Gaussian Quantum Markov Semigroups: Irreducibility and Normal Invariant States

Franco Fagnola
Politecnico di Milano
(joint work with J. Agredo and D. Poletti)

52 Symposium on Mathematical Physics
"Channels, Maps and All That"
Toruń, June 14-17, 2021

1. Gaussian QMS
   - Fock space & CCR
   - Gaussian - GKLS
   - Construction (minimal semigroup)
   - Action on Gaussian states
2. Invariant States
   - One mode $d = 1$
   - Existence: sharp conditions
   - Interpretation
   - Uniqueness & convergence
3. Irreducibility
   - Two noise operators
   - One noise operator L and the Hörmander condition
   - Irreducibility: multimode $d > 1$

$\mathsf{h} = \Gamma(\mathbb{C}^d)$ $\qquad (e_n)_n$ o.n. basis $e_\alpha = e_{n_1} \otimes \cdots \otimes e_{n_d}$
Annihilation and creation operators $a_j, a_j^\dagger$

$$\begin{aligned} a_j e_n &= \sqrt{n_j}\, e_{n_1} \otimes \cdots \otimes e_{n_j - 1} \otimes \cdots \otimes e_{n_d}, \\ a_j^\dagger e_n &= \sqrt{n_j + 1}\, e_{n_1} \otimes \cdots \otimes e_{n_j + 1} \otimes \cdots \otimes e_{n_d}. \end{aligned}$$

Domain $\{\xi = \sum_\alpha \xi_n e_n \mid \sum_n |n|\,|\xi_n|^2 < \infty\}$, CCR $\left[a_j, a_k^\dagger\right] = \delta_{jk}\mathbb{1}$
q_j, p_j the position and momentum operators on h

$$q_j = \left(a_j + a_j^\dagger\right)/\sqrt{2}, \quad p_j = \left(a_j - a_j^\dagger\right)/\sqrt{2}\,\mathrm{i}.$$

q_j, p_k selfadjoint and satisfy $[q_j, p_k] = \mathrm{i}\delta_{jk}\mathbb{1}$

$$H = \sum_{j,k=1}^{d} \left(\Omega_{jk} a_j^\dagger a_k + \frac{\kappa_{jk}}{2} a_j^\dagger a_k^\dagger + \frac{\overline{\kappa}_{jk}}{2} a_j a_k \right) + \frac{1}{2} \sum_{j=1}^{d} \left(\zeta_j a_j^\dagger + \overline{\zeta}_j a_j \right)$$

$$L_\ell = \sum_{j=1}^{d} \left(\overline{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \quad \ell = 1, \dots, m; \quad m \leq 2d$$

$v_{\ell j}, u_{\ell j}, \zeta_j \in \mathbb{C}$, $\Omega_{jk}, \kappa_{jk} \in \mathbb{C}$ m noise multiplicity

$\Omega = (\Omega_{jk})_{1 \leq j,k \leq d}$ Hermitian $K = (\kappa_{jk})_{1 \leq j,k \leq d}$ symmetric

$D := \mathrm{Lin}\{ e_n \mid n \in \mathbb{N}^d \}$ (finite linear span) $\quad \xi', \xi \in D$, $x \in \mathcal{B}(\mathsf{h})$

$$\pounds(x)[\xi', \xi] = \mathrm{i} \langle H\xi', x\xi \rangle - \mathrm{i} \langle \xi', xH\xi \rangle - \frac{1}{2} \sum_{\ell=1}^{m} \left(\langle \xi', xL_\ell^* L_\ell \xi \rangle - 2 \langle L_\ell \xi', xL_\ell \xi \rangle + \langle L_\ell^* L_\ell \xi', x\xi \rangle \right)$$

H, L_ℓ unbounded $\rightsquigarrow$ minimal qds $\mathcal{T}$ $\rightsquigarrow$ it is unique and $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$

Theorem

There exists a unique QMS, $\mathcal{T} = (\mathcal{T}_t)_t$ such that

$$\frac{d}{dt} \langle \xi', \mathcal{T}_t(x)\xi \rangle \Big|_{t=0} = \pounds(x)[\xi', \xi] \qquad \forall\, \xi, \xi' \in D$$

Action on Weyl operators

Weyl operator $$W(z) = \exp\left(\sum_{j=1}^{d}\left(z_j a_j^\dagger - \overline{z}_j a_j\right)\right)$$

define matrices $U = (u_{j\ell})$, $V = (v_{j\ell})$ & *real* linear operators on $\mathbb{C}^d$

$$\begin{aligned} Zz &= \left[\left(\overline{U^* U - V^* V}\right)/2 + \mathrm{i}\Omega\right] z + \left[\left(U^T V - V^T U\right)/2 + \mathrm{i}K\right] \overline{z} \\ Cz &= \left(\overline{U^* U + V^* V}\right) z + \left(U^T V + V^T U\right) \overline{z} \end{aligned}$$

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{Re}\,\langle \mathrm{e}^{sZ} z, C\mathrm{e}^{sZ} z\rangle \mathrm{d}s + \mathrm{i}\int_0^t \mathrm{Re}\,\langle \zeta, \mathrm{e}^{sZ} z\rangle \mathrm{d}s\right) W\left(\mathrm{e}^{tZ} z\right)$$

Definition

ρ is a gaussian state if $\forall z \in \mathbb{C}^d$

$$\hat{\rho}(z) = \mathrm{tr}(\rho W(z)) = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \omega, z\rangle - \frac{1}{2}\mathrm{Re}\,\langle z, Sz\rangle\right)$$

where $\omega \in \mathbb{C}^d$ (mean), $S : \mathbb{C}^d \to \mathbb{C}^d$ invertible, real linear.
$\rho =: \rho_{\omega,S}$.

If $z = x + \mathrm{i}y \quad x, y \in \mathbb{R}^d$

$$\hat{\rho}(z) = \mathrm{tr}(\rho W(z)) \leftrightarrow \mathbb{E}_\rho\left[e^{-\mathrm{i}x\sqrt{2}p + \mathrm{i}y\sqrt{2}q}\right]$$
$$= \exp\left\{-\mathrm{i}\left\langle \begin{pmatrix}\mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega\end{pmatrix}, \begin{pmatrix}x \\ y\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle \begin{pmatrix}x \\ y\end{pmatrix}, \mathbf{S}\begin{pmatrix}x \\ y\end{pmatrix}\right\rangle\right\}$$

Theorem

If $\rho = \rho_{(\omega,S)}$ then $\rho_t := \mathcal{T}_{*t}(\rho)$ is still a gaussian state $\rho_{(\omega_t,S_t)}$ with

$$\omega_t = e^{tZ^T}\omega - \int_0^t e^{sZ^T}\zeta ds$$

$$S_t = e^{tZ^T} S e^{tZ} + \int_0^t e^{sZ^T} C e^{sZ} ds,$$

where Z and C are the real linear operators

$$Zz = \left[\left(\overline{U^*U - V^*V}\right)/2 + \mathrm{i}\Omega\right] z + \left[\left(U^T V - V^T U\right)/2 + \mathrm{i}K\right] \overline{z}$$

$$Cz = \left(\overline{U^*U + V^*V}\right) z + \left(U^T V + V^T U\right) \overline{z}$$

Conversely, (essentially proved in

B. Demoen, P. Vanheuverzwijn, A. Verbeure, Lett. Math. Phys. 2, 161 (1977)

M. Fannes, Commun. Math. Phys 51, 55 (1976))

if $\mathcal{T}$ is a w*-continuous semigroup of normal CP maps on $\mathcal{B}(\Gamma(\mathbb{C}^d))$ s.t.,
$\forall \rho = \rho_{(\omega,S)}$ Gaussian state, $\rho_t := \mathcal{T}_{*t}(\rho)$ is still a gaussian state,
then one can find a GKLS generator with unbounded H, L_ℓ ...

Definition

QMSs, $\mathcal{T} = (\mathcal{T}_t)_t$, constructed from the above choices of H, L_ℓ are called Gaussian QMSs.

One mode Fock space $d = 1$

$$
\begin{aligned}
H &= \Omega a^\dagger a + \frac{\kappa}{2} a^{\dagger 2} + \frac{\overline{\kappa}}{2} a^2 + \frac{\zeta}{2} a^\dagger + \frac{\overline{\zeta}}{2} a \\
L_\ell &= \overline{v}_\ell a + u_\ell a^\dagger, \qquad \ell = 1, 2
\end{aligned}
$$

with $\Omega \in \mathbb{R}$, $\kappa, \zeta, v_\ell, u_\ell \in \mathbb{C}$ (not all $= 0$).

Sometimes, when stated, we consider *only one* Kraus operator L.

Definition

Invariant state: ρ positive, $\mathrm{tr}\,\rho = 1$, $\mathcal{T}_{*t}(\rho) = \rho \quad \forall t \geq 0$.

Theorem

A Gaussian semigroup has a normal invariant state if and only if

$$\gamma := \frac{1}{2}\sum_{\ell}(|v_\ell|^2 - |u_\ell|^2) > 0 \quad \textit{and} \quad \gamma^2 + \Omega^2 - |\kappa|^2 > 0$$

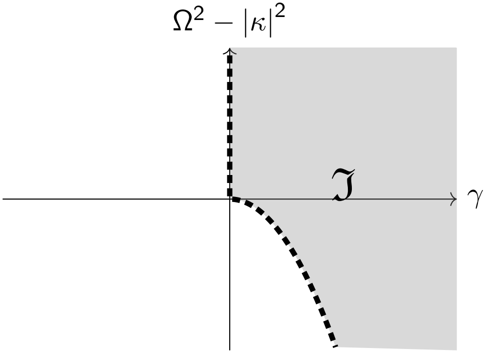
the invariant state is unique in the class of normal states and gaussian with

$$\omega = (Z^T)^{-1}\zeta = \frac{(-\gamma + \mathrm{i}\Omega)\zeta - \mathrm{i}\kappa\overline{\zeta}}{\gamma^2 + \Omega^2 - |\kappa|^2}, \quad S = \int_0^\infty e^{sZ^T} C e^{sZ} ds.$$

Note: sum on $\ell = 1, 2$ for *two* ops L_ℓ, sum on $\ell = 1$ for *only one* L_ℓ

Recall: $H = \Omega a^\dagger a + \frac{\kappa}{2} a^{\dagger 2} + \frac{\overline{\kappa}}{2} a^2 + \frac{\zeta}{2} a^\dagger + \frac{\overline{\zeta}}{2} a, \quad L_\ell = \overline{v}_\ell a + u_\ell a^\dagger$

$$H = \Omega a^\dagger a + \frac{\kappa}{2} a^{\dagger 2} + \frac{\zeta}{2} a^\dagger + \text{h.c.} \quad L_\ell = \overline{v}_\ell a + u_\ell a^\dagger \quad \gamma = \sum_\ell \frac{|v_\ell|^2 - |u_\ell|^2}{2}$$



$$\mathfrak{I} := \left\{ (\gamma, \Omega^2 - |\kappa|^2) \mid \gamma > 0,\ \gamma^2 + \Omega^2 - |\kappa|^2 > 0 \right\}$$

$$\begin{aligned} H &= \Omega a^\dagger a + \frac{\kappa}{2} a^{\dagger 2} + \frac{\overline{\kappa}}{2} a^2 + \frac{\zeta}{2} a^\dagger + \frac{\overline{\zeta}}{2} a \\ \lambda_\pm &= -\gamma \pm \sqrt{|\kappa|^2 - \Omega^2} \qquad \text{eigenvalues of } \mathbf{Z} \end{aligned}$$

An invariant state exists if and only if the parameters are in $\mathfrak{I}$

$$\gamma > 0 \text{ and } \gamma^2 + \Omega^2 - |\kappa|^2 > 0 \quad \Leftrightarrow \quad \mathrm{e}^{tZ} \to 0 \text{ as } t \to +\infty$$

- H semi-bounded (below or above) iff $\Omega^2 - |\kappa|^2 > 0$
- H semi-bounded $\Rightarrow$ existence of invariant state determined only by noise,
- H not semi-bounded $\Rightarrow$ a "strong" noise is necessary for existence of an invariant state.

Intuition behind invariant states

For an invariant state

$$\mathbf{S}_t = \mathbf{S}_0 \iff \int_0^t e^{s\mathbf{Z}^T} \left(\mathbf{Z}^T \mathbf{S}_0 + \mathbf{S}_0 \mathbf{Z} + \mathbf{C} \right) e^{s\mathbf{Z}} ds = 0$$

The operator

$$\mathbf{S} = \int_0^\infty e^{s\mathbf{Z}^T} \mathbf{C} e^{s\mathbf{Z}} ds$$

is a solution of $\mathbf{Z}^T \mathbf{S} + \mathbf{S}\mathbf{Z} = -\mathbf{C}$ if it exists.
It *does exist* when $\mathbf{Z}$ is stable

$$\mathbf{Z} = \begin{pmatrix} -\gamma - \mathrm{Im}(\kappa) & \mathrm{Re}(\kappa) - \Omega \\ \mathrm{Re}(\kappa) + \Omega & -\gamma + \mathrm{Im}(\kappa) \end{pmatrix}, \quad \lambda_{1,2} = -\gamma \pm \sqrt{|\kappa|^2 - \Omega^2}$$

Recall that an invariant state exists if the parameters are in $\mathfrak{I}$

Theorem

If

$$\gamma > 0 \ \textit{and} \ \gamma^2 + \Omega^2 - |\kappa|^2 > 0$$

and ρ_∞ *is the unique invariant state, for any initial state* ρ_0

$$\lim_{t\to+\infty} \mathcal{T}_{*t}(\rho_0) = \rho_\infty$$

Remark. ρ_∞ is either faithful or pure

Multimode $d > 1$

- All eigenvalues of Z with $\Re(\lambda) < 0 \Rightarrow \exists$ invariant state(s),
- If $\exists$ eigenvalues of Z with $\Re(\lambda) > 0 \Rightarrow$ might not $\exists$ invariant states.

If $\exists$ eigenvalues with $\Re(\lambda) = 0$ both situations.

Irreducibility

A semigroup is irreducible if and only if for all p projection

$$\mathcal{T}_t(p) \geq p \qquad \Rightarrow p = 0, \quad p = \mathbb{1}$$

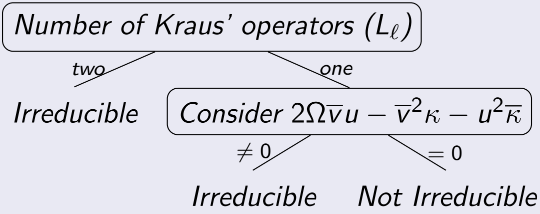
$\mathcal{T}_t : \mathcal{A} \to \mathcal{A}, \quad \mathcal{T}_t(p) \geq p$ means

- $\mathcal{T}_t(p\mathcal{A}p) \subseteq p\mathcal{A}p$
- ρ density s.t. $\rho = p\rho p \Leftrightarrow \mathcal{T}_{*t}(\rho) = p\mathcal{T}_{*t}(\rho)p$

One mode $d = 1$ case.

Theorem

Gaussian QMS



Theorem (FF & R. Rebolledo)

p projection, $Ran(p)$ its range. $\mathcal{T}_t(p) \geq p$ if and only if

1. *$Ran(p)$ is invariant for the strongly continuous contraction semigroup e^{tG}*
2. *$L_\ell u = pL_\ell u$, for $u \in Dom(G) \cap Ran(p)$*

$$G = -\mathrm{i}H - \frac{1}{2}\sum_{\ell} L_\ell^* L_\ell$$

Two linearly independent $L_\ell \Rightarrow$ a candidate invariant subspace must be invariant for $a, a^\dagger$ ... $\Rightarrow$ irreducible!

In the case of a single Kraus operator L a candidate invariant subspace should be invariant also for

$$-2\,[G, L] = [L^*, L]\, L + 2\mathrm{i}\,(\overline{v}\Omega - u\overline{\kappa})\, a + 2\mathrm{i}\,(\overline{v}\kappa - u\Omega)\, a^\dagger + 2\mathrm{i}\left(\overline{v}\zeta - u\overline{\zeta}\right)$$

We have another linear operator in creation and annihilation operators

$$\tilde{L} = (\overline{v}\Omega - u\overline{\kappa})\, a + (\overline{v}\kappa - u\Omega)\, a^\dagger$$

The condition for linear independence of L and $\tilde{L}$ is

$$\det \begin{bmatrix} \overline{v}\Omega - u\overline{\kappa} & \overline{v}\kappa - u\Omega \\ \overline{v} & u \end{bmatrix} \neq 0$$

$$L_\ell = \sum_{j=1}^{d} \left(\overline{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \rightsquigarrow \begin{bmatrix} \overline{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{bmatrix}$$

$$-2[G, L_\ell] \;=\; 2\mathrm{i}[H, L_\ell] + \sum_k [L_k^*, L_\ell] L_\ell + \sum_k L_k^* [L_k, L_\ell]$$

linear in a_j $a_j^\dagger$ $\rightsquigarrow$ vector $\begin{bmatrix} \overline{v}^{(1)}_{\ell \bullet} \\ u^{(1)}_{\ell \bullet} \end{bmatrix}$

Iterate commutators $[G, [G, L_\ell]]$ and get vectors

$$\begin{bmatrix} \overline{v}^{(n)}_{\ell \bullet} \\ u^{(n)}_{\ell \bullet} \end{bmatrix} \qquad n \geq 0.$$

Conjecture: the gaussian QMS is irreducible if and only if

$$\mathrm{Lin}_{\mathbb{R}} \left\{ \begin{bmatrix} \overline{v}_{\ell\bullet}^{(n)} \\ \\ u_{\ell\bullet}^{(n)} \end{bmatrix} \;\middle|\; n \geq 0 \right\}$$

as a real subspace of $\mathbb{C}^d$ is $2d$-dimensional.

Thank you!