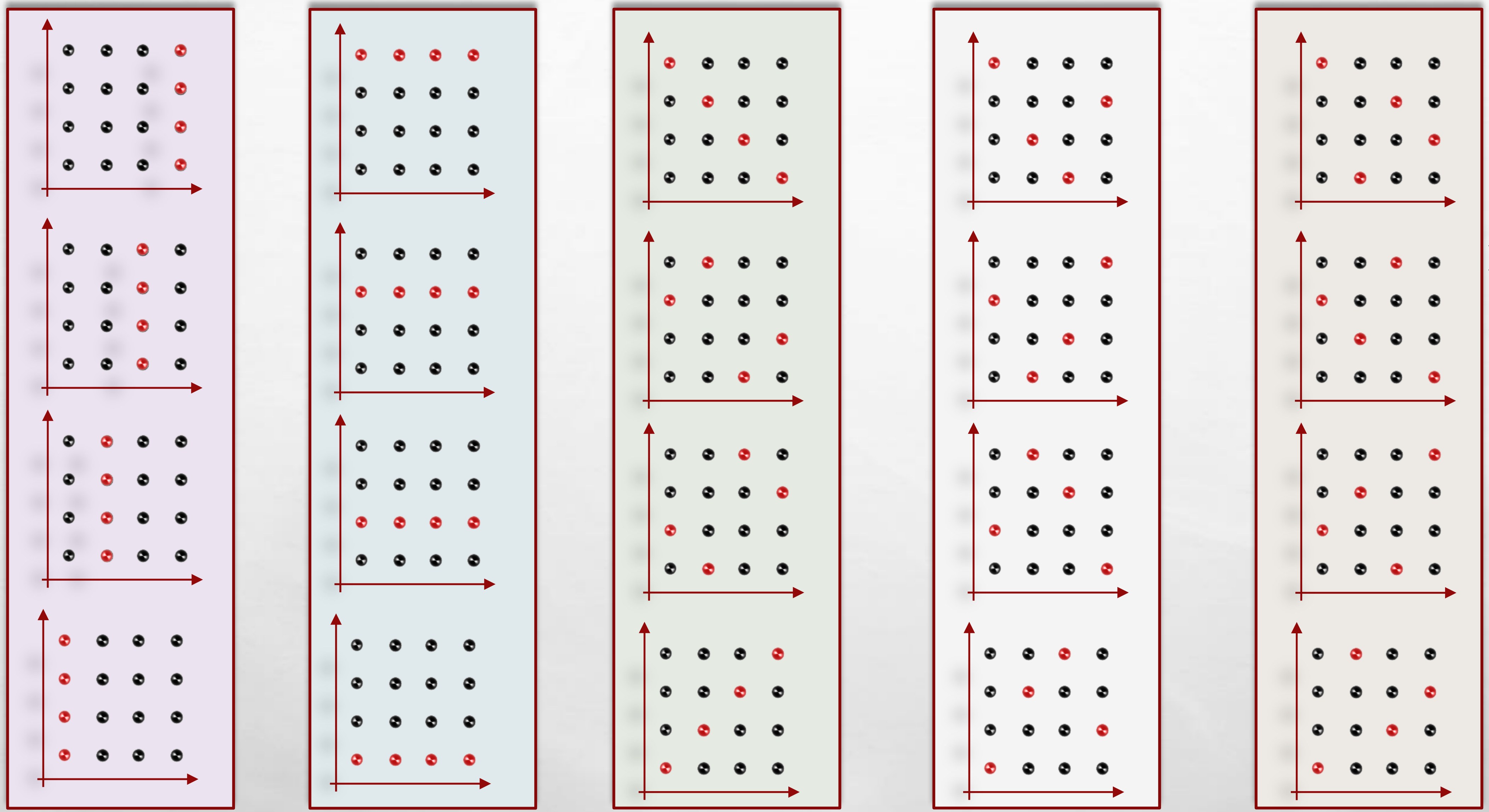
(2 qubits $\Rightarrow n = 4 = 2^2$) $\mathbb{F}_4 = (0, 1, \omega, \omega + 1 = \omega^2)$

+	0	1	ω	ω^2
0	0	1	ω	ω^2
1	1	0	ω^2	ω
ω	ω	ω^2	0	1
ω^2	ω^2	ω	1	0

$\times$	0	1	ω	ω^2
0	0	0	0	0
1	0	1	ω	ω^2
ω	0	ω	ω^2	1
ω^2	0	ω^2	1	ω







Line equation:
$aq + bp = c$

$a, b, c \in \mathbb{F}_4$

DISCRETE WIGNER FUNCTION

One to One correspondence between the Hilbert space of states and the Phase space associated to a given system.

Assign to each line λ in phase space a state as represented by a projector $Q(\lambda)$.

Impose that the probability that the system is in state $Q(\lambda)$ is given by the sum of the Wigner function over the line λ:

$$\sum_{\alpha\in\lambda} W_\alpha = Tr[\rho Q(\lambda)]$$

$$\rho = \sum_{(x_i,p_i)} W(x_i,p_i)\hat{A}(x_i,p_i)$$

$$W(x_i,p_i) = \frac{1}{d}\mathrm{Tr}\left(\rho\hat{A}(x_i,p_i)\right)$$

$$\hat{A}(x_1,p_1) = \frac{1}{2}[I + (-1)^{x_1}\sigma^z + (-1)^{p_1}\sigma^x + (-1)^{x_1+p_1}\sigma^y]$$

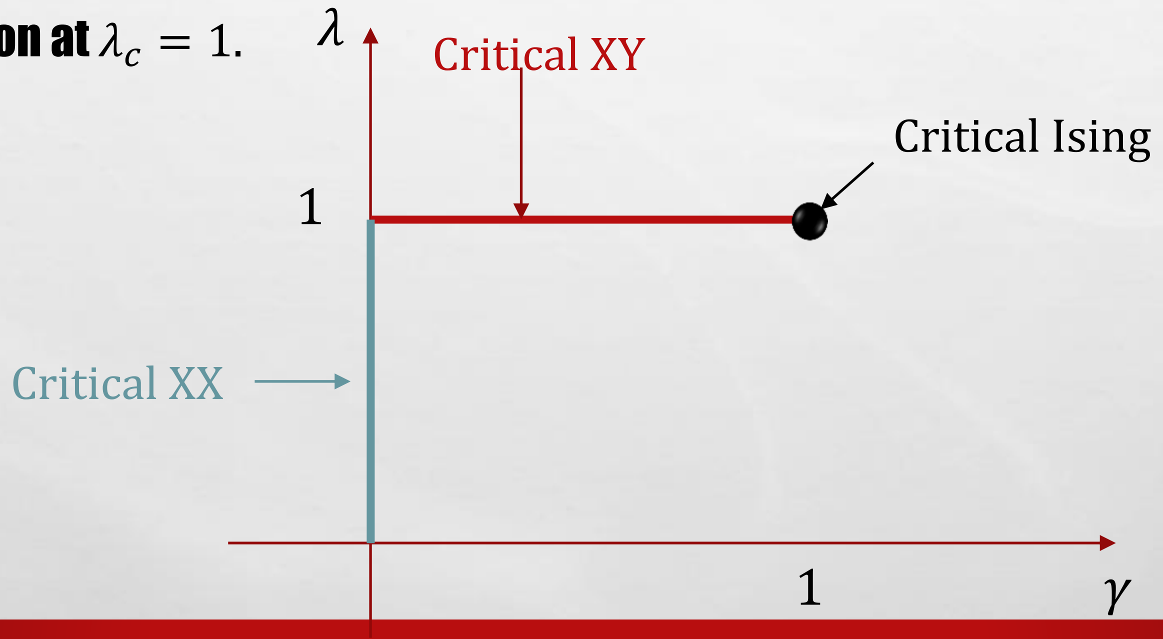
$$\hat{A}(x_1 \dots x_k, p_1 \dots p_k) = \hat{A}(x_1,p_1) \otimes \hat{A}(x_2,p_2) \otimes \cdots \otimes \hat{A}(x_k,p_k)$$

XY MODEL

Spin-$\frac{1}{2}$ anisotropic XY model with periodic boundary conditions:

$$\mathcal{H}_{XY} = -\sum_{i=0}^{N-1} \left[\frac{\lambda}{2} \{ (1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y \} + \sigma_i^z \right] \qquad \gamma \in [0,1]$$

Second order phase transition at $\lambda_c = 1$.



Single site:

Reduced density matrix

$$\rho_i = \frac{1}{2}\sum_{\alpha=0}^{3} \langle \sigma^\alpha \rangle \sigma_i^\alpha$$

DWF

$$W(x_1, p_1) = \frac{1}{2}\left(1 + (-1)^{x_1} \langle \sigma^z \rangle\right)$$

p_1

1

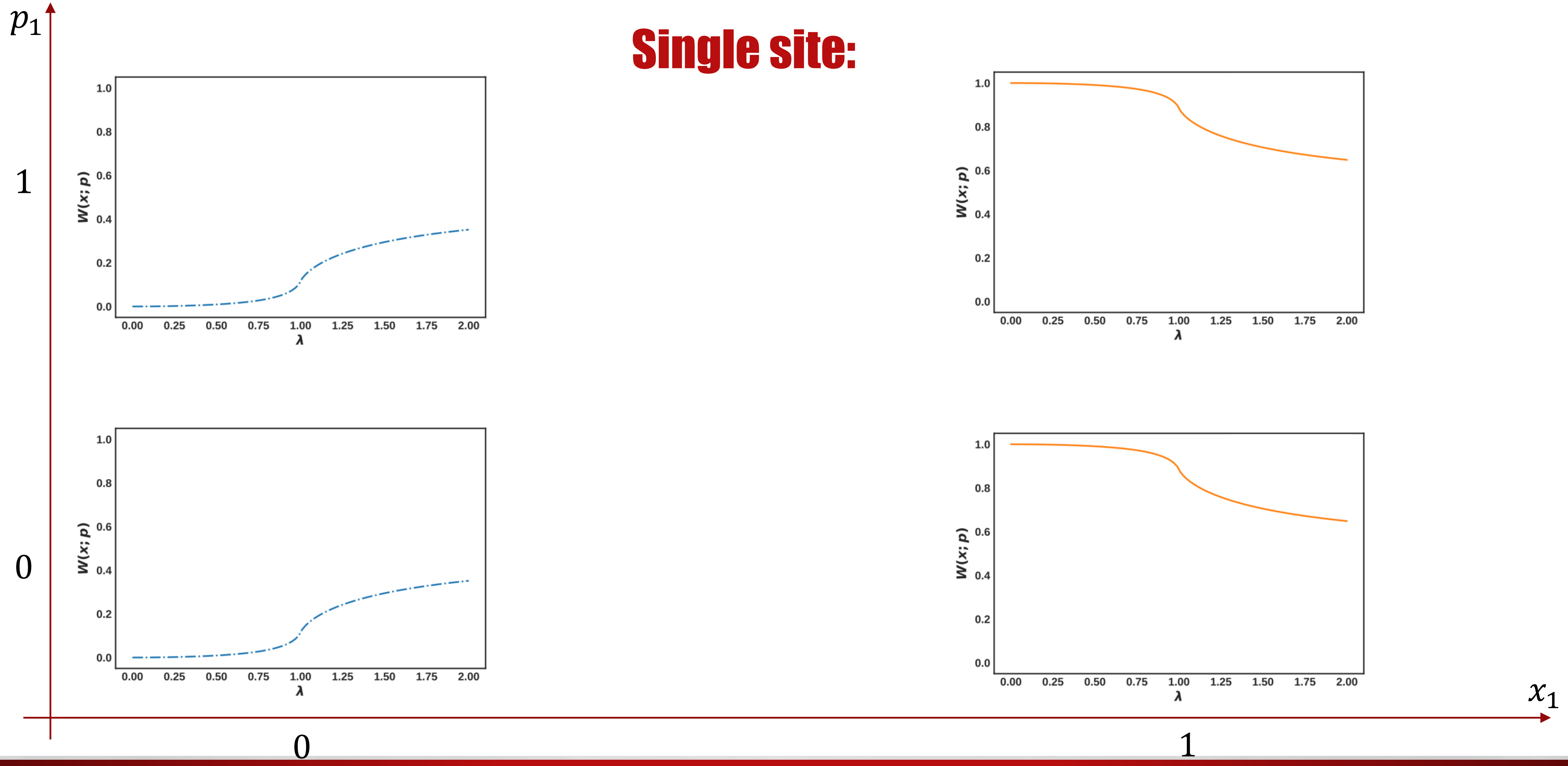
0

x_1

0

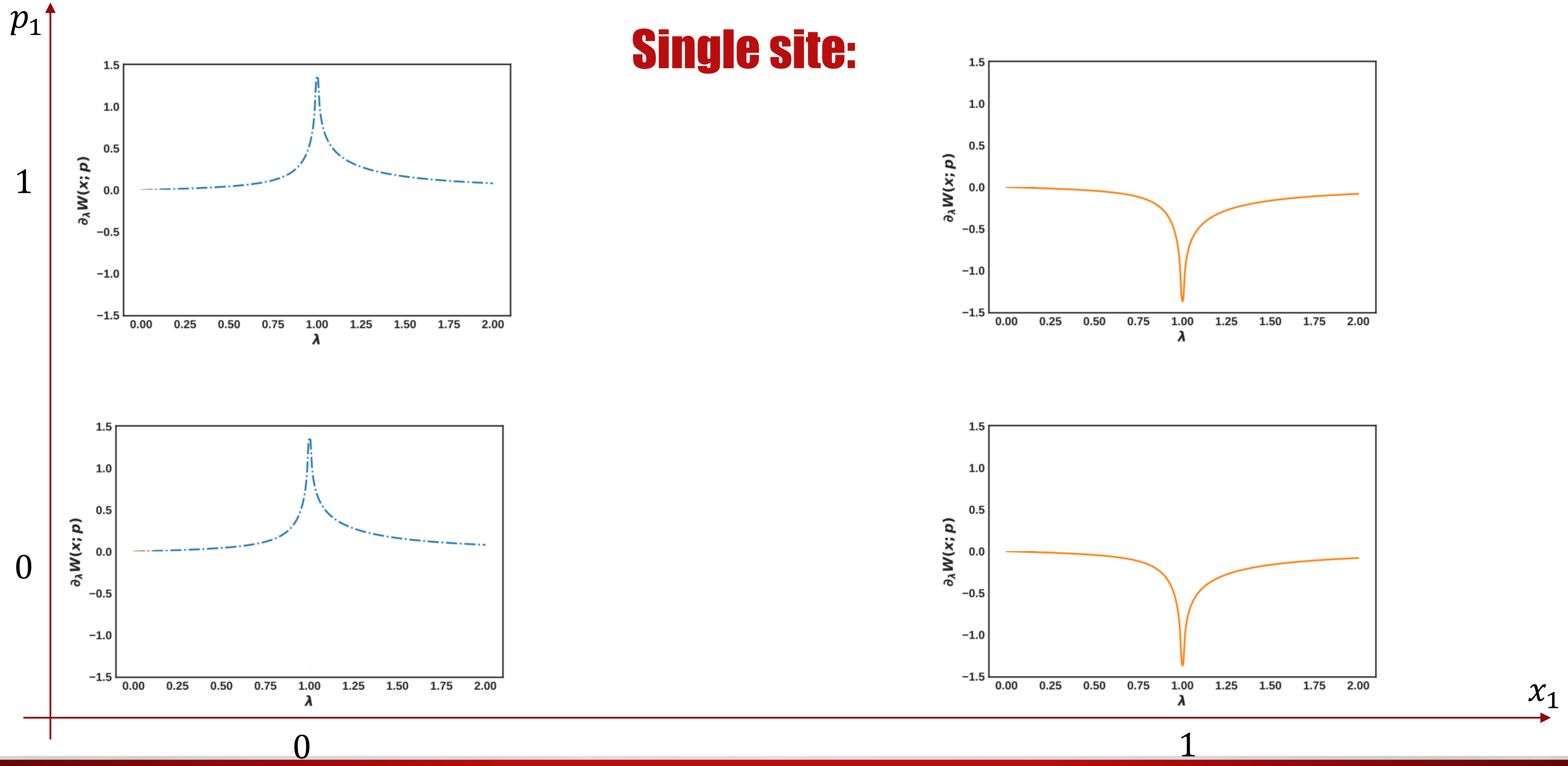
1

Single site:

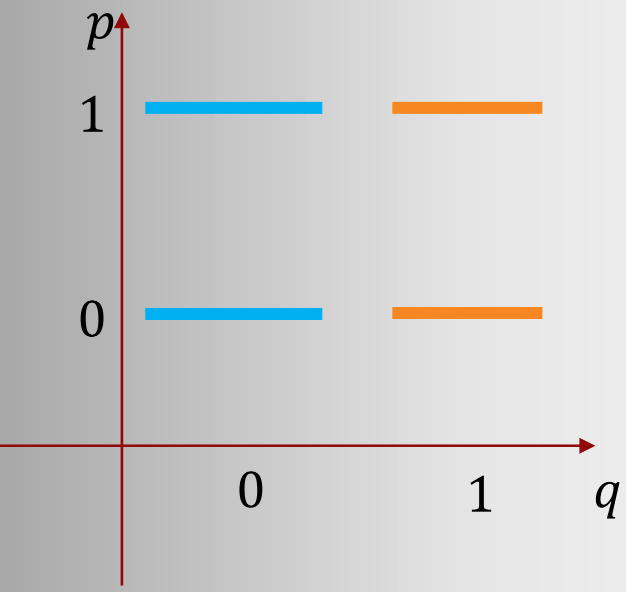


p_1
1
0
x_1
0
1
W(x; p)
λ
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

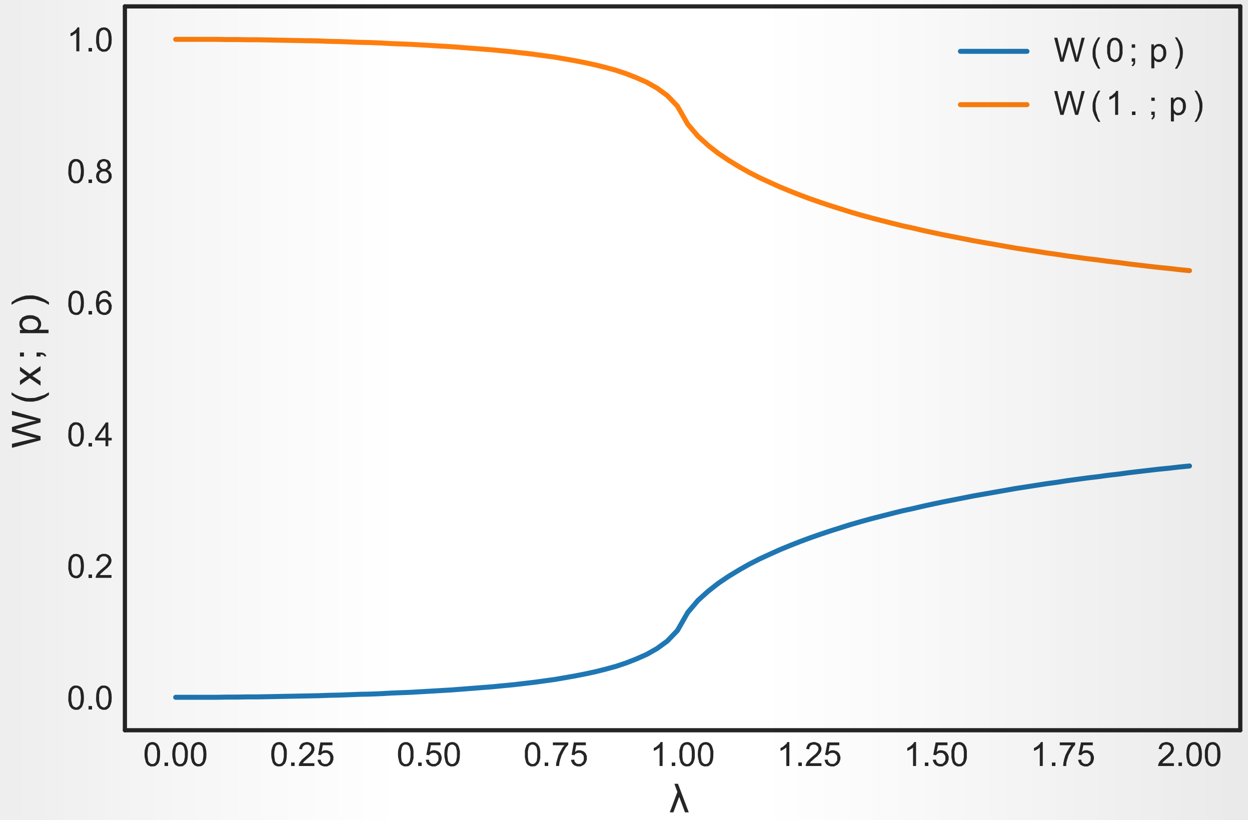
Single site:



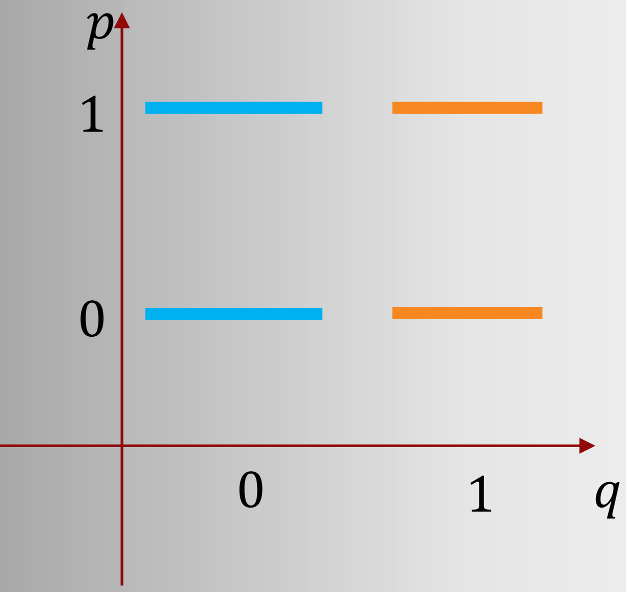
p_1
1
0
x_1
0
1
$\partial_\lambda W(x;p)$
λ
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

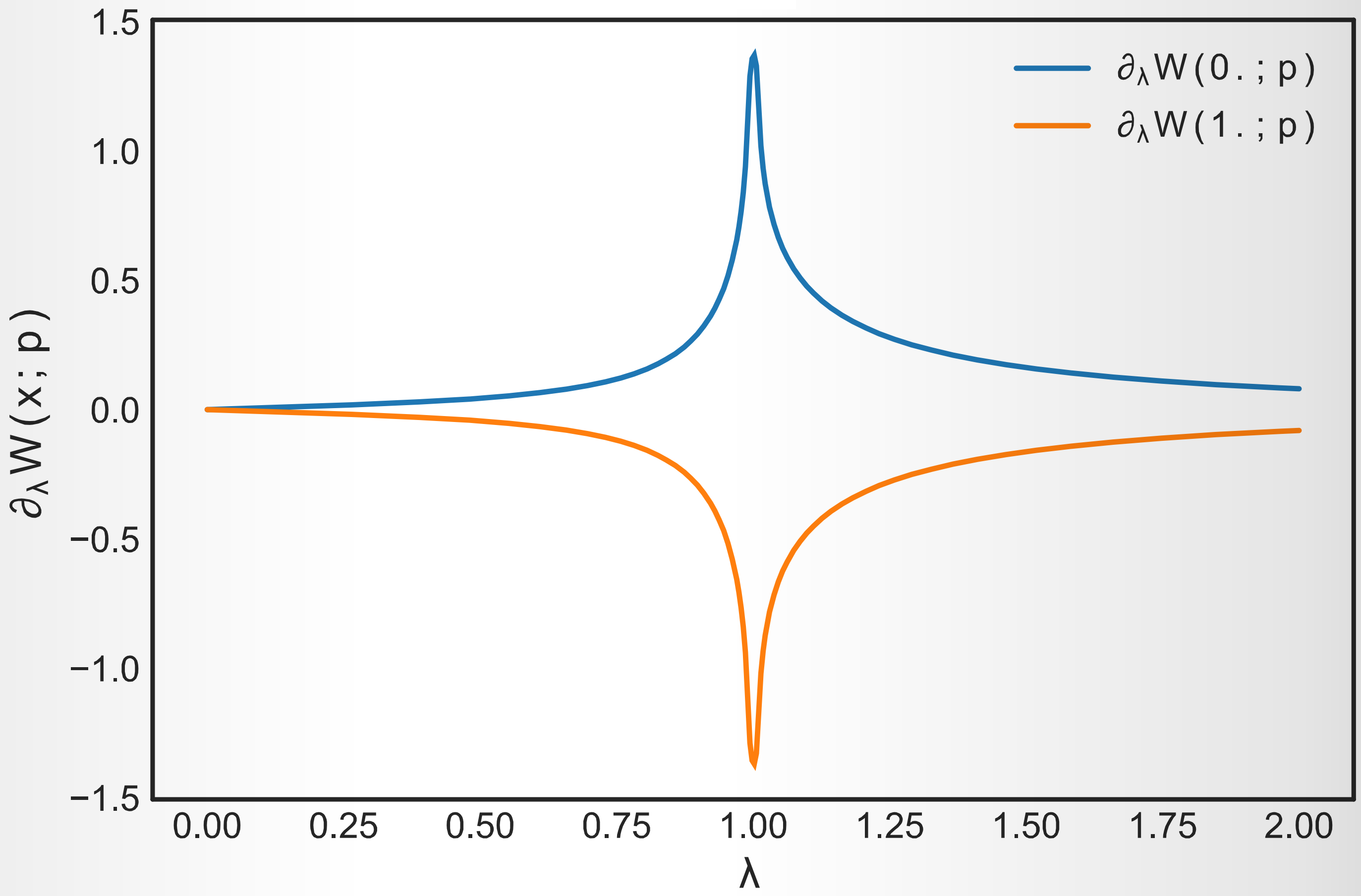

p
1
0
0
1
q

Single site:


W(0; p)
W(1. ; p)
1.0
0.8
0.6
0.4
0.2
0.0
W(x; p)
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
λ

Single site:


p
1
0
0
1
q


$\partial_\lambda W(0.;p)$
$\partial_\lambda W(1.;p)$
$\partial_\lambda W(x;p)$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
λ

Two site:

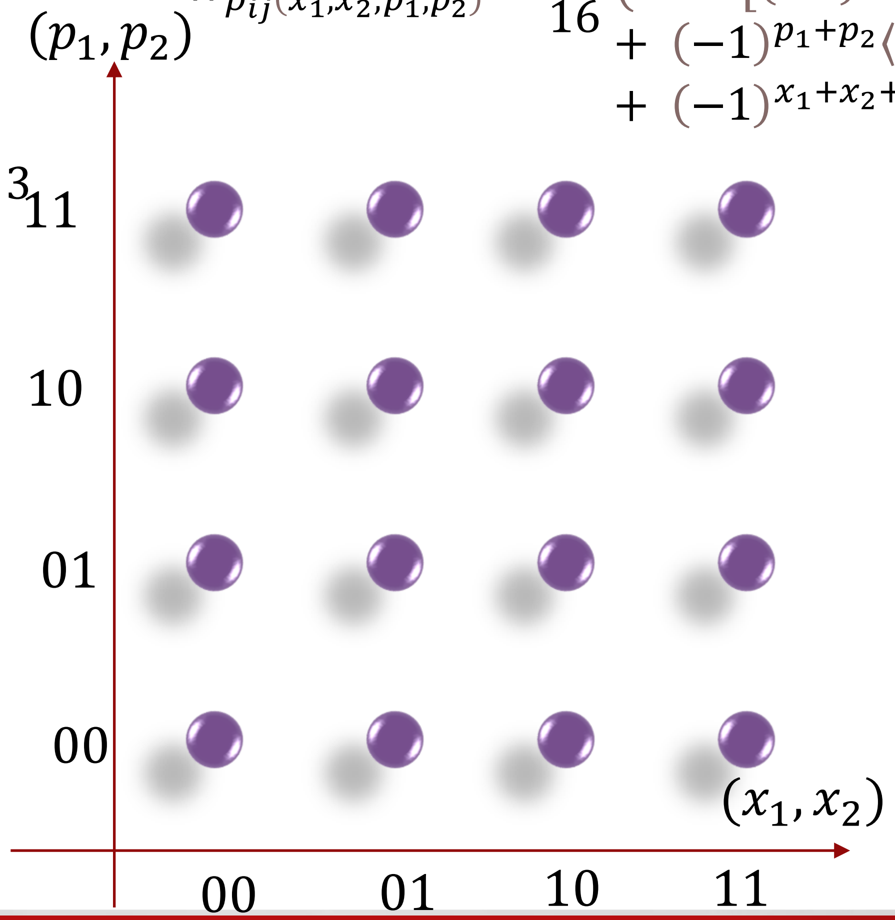
Reduced density matrix

$$\rho_{i,i+m} = \frac{1}{4}\sum_{\alpha,\beta=0}^{3} p_{\alpha\beta}\sigma_i^{\alpha} \otimes \sigma_{i+m}^{\beta}$$

with $p_{\alpha\beta} = \left\langle \sigma_i^{\alpha}\sigma_{i+m}^{\beta} \right\rangle$ $\quad (\alpha, \beta) = 0, 1, 2, 3$

DWF

$$W_{\rho_{ij}(x_1,x_2;p_1,p_2)} = \frac{1}{16}\Big(1 + [(-1)^{x_1} + (-1)^{x_2}]\langle\sigma^z\rangle + (-1)^{p_1+p_2}\langle\sigma_i^x\sigma_{i+m}^x\rangle + (-1)^{x_1+x_2}\langle\sigma_i^z\sigma_{i+m}^z\rangle + (-1)^{x_1+x_2+p_1+p_2}\langle\sigma_i^y\sigma_{i+m}^y\rangle\Big)$$



Two site:

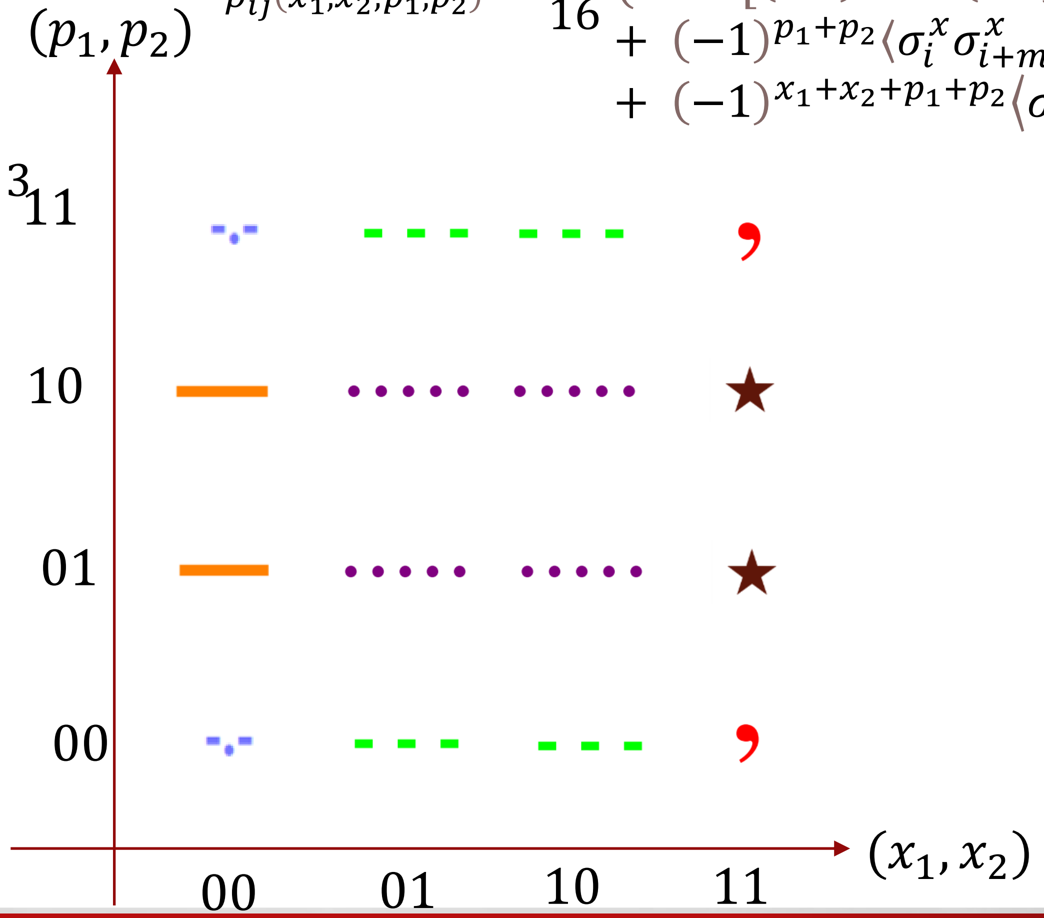
Reduced density matrix

$$\rho_{i,i+m} = \frac{1}{4}\sum_{\alpha,\beta=0}^{3} p_{\alpha\beta}\sigma_i^{\alpha} \otimes \sigma_{i+m}^{\beta}$$

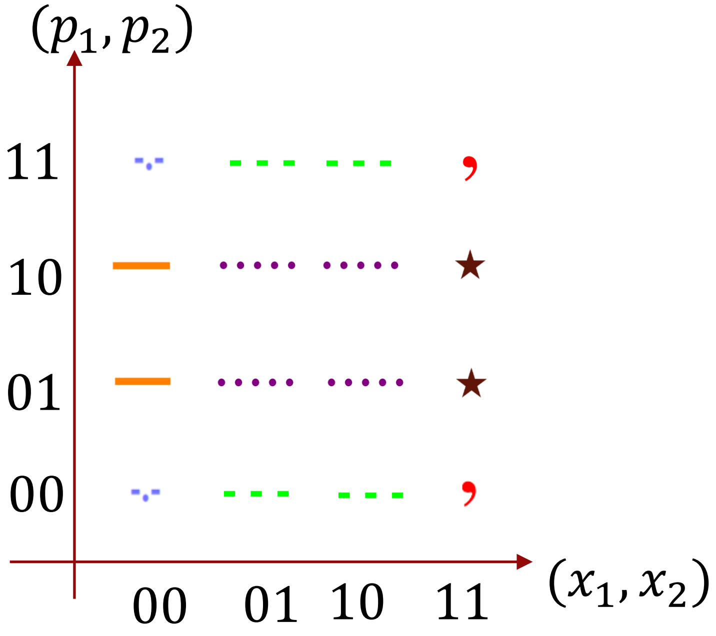
with $p_{\alpha\beta} = \left\langle \sigma_i^{\alpha}\sigma_{i+m}^{\beta} \right\rangle \qquad (\alpha,\beta) = 0, 1, 2, 3$

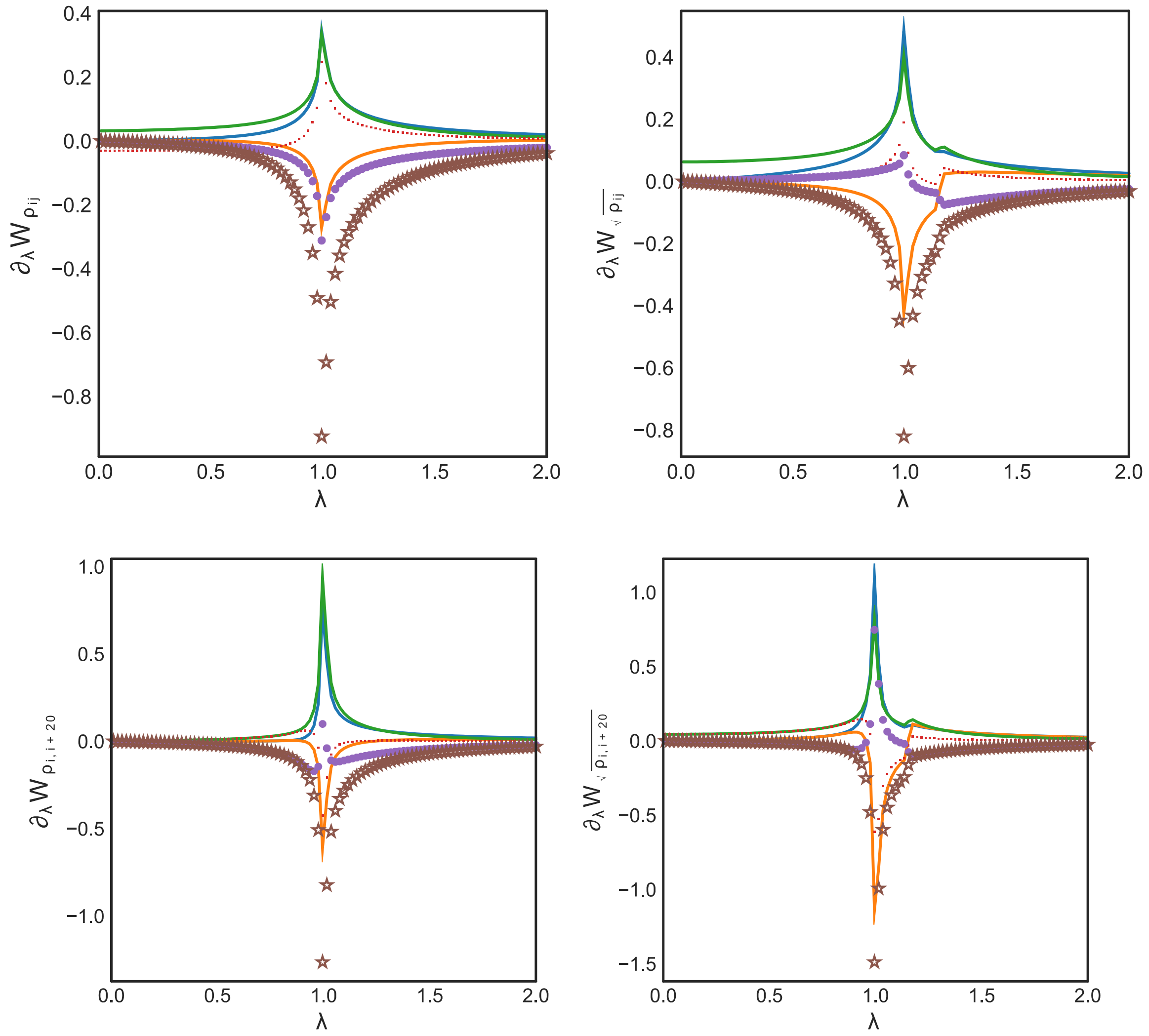
DWF

$$W_{\rho_{ij}(x_1,x_2;p_1,p_2)} = \frac{1}{16}\Big(1 + \left[(-1)^{x_1} + (-1)^{x_2}\right]\langle\sigma^z\rangle + (-1)^{p_1+p_2}\langle\sigma_i^x\sigma_{i+m}^x\rangle + (-1)^{x_1+x_2}\langle\sigma_i^z\sigma_{i+m}^z\rangle + (-1)^{x_1+x_2+p_1+p_2}\langle\sigma_i^y\sigma_{i+m}^y\rangle\Big)$$



Two site:


(p_1, p_2)
11
10
01
00
00
01
10
11
(x_1, x_2)


$\partial_\lambda W_{\rho_{ij}}$
$\partial_\lambda W_{\sqrt{\rho_{ij}}}$
$\partial_\lambda W_{\rho_{i,i+20}}$
$\partial_\lambda W_{\sqrt{\rho_{i,i+20}}}$
λ
λ
λ
λ

XXZ MODEL

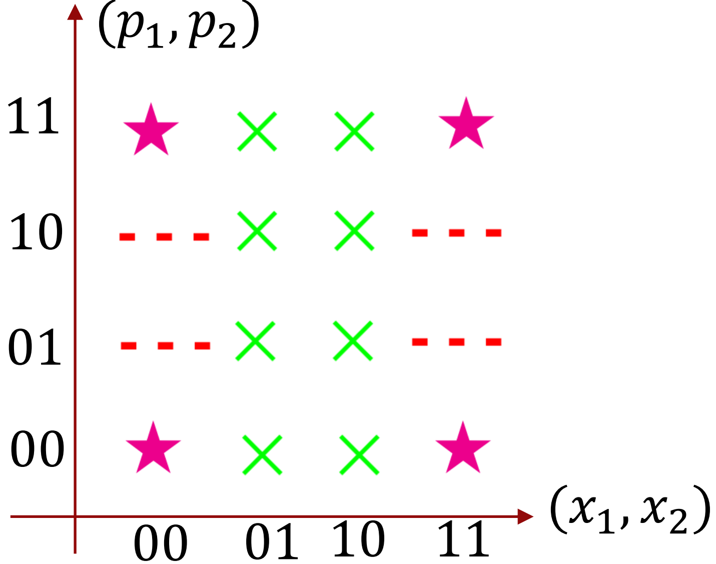
Spin-$\frac{1}{2}$ XXZ model with periodic boundary conditions:

$$\mathcal{H}_{XXZ} = \frac{1}{4}\sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z$$

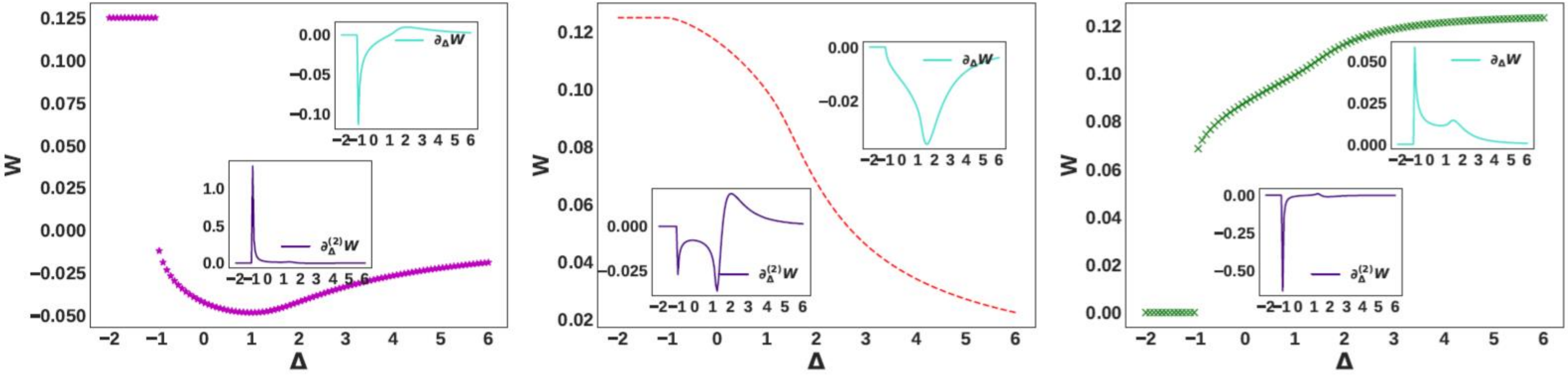
First order phase transition at $\Delta = -1$.

Continuous phase transition at $\Delta = 1$.

Two site:



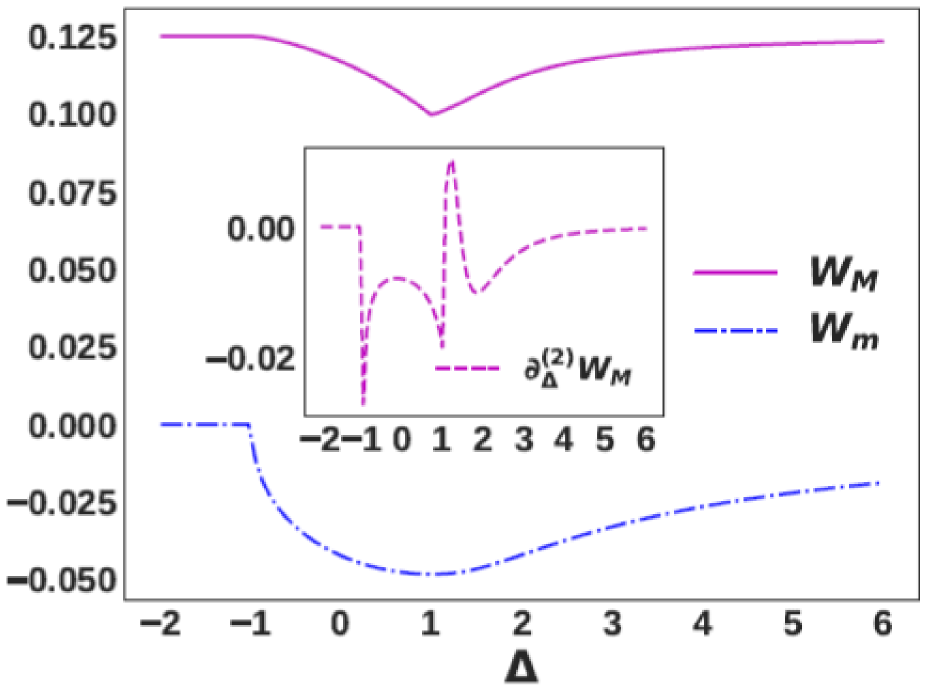
(p_1, p_2)
11
10
01
00
(x_1, x_2)
00
01
10
11



W
Δ
$\partial_\Delta W$
$\partial_\Delta^{(2)} W$

Two site:

$$W_M = max(W_{00,00}, W_{00,01}, W_{01,00})$$



$$W_m = min(W_{00,00}, W_{00,01}, W_{01,00})$$

- WHY IT WORKS?
- WHY IT DOESN'T WORK?
- OTHER MODELS?
- OTHER DIMENSIONS?

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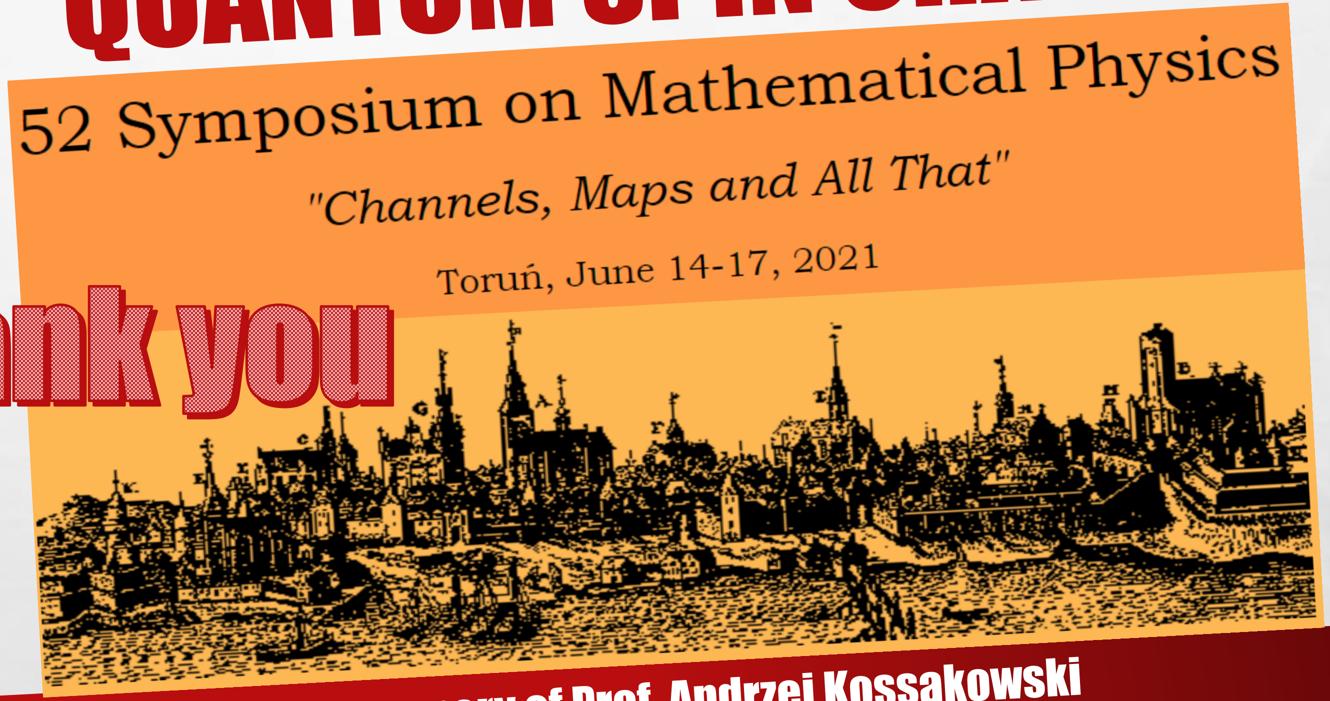
Thank you



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