Open Quantum Random Walks and Quantum Markov chains on Trees

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1. Introduction

Discovering the aspects of quantum mechanics, such as superposition and interference, has lead to the idea of quantum walks, a generalization of classical random walks [24, 41]. Recently, in [20] a quantum phase transition has been explored by means of quantum walks in an optical lattice. On the other hand, in [34] it has been showed that discrete-time quantum walks (QW) can realise topological phases in 1D and 2D for all the symmetry classes of free-fermion systems. In particular, they provide the QW protocols that simulate representatives of all topological phases, featured by the presence of robust symmetry- protected edge states [35]. In general, QW realisations are particularly useful, because, in addition to the simplicity of their mathematical description, the parameters that define them can be easily controlled in the lab.

Over the past decade, motivated largely by the prospect of superefficient algorithms, the theory of quantum Markov chains (QMC), especially in the guise of quantum walks, has generated a huge number of works, including many discoveries of fundamental importance [12, 27, 33, 50]. In [30] it has been proposed a novel approach to investigate quantum cryptography problems by means of QMC [32] where quantum effects are entirely encoded into super-operators labelling transitions, and the nodes of its transition graph carry only classical information and thus they are discrete. Recently, QMC have been applied [27, 26] to the investigations of so-called "open quantum random walks" (OQRW) [13, 18, 36, 37, 49]. We notice that OQRW are related to the study of asymptotic behavior of trace-preserving completely positive maps, which belong to fundamental topics of quantum information theory (see for instance [17, 38, 39, 40, 45, 46]).

For the sake of clarity, let us recall some necessary information about OQRW. Let $\mathcal{K}$ denote a separable Hilbert space and let $\{|i\rangle\}_{i\in\Lambda}$ be its orthonormal basis indexed by the vertices of some graph Λ (here the set Λ of vertices might be finite or countable). Let $\mathcal{H}$ be another Hilbert space, which will describe the degrees of freedom given at each point of Λ. Then we will consider the space $\mathcal{H}\otimes\mathcal{K}$. For each pair i, j one associates a bounded linear operator B^i_j on $\mathcal{H}$. This operator describes the effect of passing from $|j\rangle$ to $|i\rangle$. We will assume that for each j, one has

$$\sum_i B^{i*}_j B^i_j = \mathbb{1}, \tag{1}$$

where, if infinite, such series is strongly convergent. This constraint means: the sum of all the effects leaving site j is $\mathbb{1}$. The operators B^i_j act on $\mathcal{H}$ only, we dilate them as operators on $\mathcal{H}\otimes\mathcal{K}$ by putting

$$M^i_j = B^i_j \otimes |i\rangle\langle j| \, .$$

The operator M^i_j encodes exactly the idea that while passing from $|j\rangle$ to $|i\rangle$ on the lattice, the effect is the operator B^i_j on $\mathcal{H}$.

According to [13] one has

$$\sum_{i,j} M^{i*}_j M^i_j = \mathbb{1}. \tag{2}$$

Therefore, the operators $(M_j^i)_{i,j}$ define a completely positive mapping

$$\mathcal{M}(\rho) = \sum_i \sum_j M_j^i \, \rho \, M_j^{i*} \tag{3}$$

on $\mathcal{H} \otimes \mathcal{K}$.

In what follows, we consider density matrices on $\mathcal{H} \otimes \mathcal{K}$ which take the form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|, \tag{4}$$

assuming that $\sum_i \mathrm{Tr}(\rho_i) = 1$.

For a given initial state of such form, the *Open Quantum Random Walk (OQRW)* is defined by the mapping $\mathcal{M}$, which has the following form

$$\mathcal{M}(\rho) = \sum_i \Big(\sum_j B_j^i \rho_j B_j^{i*} \Big) \otimes |i\rangle\langle i|. \tag{5}$$

If the evolution is performed two times we have

$$\mathcal{M}^2(\rho) = \sum_i \sum_j \sum_k B^i_j B^j_k \, \rho_k \, {B^j_k}^* {B^i_j}^* \otimes |i\rangle\langle i| \, .$$

Hence measuring the position after two steps, we get the site $|i\rangle$ with probability

$$\sum_j \sum_k \mathrm{Tr}\left(B^i_j B^j_k \, \rho_k \, {B^j_k}^* {B^i_j}^* \right) \, .$$

By means of the map $\mathcal{M}$ one defines a family of classical random process on $\Omega = \Lambda^{\mathbb{Z}_+}$. Namely, for any density operator ρ on $\mathcal{H} \otimes \mathcal{K}$ (see (4)) the probability distribution is defined by

$$(6) \qquad \mathbb{P}_\rho(i_0, i_1, \ldots, i_n) = \mathrm{Tr}(B^{i_n}_{i_{n-1}} \cdots B^{i_2}_{i_1} B^{i_1}_{i_0} \rho_{i_0} B^{i_1 *}_{i_0} B^{i_2 *}_{i_1} \cdots B^{i_n *}_{i_{n-1}}).$$

We point out that this distribution is not a Markov measure [15].

On the other hand, it is well-known [44] that to each classical random walk one can associate certain Markov chain and some properties of the walk can be explored by the constructed chain. Therefore, it is natural to construct Quantum Markov chain associated with OQRW and investigate its properties.

Let us denote $\Omega_{\mathbb{Z}_+} = \Lambda^{\mathbb{Z}_+}$, $\Omega_{\mathbb{Z}} = \Lambda^{\mathbb{Z}}$, here $\mathbb{Z}_+$ denotes the set of all non negative integers. A subset of $\Omega_{\mathbb{Z}_+}$ (resp. $\Omega_{\mathbb{Z}}$) given by

$$A^{[l,m]}(i_l, i_{l+1}, \dots, i_m) = \{\omega \in \Omega_{\mathbb{Z}_+}\, w_l = i_l, \dots, \omega_m = i_m\}.$$

is called *thin cylindrical set*, where $i_k \in \Lambda$, $k \in \mathbb{Z}_+$. By $\mathfrak{F}$ we denote the σ-algebra generated by thin cylindrical sets.

Since the finite disjoint unions of thin cylinders form an algebra which generates $\mathfrak{F}$, therefore a measure μ on $\mathfrak{F}$ is uniquely determined by the values:

$$\mu_n(A^{[l,n]}(i_l, i_{l+1}, \dots, i_n)).$$

which should satisfy the compatibility conditions, i.e.

$$\sum_{j\in\Lambda} \mu_{n+1}(A^{[0,n+1]}(i_0, i_1, \dots, i_n, j)) = \mu_n(A^{[0,n]}(i_0, i_1, \dots, i_n)) \tag{7}$$

The Kolmogorov's Theorem ensures the existence of the measure μ on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$.

Now for a given $\mathcal{M}$ (see (3)) and a fixed ρ (see (4)), for every $n \in \mathbb{N}$, we define a measure $\mathbb{P}_{\rho,n}$ on $\Omega_n := \Lambda^{[0,n]}$ as the distribution of the OQRW, i.e.

(8)

$$\mathbb{P}_{\rho,n}(A^{[0,n]}(i_0, i_1, \ldots, i_n)) = \mathrm{Tr}(B^{i_n}_{i_{n-1}} \cdots B^{i_2}_{i_1} B^{i_1}_{i_0} \rho_{i_0} B^{i_1 *}_{i_0} B^{i_2 *}_{i_1} \cdots B^{i_n *}_{i_{n-1}}).$$

Proposition 1.1. *Let $\mathbb{P}_\rho$ be a measure defined on $(\Omega_{\mathbb{Z}_+}, \mathfrak{F})$ associated with OQRW $\mathcal{M}$ and an initial density operator ρ. If $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$ is an invariant density operator w.r.t. $\mathcal{M}$, then the measure $\mathbb{P}_\rho$ can be extended to $(\Omega_{\mathbb{Z}}, \mathfrak{F})$.*

Recently, in [27], we have found a quantum Markov chain (QMC)[1] (or finitely correlated state (FCS)[28]) φ on the algebra $\mathcal{A} = \otimes_{i\in\mathbb{Z}_+}\mathcal{A}_i$, where $\mathcal{A}_i$ is isomorphic to $B(\mathcal{H}) \otimes B(\mathcal{K})$, $i \in \mathbb{Z}_+$, such that the transition operator P equals to the mapping $\mathcal{M}^*$ and the restriction of φ to the commutative subalgebra of $\mathcal{A}$ coincides with the distribution $\mathbb{P}_\rho$, i.e.

$$\varphi\Big((\mathbb{1}\otimes|i_0><i_0|)\otimes\cdots\otimes(\mathbb{1}\otimes|i_n><i_n|)\Big) = \mathbb{P}_\rho(i_0,i_1,\dots,i_n). \tag{9}$$

Hence, this result allows us to interpret the distribution $\mathbb{P}_\rho$ as a QMC, and to study further properties of $\mathbb{P}_\rho$.

[1] We note that a Quantum Markov Chain is a quantum generalization of a Classical Markov Chain where the state space is a Hilbert space, and the transition probability matrix of a Markov chain is replaced by a transition amplitude matrix, which describes the mathematical formalism of the discrete time evolution of open quantum systems, see [3]-[**?**],[28, 31] for more details.

In the present paper, we are going to look at the probability distribution (6) as a Markov field over the Cayley tree Γ^k. Roughly speaking, $(i_0, i_1, \ldots, i_n)$ is considered as a configuration on $\Omega = \Lambda^{\Gamma^k}$. Such kind of consideration, allows us to investigated phase transition phenomena associated for OQRW within QMC scheme [52, **?**]. We stress that, in physics, a spacial classes of QMC, called "Matrix Product States" (MPS) and more generally "Tensor Network States" [21, 47] were used to investigate quantum phase transitions for several lattice models. This method uses the density matrix renormalization group (DMRG) algorithm which opened a new way of performing the renormalization procedure in 1D systems and gave extraordinary precise results. This is done by keeping the states of subsystems which are relevant to describe the whole wave-function, and not those that minimize the energy on that subsystem.

In [7, 8, 7, 10] it has been used a QMC approach to investigate models defined over the Cayley trees. Furthermore, in [52, 53, 54, 58, 59] we have established that Gibbs measures of the Ising model with competing (Ising) interactions (with commuting interactions) on a Cayley trees, can be considered as QMC.

In this paper, we first propose new construction of QMC on trees, which is an extension of QMC considered in [10]. Using such a construction, we are able to construct QMC on tress associated with OQRW. Furthermore, our investigation leads to the detection of the phase transition phenomena within the proposed scheme.

2. Preliminaries

Let $T = (V, E)$ be a locally finite tree. We fix a root $o \in V$. Two vertices x and y are *nearest neighbors* (denoted $x \sim y$) if they are joined through an edge (i.e. $< x, y >\in E$). A list $x \sim x_1 \sim \cdots \sim x_{d-1} \sim y$ of vertices is called a *path* from x to y. The distance on the tree between two vertices x and y (denoted $d(x, y)$) is the length of the shortest edge-path joining them.
The set of *direct successors* for a given vertex $x \in V$ is defined by

$$S(x) := \{y \in V \ : \ x \sim y \text{ and } d(y, o) > d(x, o)\} . \tag{10}$$

Let $o = x_0 \sim x_1 \sim \cdots x_n = x$ be the shortest edge-path joining o and x. The set

$$P_x := \{o = x_0, x_1, \cdots , x_n = x\} \tag{11}$$

represents the "past" of the vertex x w.r.t. the root o.
Define

$$V_n := \{x \in V \quad | \quad d(x, o) = n\}$$

$$V_{n]} := \bigcup_{j \leq n} V_j; \quad V_{[m,n]} = \bigcup_{j=m}^{n} V_j.$$

To each vertex x, we associate a C*–algebra $\mathcal{A}_x$ with identity $1\!\!1_x$. For a given bounded region $V' \subset V$, we consider the algebra $\mathcal{A}_{V'} = \bigotimes_{x\in V'} \mathcal{A}_x$. One can consider the following embedding

$$\mathcal{A}_{V_{n]}} \equiv \mathcal{A}_{V_{n]}} \otimes 1\!\!1_{V_{n+1}} \subset \mathcal{A}_{V_{n+1]}}.$$

The algebra $\mathcal{A}_{V_{n]}}$ can be viewed as a subalgebra of $\mathcal{A}_{V_{n+1]}}$. It follows the quasi-local algebra.

$$\mathcal{A}_{V;\,loc} := \bigcup_{n\in\mathbb{N}} \mathcal{A}_{n]} \tag{12}$$

and the quasi-local algebra

$$\mathcal{A}_V := \overline{\mathcal{A}_{V;\,loc}}^{C^*}.$$

The set of states on a C*–algebra $\mathcal{A}$ will be denoted $\mathcal{S}(\mathcal{A})$.

3. Quantum Markov chains on trees

3.1. **QMC on $\mathbb{Z}_+$.** In this section, we recall the definition of quantum Markov chain.

For each $i \in \mathbb{Z}_+$, (here $\mathbb{Z}_+$ denotes the set of all non negative integers) let us associate identical copies of a separable Hilbert space $\mathcal{H}$ and C^*-subalgebra M_0 of $\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$:

$$\mathcal{H}_{\{i\}} = \mathcal{H},$$

$$\mathcal{A}_{\{i\}} = M_0 \subset \mathcal{B}(\mathcal{H}) \text{ for each } i \in \mathbb{Z}_+$$

We assume that any minimal projection in M_0 is one dimensional.

For any bounded $\Lambda \subset \mathbb{Z}_+$, let

$$\mathcal{A}_\Lambda = \bigotimes_{i\in\Lambda} \mathcal{A}_i, \quad \mathcal{A}_{loc} = \bigcup_{\Lambda\subset\mathbb{Z}_+, |\Lambda|<\infty} \mathcal{A}_\Lambda$$

$$\mathcal{A} = \overline{\mathcal{A}_{loc}} =: \bigotimes_{i\in\mathbb{Z}_+} \mathcal{A}_i$$

where the bar denotes the norm closure.

For each $i \in \mathbb{Z}_+$, let J_i be the canonical injection of M_0 to the i-th component of $\mathcal{A}$. For each $\Lambda \subset \mathbb{Z}_+$ we identity $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}$.

The basic ingredients in the construction of a stationary generalized quantum Markov chain in the sense of Accardi [2, 12] consist of a *transition expectation* $\mathcal{E} : M_0 \otimes M_0 \to M_0$ which is completely positive unital map (i.e. $\mathcal{E}(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1})$), and a state ϕ_0 on M_0. In what follows, a pair $(\phi_0, \mathcal{E})$ is called a *Markov pair*.

A state φ defined on $\mathcal{A}$ associated with a Markov pair $(\phi_0, \mathcal{E})$, is called *Quantum Markov Chain (QMC)* if

$$\varphi(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = \phi_0(\mathcal{E}(x_0 \otimes \mathcal{E}(x_1 \otimes \cdots \otimes \mathcal{E}(x_n \otimes \mathbb{1}) \cdots)))). \tag{13}$$

Let $\sigma : M_0 \otimes M_0 \to M_0 \otimes M_0$ be the flipping automorphism defined by $\sigma(x \otimes y) = y \otimes x$. For every transition expectation $\mathcal{E}$ one can associate its *transpose* by $\mathcal{E}^t = \mathcal{E} \circ \sigma$. Hence, given a Markov pair $(\phi_0, \mathcal{E})$ we naturally associate its *transpose Markov pair* $(\phi_0, \mathcal{E}^t)$. The QMC corresponding to the pair $(\phi_0, \mathcal{E}^t)$ is called *transpose QMC* of φ, and it is denoted by φ^t.

To every transition expectation one associates two kinds of Markov operators (i.e. completely positive, identity preserving map) from M_0 into itself:

$$P(a) = \mathcal{E}(\mathbb{1} \otimes a), \quad \text{(backward transition operator)} \tag{14}$$

$$T(a) = \mathcal{E}(a \otimes \mathbb{1}), \quad \text{(forward transition operator).} \tag{15}$$

Remark 3.1. *It is known* [3] *that in the classical setting T is the identity operator, and P coincides with usual Markov transition operator.*

Remark 3.2. *We point out that the quantum Markov chain can be also treated as a special case of finitely correlated states (FCS) which were introduced in* [28]. *Let us recall the well-known construction. Let $\mathfrak{A}, \mathfrak{B}$ be two C^*-algebras with units $\mathbb{1}_{\mathfrak{A}}, \mathbb{1}_{\mathfrak{B}}$, respectively, φ_0 be a state on $\mathfrak{B}$, and $\mathcal{E} : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B}$ be a completely positive unital map such that for all $b \in \mathfrak{B}$ one has*

$$\varphi_0(\mathcal{E}(\mathbb{1}_{\mathfrak{A}} \otimes b)) = \varphi_0(b).$$

For each $a \in \mathfrak{A}$ one defines a map $\mathcal{E}_a : \mathfrak{B} \to \mathfrak{B}$ by setting $\mathcal{E}_a(b) = \mathcal{E}(a \otimes b)$. The functional

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi_0(\mathcal{E}_{x_1} \cdots \mathcal{E}_{x_n}(\mathbb{1}_{\mathfrak{B}}))$$

uniquely defines a state on the C^*-algebra $\bigotimes_{i \in \mathbb{N}} \mathfrak{A}_i$, where $\mathfrak{A}_i$ is a copy of $\mathfrak{A}$. The state φ is the finitely correlated state associated to $(\mathfrak{A}, \mathfrak{B}, \mathcal{E}, \varphi_0)$. In case, $\mathfrak{A} = \mathfrak{B}$ we will recover QMC. On the other hand, we stress that, in general, we cannot define the transpose FCS on the same algebra with the initial one. Therefore, in what follows, we will work within QMC scheme.

In what follows, by $\mathcal{A}_{n]}$ we denote the subalgebra of $\mathcal{A}$, generated by the first $(n+1)$ factors, i.e.

$$a_{n]} = a_0 \otimes a_1 \otimes \cdots a_n \otimes \mathbb{1}_{[n+1} = J_0(a_0)J_1(a_1)\cdots J_n(a_n),$$

with $a_0, a_1, \ldots, a_n \in M_0$. It is well known [?] that for each $n \in \mathbb{N}$ there exists a unique completely positive identity preserving mapping $E_{n]} : \mathcal{A} \to \mathcal{A}_{n]}$ such that

$$E_{n]}(a_{m]}) = a_0 \otimes \cdots \otimes a_{n-1} \otimes \mathcal{E}(a_n \otimes \mathcal{E}(a_{n+1} \otimes \cdots \otimes \mathcal{E}(a_m \otimes \mathbb{1})\cdots)), \quad m > n \tag{16}$$

Remark 3.3. *We notice that if the state ϕ_0 satisfies the following condition:*

$$\phi_o(\mathcal{E}(\mathbb{1} \otimes x)) = \phi_0(x), \quad x \in M_0 \tag{17}$$

then the Markov pair $(\phi_0, \mathcal{E})$ defines local states

$$\varphi_{[i,n]}(x_i \otimes x_{i+1} \otimes \ldots \otimes x_n) = \phi_0(\mathcal{E}(x_i \otimes \mathcal{E}(x_{i+1} \otimes \cdots \otimes \mathcal{E}(x_n \otimes \mathbb{1})\cdots)))). \tag{18}$$

The family of local states $\{\varphi_{[i,n]}\}$, due to (17), satisfies a compatibility condition, and therefore, the state φ is well defined on $A_{\mathbb{Z}} := \bigotimes_{i\in\mathbb{Z}} \mathcal{A}_i$. Moreover, φ is translation invariant, i.e. it is invariant with respect to the shift α, i.e. $\alpha(J_n(a)) = J_{n+1}(a)$.

3.2. **Tree-homogeneous quantum Markov chains.** Let $T^{'} = (V^{'}, E^{'})$ be a subtree of the tree. There exists a unique vertex $o' \in V'$ such that $d(o, V') = d(o, o')$. This vertex o' will be referred as root of the subtree $T^{'}$. In the sequel, we reduce ourselves to the case of regular trees (known as Cayley trees). The Cayley tree of order k is characterized by being a tree for which every vertex has exactly $k+1$ nearest-neighbors. We consider the semi-infinite Cayley tree $\Gamma_+^k = (V, E)$ with root o. In this case, any vertex has exactly k direct successors denoted $(x, i), i = 1, 2, \cdots, k$.

$$S(x) = \{(x,1),\ (x,2),\ \cdots,\ (x,k)\}.$$

A coordinate structure on Γ_+^k is given by

$$V_n = \{(i_1, i_2, \cdots, i_n); \quad i_j = 1, 2, \cdots, k\}.$$

For $x = (i_1, i_2, \cdots, i_n) \in V_n$, we define k shifts on the tree as follows

$$\alpha_j(x) = (j, x) = (j, i_1, i_2, \cdots, i_n) \in V_{n+1}. \tag{19}$$

Moreover, we have

$$\alpha_x := \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_n}.$$

The shift α_x maps the Cayley tree Γ^k_+ onto its subtree T_x having root at x.

One has $\alpha_x(o) = x$ and $\alpha_x(V_n) = S_n(x)$. The shifts α_j can be extended to the algebra $\mathcal{A}_V$ as follows:

$$\alpha_j\left(\bigotimes_{x\in V_{n]}} a_x\right) := \mathbb{1}^{(o)} \otimes \bigotimes_{i=0}^{n} a_x^{(j,x)}. \tag{20}$$

Consider a triplet $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ of C*–algebras. A *quasi-conditional expectation* [4] is a completely positive identity preserving linear map $E : \mathcal{A} \to \mathcal{B}$ such that $E(ca) = cE(a)$, for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

Definition 3.4. *A (backward) quantum Markov chain on $\mathcal{A}_V$ is a triplet $(\phi_o, (E_{V_{n]}})_{n\geq 0}, (h_n)_n)$ of initial state $\phi_o \in \mathcal{S}(\mathcal{A}_o)$, a sequence of quasi-conditional expectations $(E_{V_{n]}})_n$ w.r.t. the triple $\mathcal{A}_{\Lambda_{n-1]}} \subseteq \mathcal{A}_{V_{n]}} \subseteq \mathcal{A}_{V_{n+1]}}$ and a sequence $h_n \in \mathcal{A}_{V_n,+}$ of boundary conditions such that for each $a \in \mathcal{A}_V$ the limit*

$$\varphi(a) := \lim_{n\to\infty} \phi_0 \circ E_{V_{0]}} \circ E_{V_{1]}} \circ \cdots \circ E_{V_{n]}}(h_{n+1}^{1/2} a h_{n+1}^{1/2}) \tag{21}$$

exists in the weak--topology and defines a state. In this case the state φ defined by (21) is also called quantum Markov chain (QMC).*

Remark 3.5. *The above definition introduce quantum Markov chains on trees as a triplet generalizing the definitions considered in* [10], [52] *by adding the boundary conditions. On the other hand it extends to trees the recent unifying definition for quantum Markov chains on the one-dimensional case* [6].

Definition 3.6. *A quantum Markov chain* $\varphi \equiv (\phi_o, (E_{V_{n]}})_{n\geq 0}, (h_n)_n)$ *is said to be **tree-homogeneous** if there exists a transition expectation* $\mathcal{E} : \mathcal{A}_{\{o\}\cup S(o)} \to \mathcal{A}_o$ *such that for each* n

$$E_{V_{n]}} = id_{\mathcal{A}_{V_{n-1]}}} \otimes \bigotimes_{x\in V_n} \alpha_x \circ \mathcal{E} \circ \alpha_x^{-1} \tag{22}$$

where $id_{\mathcal{A}_{V_{n-1]}}}$ *denotes the identity map on* $\mathcal{A}_{V_{n-1]}}$ *and*

$$h_n = \bigotimes_{u\in V_n} \alpha_u(h) \tag{23}$$

for some density operator $h \in \mathcal{A}_o$.
For the sake of simplicity, the triplet $(\phi_o, \mathcal{E}, h)$ *will be referred as the tree-homogeneous QMC* φ.

Remark 3.7. *Notice that if* $\mathcal{E}$ *is a transition expectation from* $\mathcal{A}_{V_{1]}} = \mathcal{A}_{S(o)}$ *into* $\mathcal{A}_0$, *then for any* $u \in \Lambda$ *the map*

$$\mathcal{E}_u := \alpha_u \circ \mathcal{E} \circ \alpha_u^{-1}$$

defines a transition expectation from $\mathcal{A}_{\{u\}\cup S(u)}$ *into* $\mathcal{A}_u$. *It follows that* $E_u := id_{\mathcal{A}_{P_u\setminus\{u\}}} \otimes \mathcal{E}_u$ *is a quasi-conditional expectation with respect to the following triplet* $\mathcal{A}_{P_u\setminus\{u\}} \subset \mathcal{A}_{P_u} \subset \mathcal{A}_{P_u\cup S(u)}$, *where* P_x *is given by (11). Moreover, for any* $n \in \mathbb{N}$ *if* $u, v \in V_n$ *such that* $u \neq v$ *then* $E_uE_v = E_vE_u$. *Therefore, the map* $E_{V_{n]}} = \prod_{u\in V_n} E_u$ *is a quasi-conditional expectation with respect to the triplet* $\mathcal{A}_{n-1]} \subset \mathcal{A}_{n]} \subset \mathcal{A}_{n+1]}$.

3.3. **General construction of QMC on trees.** Let $(\kappa_i)_{i\in I}$ be a finite family conditional density amplitude. Define

$$\kappa_{\{u\}\cup S(u)} = \sum_{i\in I} \alpha_u(\kappa_i), \quad u \in V \tag{24}$$

$$\kappa_{[n,n+1]} = \bigotimes_{u\in V_n} \kappa_{\{u\}\cup S(u)} \tag{25}$$

$$h_n = \bigotimes_{u\in V_n} h^{(u)}. \tag{26}$$

$$\kappa_{n]} = \prod_{j=0}^{n-1} \kappa_{[j,j+1]} h_n^{1/2} \tag{27}$$

$$w_{n]} = w_0^{1/2} \kappa_{n]} {\kappa_{n]}}^* w_0^{1/2} \tag{28}$$

where $h^{(u)} \in \mathcal{A}_{u,+}$ is a positive boundary condition on $\mathcal{A}_u$ for every $u \in \Lambda$ and $w_0 \in \mathcal{A}_o^+$ be an initial density matrix. Define

$$\varphi_{V_{n]}}(a) := \mathrm{Tr}(w_{n+1]} a \otimes 1\!\!1) \tag{29}$$

for every $n \geq 1$.

Theorem 3.8. *With these notations in mind , if*

$$\operatorname{Tr}(w_0 h^{(o)}) = 1 \tag{30}$$

and

$$\operatorname{Tr}_{u]}\left(\left(\sum_{i\in I}\kappa_i^{(u)}\right)\mathbb{1}\otimes\bigotimes_{v\in S(u)}h^{(v)}\left(\sum_{i\in I}\kappa_i^{(u)^*}\right)\right) = h^{(u)}. \tag{31}$$

Then the limit

$$\varphi := \lim_{n\to\infty}\varphi_{V_{n]}} \tag{32}$$

exists in the weak--topology and defines a QMC on $\mathcal{A}_V$. Moreover, if the boundary condition $(h^{(u)})_{u\in V}$ is translation invariant then φ is tree-homogeneous.*

Remark 3.9. *The conditional density amplitude $\sum_{i\in I}\kappa_i$ considered in Theorem 3.8 is a finite sum of amplitudes. This generalizes the conditional expectations on the Cayley tree considered in the previous works, see for instance* [10, 7, 8, 9, 52, 53].

4. QMC associated with OQRW on trees

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces. Let $\{|i\rangle\}_{i\in\Lambda}$ be an ortho-normal basis of $\mathcal{K}$ indexed by a graph Λ. To each $x \in V$ we associate the algebra $\mathcal{A}_x = \mathcal{A} := \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.
For each $(i,j) \in \Lambda^2$ one associates an operator $B^i_j \in \mathcal{B}(\mathcal{H})$ to describe the transition from the state $|j\rangle$ to the state $|i\rangle$ such that

$$\sum_{i\in\Lambda} B^{i*}_j B^i_j = 1\!\!1_{\mathcal{B}(\mathcal{H})}. \tag{33}$$

Consider the density operator $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, of the form

$$\rho = \sum_{i\in\Lambda} \rho_i \otimes |i\rangle\langle i|; \quad \rho_1 \in \mathcal{B}(\mathcal{H})^+.$$

Let us consider

$$M^i_j = B^i_j \otimes |i\rangle\langle j| \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}). \tag{34}$$

Put

$$A^i_j := \frac{1}{\mathrm{Tr}(\rho_j)^{1/2}} \rho_j^{1/2} \otimes |i\rangle\langle j|, \quad i,j \in \Lambda. \tag{35}$$

For each $u \in V$, we set

$$\kappa^{(\{u\}\cup S(u))}_{(i,j)} = K^{i(\{u\}\cup S(u))}_j := M^{i*(u)}_j \otimes \bigotimes_{v\in S(u)} A^{i\,(v)}_j \in \mathcal{A}_{\{u\}\cup S(u)}. \tag{36}$$

Consider $\mathfrak{S}_{k+1}$ be the symmetric group of the set $\{o\} \cup S(o)$. For $u \in V$, the map

$$T_u : \sigma \in \mathfrak{S}_{k+1} \mapsto \alpha_u \circ \sigma \circ \alpha_u^{-1} \tag{37}$$

defines a group isomorphism from $\mathfrak{S}_{k+1}$ onto the symmetry group of $\{u\} \cup S(u)$. A permutation $\sigma \in \mathfrak{S}_{k+1}$ leaves o invariant if and only if $T_u(\sigma)$ leaves u invariant.

Define

(38)

$$\mathcal{E}^{\sigma}(a_o \otimes a_{(o,1)} \otimes \cdots \otimes a_{(o,k)}) = \sum_{(i,j),(i',j') \in \Lambda^2} \mathrm{Tr}_{o]} \left(K^i_j a_{\sigma(o,1)} \otimes \cdots \otimes a_{\sigma(o,k)} K^{i'^*}_{j'} \right)$$

Definition 4.1. *A tree-homogeneous quantum Markov chain $\varphi \equiv (\varphi_0, \mathcal{E}, h)$ on $\mathcal{A}_V$ is said to be* ***associated wit the open quantum random walk (OQRW)*** *if the transition expectation $\mathcal{E}$ is a convex combination of the maps $(\mathcal{E}^{(\sigma)})_{\sigma\in\mathfrak{S}_{k+1}}$ i.e.*

$$\mathcal{E} = \sum_{\sigma\in\mathfrak{S}_{k+1}} \lambda_\sigma \mathcal{E}^{(\sigma)} \tag{39}$$

where $\lambda_\sigma \geq 0$ and $\sum_{\sigma\in\mathfrak{S}_{k+1}} \lambda_\sigma = 1$.

Remark 4.2. *In the above definition if $\lambda_\sigma = 1$ for some $\sigma \in \mathfrak{S}_{k+1}$ the homogeneous quantum Markov chain $\varphi^{(\sigma)}$ associated with a transition expectation $\mathcal{E}^{(\sigma)}$ in the sense of Definition 3.6 is a QMC associated with OQRW on the Cayley tree.*
Notice that, in [27] some examples of ome-dimensional quantum Markov chains associated with OQRW were studied. Therein $k = 1$ then the symmetry group is $\mathfrak{S}_2 = \{id, t\}$ and the studied Markov chains were exactly $\varphi^{(id)}$ and $\varphi^{(t)}$.

Remark 4.3. *The above definition gives rise to a new class of QMC in connection with OQRW. We forecast that these quantum have rich ergodic properties. Namely the two examples $\varphi^{(id)})$ and $\varphi^{(t)}$ studied in [27] were proven to admit different structures. However, it is possible to consider the QMC associated with an arbitrary convex combination of the associated transition expectations $\mathcal{E}$ and $\mathcal{E}^t$ of the form $\mathcal{E}_\lambda := \lambda\mathcal{E} + (1-\lambda)\mathcal{E}^t$.*

Theorem 4.4. *For every $\sigma \in \mathfrak{S}_{k+1}$ leaving invariant o, the tree-homogeneous QMC $\varphi^{(\sigma)} \equiv (\phi_o, \mathcal{E}^{(\sigma)}, h)$ satisfies*

(40)

$$\varphi^{(\sigma)}(a) = \sum_{i_o,j,j'\in\Lambda} \mathrm{Tr}\left(w_0 {M^{i_o}_{j_o}}^* a_o M^{i_o}_{j_o}\right) \prod_{u\in V_{[1,n]}} \psi_{j,j'}(a_u) \prod_{v\in V_{n+1}} \varphi_{j,j'}(h^{(v)})$$

where

$$\varphi_{j,j'}(b) = \frac{1}{\mathrm{Tr}(\rho_j)^{1/2}\mathrm{Tr}(\rho_{j'})^{1/2}} \mathrm{Tr}\left(\rho_j^{1/2}\rho_{j'}^{1/2} \otimes |j'\rangle\langle j| b\right) \tag{41}$$

and

(42)

$$\psi_{j,j'}(a_v) = \frac{1}{\mathrm{Tr}(\rho_j)^{1/2}\mathrm{Tr}(\rho_{j'})^{1/2}} \sum_{i_v\in\Lambda} \mathrm{Tr}\left(B^{i_v}_{j'}\rho_{j'}^{1/2}\rho_j^{1/2}{B^{i_v}_j}^* \otimes |i_v\rangle\langle i_v| a_v\right).$$

for every $a = \bigotimes_{u\in V_n} a_u \in \mathcal{A}_{V_{n]}}$.

Remark 4.5. *The maps $\varphi_{j,j'}$ and $\psi_{j,j'}$ are linear functionals. If $j = j'$ then $\varphi_{j,j'}$ and $\psi_{j,j'}$ are states, we write*

$$\varphi_j(b) = \frac{1}{\mathrm{Tr}(\rho_j)}\mathrm{Tr}\left(\rho_j \otimes |j\rangle\langle j| b\right). \tag{43}$$

and

$$\psi_j(b) = \frac{1}{\mathrm{Tr}(\rho_j)}\mathrm{Tr}\left(\rho_j \otimes |j\rangle\langle j| b\right). \tag{44}$$

Remark 4.6. *The Markov chain (40) generalizes the Markov chains associated with open quantum random walks studied in* [27] *to trees. But even un the one dimensional they propose a more general class. Moreover, if the density operators $(\rho_j)_{j\in\Lambda}$ are mutually orthogonal one gets the Markov chain $\varphi^{(id)}$ considered in* [27].

5. Exampes

Let $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$ with canonical basis $(|1\rangle, |2\rangle)$. Let $\mathcal{A}_u = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \equiv M_4(\mathbb{C})$. Let $\Lambda = \{1, 2\}$. The interactions are given by

$$B_1^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B_2^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1^2 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad B_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1, ac \neq 0$.

Let $\sigma \in \mathfrak{S}_{k+1}$ such that $\sigma(o) = o$. Then (31) becomes

$$h^{(u)} = \sum_{i,j,i',j'=1,2} M_j^{i*(u)} M_{j'}^{i'(u)} \prod_{\ell=1}^{k} \mathrm{Tr}(A_j^i h^{(u,\ell)} A_{j'}^{i'*})$$

For the sake of simplicity we assume that the boundary condition is translation invariant $h^{(u)} = h$ for all $u \in V$ one gets

$$h = \sum_{i,j,i',j'=1,2} M_j^{i*} M_{j'}^{i'} \mathrm{Tr}(A_j^i h A_{j'}^{i'\,*})^k. \tag{45}$$

where

$$M_j^{i*} M_{j'}^{i'} = B_j^{i*} B_{j'}^{i'} \otimes |j\rangle\langle j'|\delta_{i,i'}$$

and

$$\mathrm{Tr}(A_j^i h A_{j'}^{i'\,*}) = \mathrm{Tr}(A_{j'}^{i'\,*} A_j^i h) = \frac{1}{\sqrt{\mathrm{Tr}(\rho_j)\mathrm{Tr}(\rho_j')}} \mathrm{Tr}(\rho_{j'}^{1/2} \rho_j^{1/2} \otimes |j'\rangle\langle j|h)\delta_{i,i'}.$$

Thus, (45) becomes

$$h = \sum_{i,j,j'=1,2} \left(\frac{\mathrm{Tr}(\rho_{j'}^{1/2} \rho_j^{1/2} \otimes |j'\rangle\langle j|h)}{\sqrt{\mathrm{Tr}(\rho_j)\mathrm{Tr}(\rho_j')}} \right)^k B_j^{i*} B_{j'}^{i} \otimes |j\rangle\langle j'|.$$

By identifying entries, we are led to

$$\begin{cases} h_{11} = & (|a|^2+|c|^2)\frac{\mathrm{Tr}(\rho_1 \otimes |1\rangle\langle 1|h)^k}{\mathrm{Tr}(\rho_1)^k} \\ h_{22} = & \frac{\mathrm{Tr}(\rho_2 \otimes |2\rangle\langle 2|h)^k}{\mathrm{Tr}(\rho_2)^k} \\ h_{33} = & (|b|^2+|d|^2)\frac{\mathrm{Tr}(\rho_1 \otimes |1\rangle\langle 1|h)^k}{\mathrm{Tr}(\rho_1)^k} \\ h_{44} = & \frac{\mathrm{Tr}(\rho_2 \otimes |2\rangle\langle 2|h)^k}{\mathrm{Tr}(\rho_2)^k} \\ h_{12} = & \overline{c}\frac{\mathrm{Tr}(\rho_2^{1/2}\rho_1^{1/2} \otimes |2\rangle\langle 1|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \\ h_{21} = & c\frac{\mathrm{Tr}(\rho_1^{1/2}\rho_2^{1/2} \otimes |1\rangle\langle 2|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \\ h_{14} = & \overline{a}\frac{\mathrm{Tr}(\rho_2^{1/2}\rho_1^{1/2} \otimes |2\rangle\langle 1|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \\ h_{41} = & a\frac{\mathrm{Tr}(\rho_1^{1/2}\rho_2^{1/2} \otimes |1\rangle\langle 2|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \end{cases}$$

and $h_{ij} = 0$ otherwise.

Then, using

$$|a|^2+|c|^2=|b|^2+|d|^2=1,$$

one gets

$$\left\{\begin{array}{lll} h_{11}= & h_{33}= & \frac{\mathrm{Tr}(\rho_1\otimes|1\rangle\langle 1|h)^k}{\mathrm{Tr}(\rho_1)^k} \\ h_{22}= & h_{44}= & \frac{\mathrm{Tr}(\rho_2\otimes|2\rangle\langle 2|h)^k}{\mathrm{Tr}(\rho_2)^k} \\ h_{12}= & \frac{\overline{c}}{\overline{a}}h_{1,4}= & \overline{c}\frac{\mathrm{Tr}(\rho_2^{1/2}\rho_1^{1/2}\otimes|2\rangle\langle 1|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \\ h_{21}= & \frac{c}{a}h_{4,1}= & c\frac{\mathrm{Tr}(\rho_1^{1/2}\rho_2^{1/2}\otimes|1\rangle\langle 2|h)^k}{(\mathrm{Tr}(\rho_1)\mathrm{Tr}(\rho_2))^{k/2}} \end{array}\right. \tag{46}$$

and $h_{ij}=0$ otherwise.

Example 5.1. *Consider*

$$\rho_j = |j\rangle\langle j|, \quad j \in \{1,2\}.$$

Then one has

$$\frac{\mathrm{Tr}(\rho_1 \otimes |1\rangle\langle 1|h)}{\mathrm{Tr}(\rho_1)} = \mathrm{Tr}(|1\rangle\langle 1| \otimes |1\rangle\langle 1|h) = h_{11},$$

$$\frac{\mathrm{Tr}(\rho_2 \otimes |2\rangle\langle 2|h)}{\mathrm{Tr}(\rho_2)} = \mathrm{Tr}(|2\rangle\langle 2| \otimes |2\rangle\langle 2|h) = h_{44}$$

and

$$\mathrm{Tr}(\rho_1^{1/2}\rho_2^{1/2} \otimes |1\rangle\langle 2|h) = \mathrm{Tr}(\rho_2^{1/2}\rho_1^{1/2} \otimes |2\rangle\langle 1|h) = 0.$$

Then (46) *becomes,*

$$\begin{cases} h_{11} = & h_{33} = & h_{11}^k \\ h_{22} = & h_{44} = & h_{44}^k \end{cases}$$

and $h_{ij} = 0$ *otherwise. Denote*

$$\mathcal{U}_k := \{z \in \mathbb{C}; z^k = 1\}$$

the set of k*-th roots of unity. For* $k \geq 2$*, one has* $h_{11}, h_{44} \in \mathcal{U}_{k-1} \cup \{0\}$ *and therefore one gets* $2^k - 1$ *non-trivial solutions. Since* h *is positive and does not vanish, one gets the following three solutions:*

$$h_0 = \mathbb{1}_{M_4}, \quad h_1 = \mathbb{1}_{M_2} \otimes |1\rangle\langle 1|, \quad h_2 = \mathbb{1}_{M_2} \otimes |2\rangle\langle 2|.$$

Example 5.2. *Now let us consider*

$$\rho_1 = \rho_2 = |1\rangle\langle 1|.$$

Then (46) *becomes,*

$$\begin{cases} h_{11} = & h_{33} = & h_{11}^k \\ h_{22} = & h_{44} = & h_{22}^k \\ h_{12} = & \frac{\overline{c}}{\overline{a}} h_{1,4} = & \overline{c}\, h_{12}^k \\ h_{21} = & \frac{c}{a} h_{4,1} = & c\, h_{21}^k \end{cases}$$

and $h_{ij} = 0$ *otherwise. Moreover,* h *is hermitian.*

Then, if $k \geq 1$, one has $h_{11}, h_{44} \in \mathcal{U}_{k-1} \cup \{0\}$ and $h_{12} \in \mathcal{U}_{k-1}(1/\overline{c}) \cup \{0\}$, where $\mathcal{U}_{k-1}(1/\overline{c})$ denotes the set $(k-1)$-th roots of the complex $1/\overline{c}$. Therefore, one gets $4k-1$ non-trivial solutions. In particular, for $k=2$, one gets the solutions

$$h_0 = \mathbb{1}_{M_4}, \quad h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_3 = \begin{pmatrix} 1 & \frac{1}{\overline{c}} & 0 & \frac{\overline{a}}{\overline{c}^2} \\ \frac{1}{c} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a}{c^2} & 0 & 0 & 1 \end{pmatrix}, \quad h_4 = \begin{pmatrix} 1 & \frac{1}{\overline{c}} & 0 & \frac{\overline{a}}{\overline{c}^2} \\ \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a}{c^2} & 0 & 0 & 0 \end{pmatrix}, \quad h_5 = \begin{pmatrix} 0 & \frac{1}{\overline{c}} & 0 & \frac{\overline{a}}{\overline{c}^2} \\ \frac{1}{c} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{c^2} & 0 & 0 & 1 \end{pmatrix}$$

$$h_6 = \begin{pmatrix} 0 & \frac{1}{\overline{c}} & 0 & \frac{\overline{a}}{\overline{c}^2} \\ \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a}{c^2} & 0 & 0 & 0 \end{pmatrix}$$

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