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A QUANTUM GIBBS PARADOX

Benjamin Yadin, Benjamin Morris, Gerardo Adesso

NATURE COMMUNICATIONS 12, 1471 (2021)

VolkswagenStiftung

Entanglement, Information & Noise

June 14 - 20, 2004
to be held in Krzyzowa (Lower Silesia), Poland

Preliminary programme

Sunday, June 13	
20:00 -	Welcome (providing food and beverages)
Monday, June 14	
09:00 - 09:15	Opening
	Chairman: Andrzej Jamiolkowski
09:15 - 10:00	Andrzej Kossakowski
	Positive maps and classification of states
10:00 - 10:45	Reinhard F. Werner
	Proving Heisenberg's first uncertainty relation
10:45 - 11:25	Coffee break

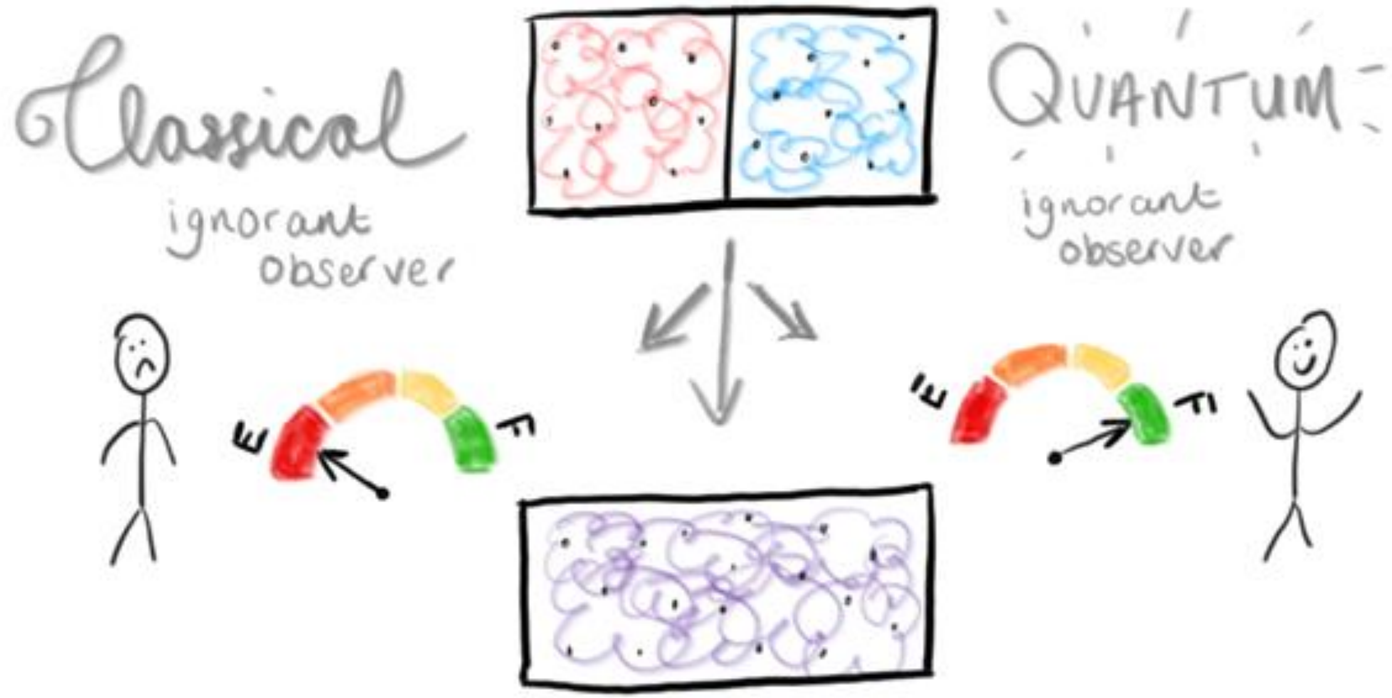
Scientific Coordinators:

- Andreas Buchleitner (MPIPKS Dresde
- Karol Zyczkowski (Smoluchowski Ins ... Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, Poland)


THE BORDER TERRITORY
QUANTUM DOMAIN
CLASSICAL DOMAIN
PHOTONS
ELECTRONS
ATOMS
GRAVITY WAVE DETECTOR
SUN
PLANETS
US
QUANTUM FLUIDS
QUANTUM
CLASSICAL
DO NOT CROSS WITHOUT AMPLIFIER
N. BOHR
QUANTUM BILL OF RIGHTS
INTERFERE IF YOU CAN!!!
SCHRODINGER'S EQUATION
CLASSICAL LAW AND ORDER
DO NOT INTERFERE!!!
NEWTON'S EQUATIONS
SECOND LAW OF THERMODYNAMICS

Our Research

- Identifying **quantumness** by its most genuine signatures in general composite systems
- Providing novel operational interpretations and satisfactory measures for quantum **resources**


Classical
ignorant observer
QUANTUM
ignorant observer
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E
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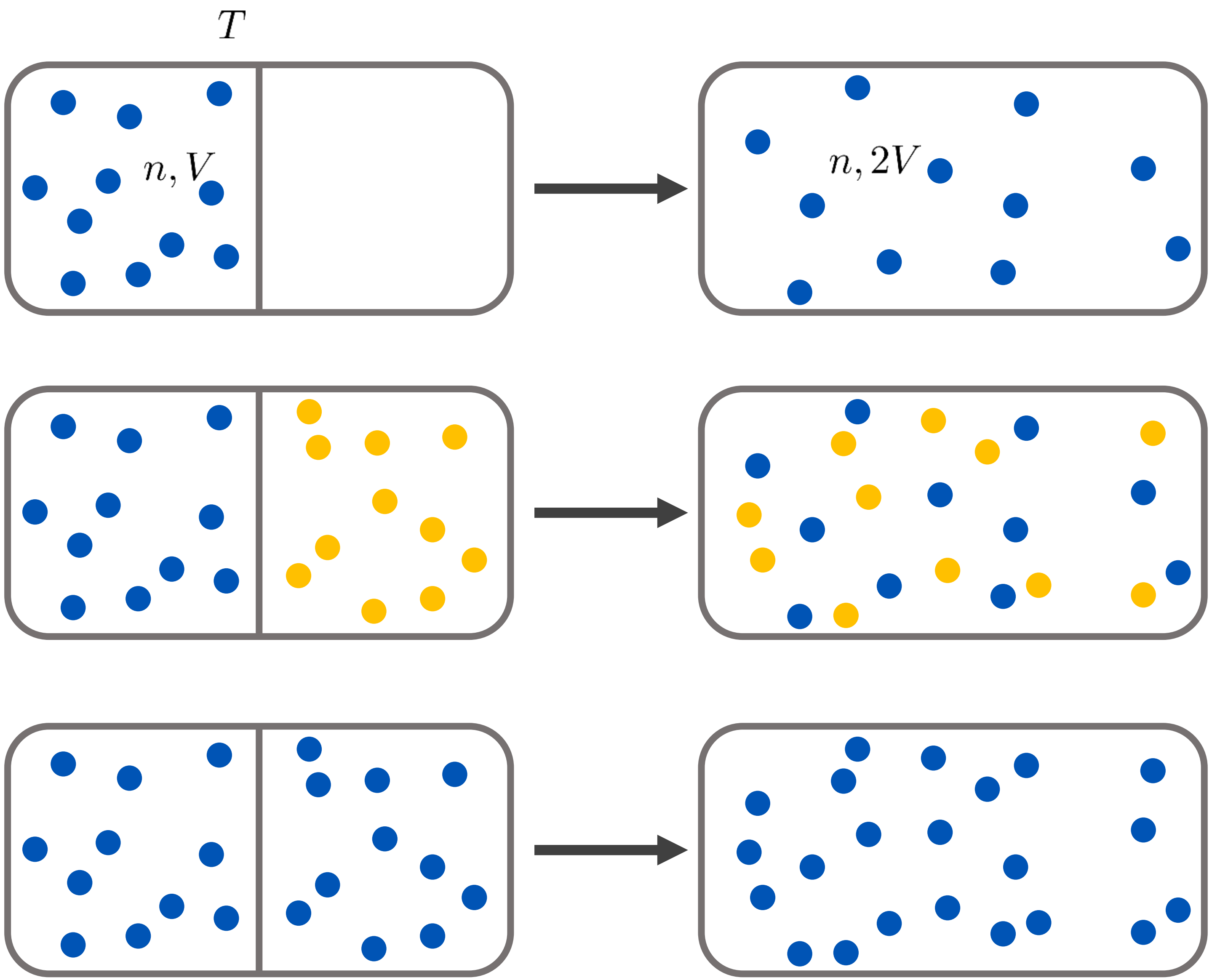
This talk

GIBBS PARADOX

Ideal gas in a box, expanding isothermally to twice its original volume

Entropy change depends on whether the gases are identical



$\Delta S = n \ln 2$

$\Delta S = 2n \ln 2$

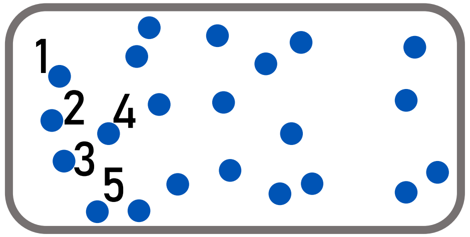
contradiction?

$\Delta S = 0$

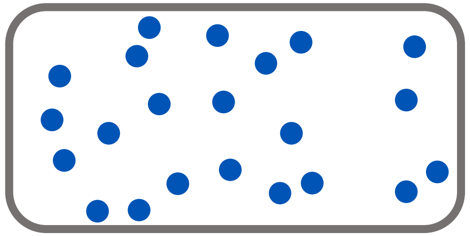
GIBBS PARADOX

Hstorically, seen as a problem with making entropy extensive

Gibbs + Boltzmann introduced a correction factor into microstate counting



$$\Omega = V^N,$$
$$S = \ln \Omega = N \ln V$$



$$\Omega = \frac{V^N}{N!},$$
$$S = \ln \Omega \approx N \ln \left(\frac{V}{N} \right) + N$$

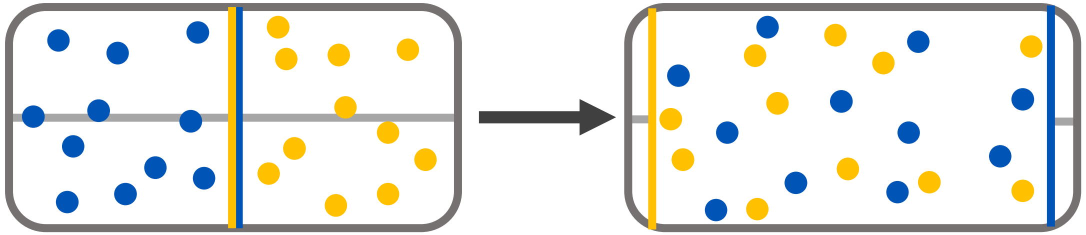
Removing particle labels means phase space volume is divided by $N!$

Simon Saunders, "The Gibbs Paradox," Entropy 20, 552 (2018)
Dennis Dieks, "The Gibbs Paradox and Particle Individuality," Entropy 20, 466 (2018)

ROLE OF THE OBSERVER

INFORMED OBSERVER

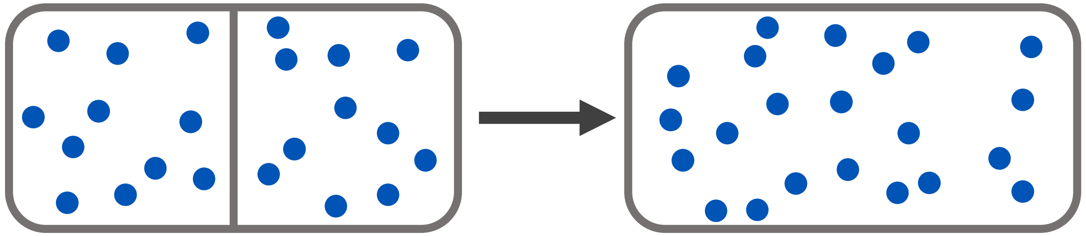
$\Delta S = 2n \ln 2$



Can extract work from each gas independently
(semi-permeable membrane)

IGNORANT OBSERVER

$\Delta S = 0$



Can't extract work – apparatus couples identically to both gases

"observer" = designation of which degrees of freedom can be operated upon

Edwin T Jaynes, "The Gibbs paradox," in Maximum entropy and Bayesian methods (Springer, 1992)

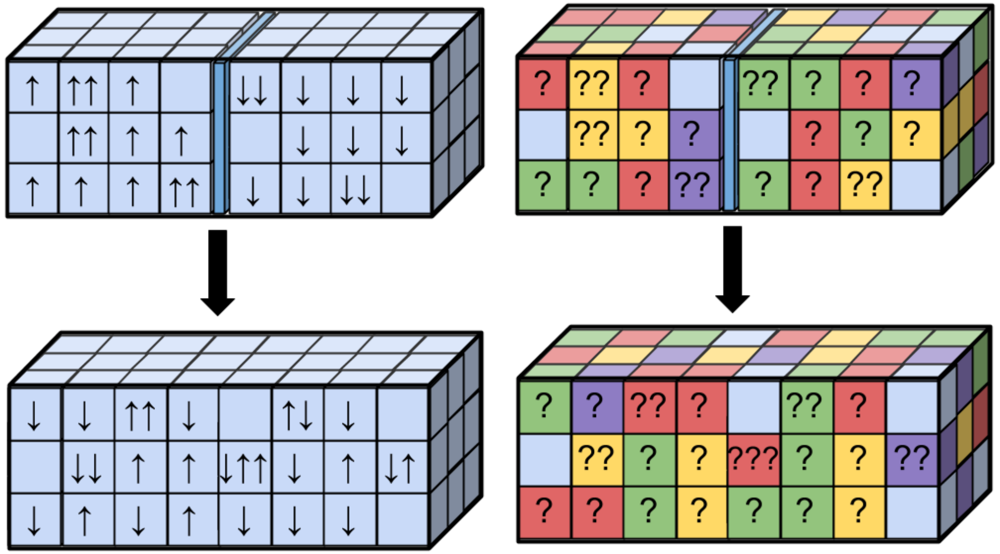
The amount of useful work that we can extract from any system depends – obviously and necessarily – on how much "subjective" information we have about its microstate, because that tells us which interactions will extract energy and which will not; this is not a paradox, but a platitude. If the entropy we ascribe to a macrostate did not represent some kind of human information about the underlying microstates, it could not perform its thermodynamic function of determining the amount of work that can be extracted reproducibly from that macrostate.

OVERVIEW

We analyse Gibbs mixing in a **fully quantum** fashion using bosons / fermions

GOAL: Find the **fundamental limits** on work extractable by different observers

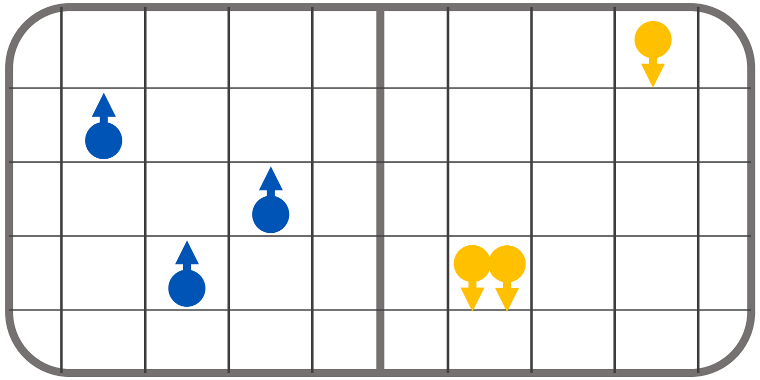
- Toy model
- Classical analysis and state-counting
- Hlbert space structure and observers' operations
- Entropy changes and interesting limits



THE MODEL

TOY MODEL

- 2 sides of a box, each with $\frac{d}{2}$ "cells"
- Start with n particles on each side
- Distinguish the gases by a "spin" degree of freedom ($\uparrow$ or $\downarrow$)
- $H = 0$ (all cells are degenerate in energy)
- Both sides initially thermalised



Difference between the observers:

- *INFORMED OBSERVER CAN INTERACT WITH THE SPIN DEGREE OF FREEDOM*
- *IGNORANT OBSERVER CANNOT – DYNAMICS MUST BE SPIN-INDEPENDENT*

TOY MODEL

Is the model too naïve?

- **How can you extract work if $H = 0$?**

  Couple the system to a heat bath B at temperature T and a work battery W
  Total energy is conserved (resource theory framework of *thermal operations*)

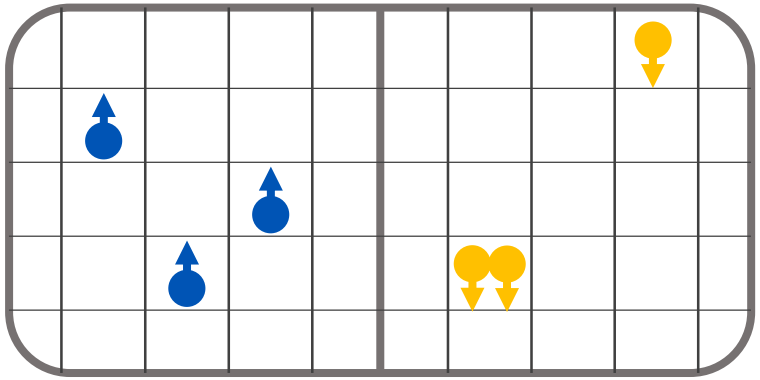
  Extractable work is proportional to entropy change ΔS:

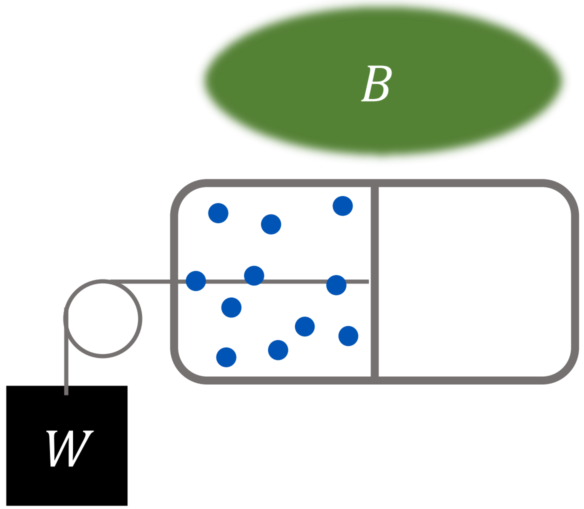
$$\boxed{\Delta W = k_B T\left[S(\rho') - S(\rho)\right]}$$

$$S(\rho) = -\operatorname{Tr}(\rho \ln \rho)$$

- **Does this really model an ideal gas?**

  Yes – recovers the correct classical entropy changes





Simon Saunders, "The Gibbs Paradox," Entropy 20, 552 (2018)
Dennis Dieks, "The Gibbs Paradox and Particle Individuality," Entropy 20, 466 (2018)

CLASSICAL CASE

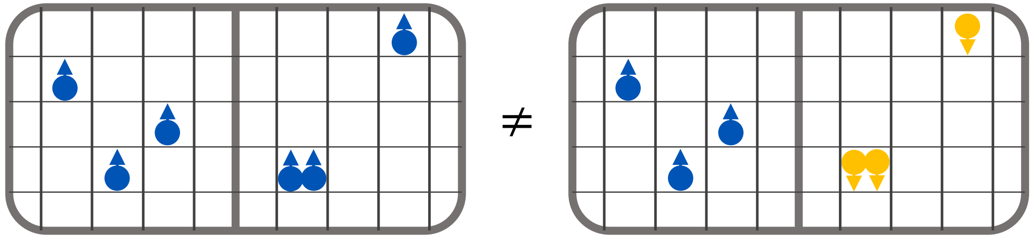
CLASSICAL CASE

Initial state: on each side, we have a uniform distribution of n identical particles over $d/2$ cells

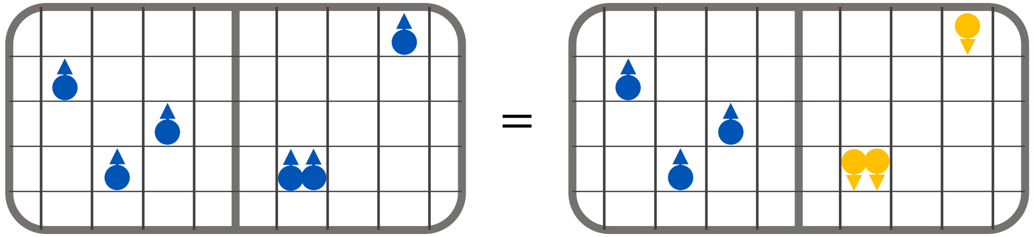
Final state: uniform distribution over **all** configurations of $2n$ particles over d cells

Entropy calculation boils down to a simple counting of microstates

- ***INFORMED OBSERVER:*** counting depends on whether the spins are the same or different



- ***IGNORANT OBSERVER:*** different spin configurations are counted as the same



CLASSICAL CASE

Entropy change ΔS from microstate counting

	INFORMED	IGNORANT
Identical gases	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$
Different gases	$2\ln\binom{n+d-1}{n} - 2\ln\binom{n+d/2-1}{n}$	$\ln\binom{2n+d-1}{2n} - 2\ln\binom{n+d/2-1}{n}$

CLASSICAL CASE: MACROSCOPIC LIMIT

Entropy change ΔS from microstate counting

Take the limit $d \gg n \gg 1$ **(large particle number and low density)**

	INFORMED	IGNORANT
Identical gases	≈ 0	≈ 0
Different gases	$2n \ln 2$	≈ 0

≈ 0 means $O(\ln n)$

Recovers the ideal gas results

This changes in the quantum case

QUANTUM CASE

HILBERT SPACE STRUCTURE

Our task is to describe the effective state space seen by the ignorant observer

Single-particle Hilbert space:

$$\mathcal{H}_1 = \mathcal{H}_x \otimes \mathcal{H}_s$$

"spatial" (cell)

$|i\rangle\,,\ i = 1, \ldots, d$

"spin"

$|\uparrow\rangle\,, |\downarrow\rangle$

N-particle Hilbert space for bosons / fermions:

$$\mathcal{H}_N = P_\pm \left(\mathcal{H}_x^{\otimes N} \otimes \mathcal{H}_s^{\otimes N} \right)$$

$P_\pm$ is projector onto (anti-)symmetric subspace

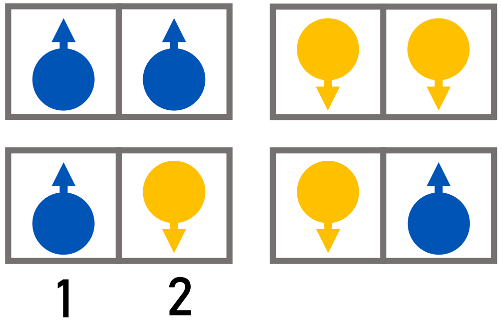
Permutations of particle labels act simultaneously on space and spin:

$$\Pi = \Pi_x \otimes \Pi_s$$

COUPLING SPIN AND SPATIAL SYMMETRY

Spatial and spin permutation symmetries must combine to give overall (anti-)symmetry

Familiar example from atomic physics:
(2 particles, each in its own cell)



$$\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}} \quad \xleftrightarrow{\text{bosons}} \quad |\uparrow\uparrow\rangle\,,\ |\downarrow\downarrow\rangle\,,\ \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad J = 1$$

fermions

$$\frac{|1\,2\rangle - |2\,1\rangle}{\sqrt{2}} \quad \longleftrightarrow \quad \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad J = 0$$

A particular symmetry of the spatial wavefunction comes with each J

$$J(J+1) \text{ = eigenvalue of } \left(\sum_{i=1}^{N} J_x^{(i)}\right)^2 + \left(\sum_{i=1}^{N} J_y^{(i)}\right)^2 + \left(\sum_{i=1}^{N} J_z^{(i)}\right)^2$$

STATE SPACE OF QUANTUM IGNORANT OBSERVER

Spatial and spin representations are linked via *J*

In general:

$$\mathcal{H}_N = \bigoplus_{J=0}^{N/2} \underbrace{\mathcal{H}_x^J}_{\text{U}(d)\text{ irrep}} \otimes \underbrace{\mathcal{H}_s^J}_{\text{U}(2)\text{ irrep}}$$

Ignorant observer acts on this part only

(Schur-Weyl duality for groups $\mathrm{U}(d)$ and S_N, used twice)

Adamson et al., PRA 78, 033832 (2008)

Conditions on the global unitary U (coupling system, heat bath, work battery):

- U acts only on $\mathcal{H}_x^J$ factors
- Must preserve exchange symmetry: $[U, \Pi] = 0 \ \forall \Pi$

J is conserved

Tracing out the spin part, the ignorant observer works with the state $\rho_x = \mathrm{Tr}_s\, \rho = \bigoplus_J p_J \rho_x^J$

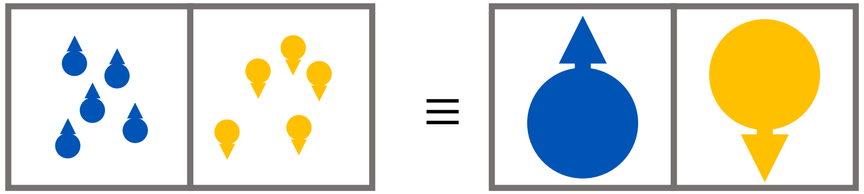
Each component ρ_x^J in the ensemble evolves independently in the space $\mathcal{H}_x^J$ of dimension d_J

THERMALISATION BY QUANTUM IGNORANT OBSERVER

State evolution described by quantum ignorant observer

$$\rho_x = \mathrm{Tr}_s\, \rho = \bigoplus_J p_J \rho_x^J \to \bigoplus_J p_J \frac{I_x^J}{d_J}$$

not maximally mixed (due to conservation law)



$$\Delta S_{\text{igno}} = \sum_J p_J \ln d_J - 2 \ln \binom{n + d/2 - 1}{n}$$

p_J from Clebsch-Gordan coefficients (two large spins);
d_J from representation theory formulas

Never larger than the **informed** observer:

$$\Delta S_{\text{igno}} \leq \Delta S_{\text{info}} = 2 \ln \binom{n+d-1}{n} - 2 \ln \binom{n + d/2 - 1}{n}$$

But can be above the **classical ignorant** observer!

Translates into an average extracted work $\sum_J p_J W_J = k_B T \Delta S$ (achievable with thermal operations)

Horodecki & Oppenheim, Nat. Comm 4, 2059 (2013)

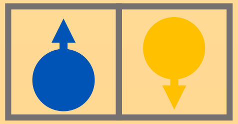
THERMALISATION BY QUANTUM IGNORANT OBSERVER: EXAMPLE

State evolution described by quantum ignorant observer

$$\rho_x = \mathrm{Tr}_s\, \rho = \bigoplus_J p_J \rho_x^J \to \bigoplus_J p_J \frac{I_x^J}{d_J}$$

Example: $\boldsymbol{n = 1, d = 2}$

initially $$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot \overbrace{\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}}^{J=1} + \frac{1}{\sqrt{2}} \cdot \overbrace{\frac{|1\,2\rangle - |2\,1\rangle}{\sqrt{2}} \cdot \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}}^{J=0}$$

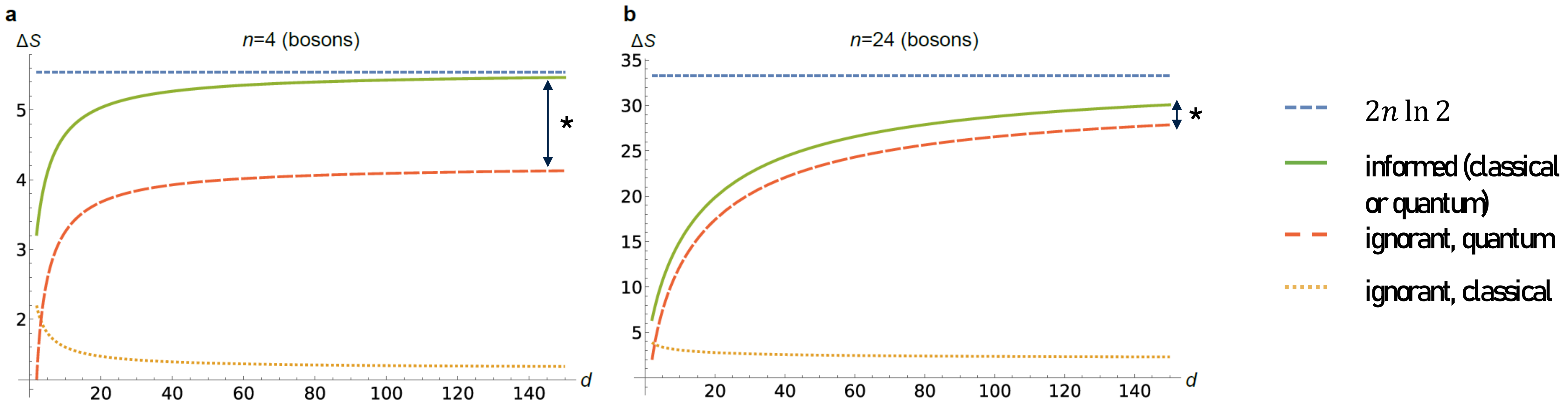


(unchanged)

mixture of $$\frac{|1\,2\rangle + |2\,1\rangle}{\sqrt{2}},\ |1\,1\rangle,\ |2\,2\rangle$$

$$\Delta S_{\text{igno}} = \frac{1}{2}\ln 3 + \frac{1}{2}\ln 1$$

INTERESTING LIMITS



Low density limit: $d \to \infty$

$$\Delta S_{\text{info}} - \Delta S_{\text{igno}} \approx H(\mathbf{p}) - \frac{n^2}{2d^2} \qquad (*)$$

$$H(\mathbf{p}) = -\sum_J p_J \ln p_J \approx \frac{1}{2} \ln n$$ for large n

Also taking **large n**: $\boxed{\Delta S_{\text{info}} \approx \Delta S_{\text{igno}} \approx 2n \ln 2}$ (and negligible fluctuations)

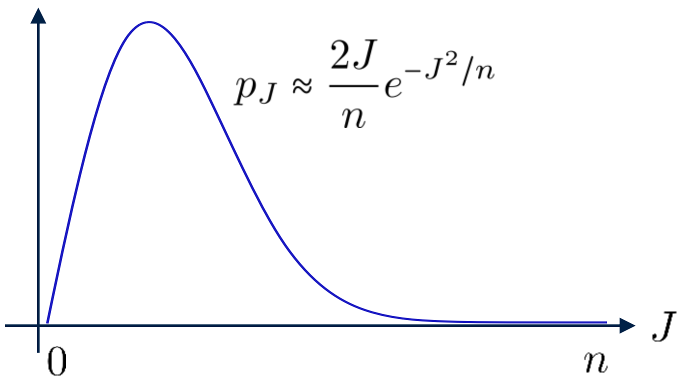
WORK FLUCTUATIONS

Work isn't extracted deterministically

Each J occurs with probability p_J and results in entropy change $\quad \Delta S_{\text{igno}}(J) = \ln d_J - 2\ln\binom{n+d/2-1}{n}$

with average value $\quad \Delta S_{\text{igno}} = \sum_J p_J \Delta S_{\text{igno}}(J)$

Look at variance: $\quad \boxed{V(\Delta S_{\text{igno}}) = \sum_J p_J \Delta S_{\text{igno}}(J)^2 - \Delta S_{\text{igno}}^2}$



Large number, low density limit:

Mean is $\quad \Delta S_{\text{igno}} \approx 2n\ln 2$

Variance is $\quad V(\Delta S_{\text{igno}}) \approx \frac{\pi^2}{24} \approx 0.411$ **negligible compared with mean**

QUANTUM VS CLASSICAL CASE: MACROSCOPIC LIMIT

Entropy change ΔS from our analysis

Take the limit $d \gg n \gg 1$ **(large particle number and low density)**

	INFORMED	*IGNORANT*
Identical gases	≈ 0	≈ 0
Different gases	$2n \ln 2$	CLASSICAL ≈ 0 **QUANTUM** $\approx 2n \ln 2$

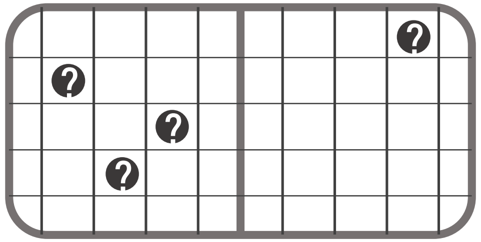
≈ 0 means $O(\ln n)$

Maximum divergence with the classical case in the **macroscopic** low density limit!

WHY DOES IT WORK?

Low density means two particles almost never sit in the same cell

There are $\binom{d}{2n}$ cell configurations



... this is all that the classical ignorant observer sees

For each cell configuration, there are $\binom{2n}{n}$ spin configurations: choices of where to put the ↑ and ↓ particles

WLOG, choose cells $1, \ldots, 2n$ to be occupied, then a spin configuration is a permutation of $|\uparrow\rangle_1 \cdots |\uparrow\rangle_n |\downarrow\rangle_{n+1} \cdots |\downarrow\rangle_{2n}$

How much information is lost when we trace out the spin part?

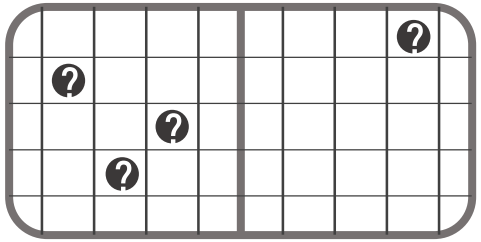
WHY DOES IT WORK?

A slightly different application of Schur-Weyl duality:

$$(\mathbb{C}^2)^{\otimes 2n} = \bigoplus_J \mathcal{H}^J \otimes \mathcal{K}^J$$

$SU(2)$ irrep under $u_s^{\otimes 2n}$

S_N irrep under cell permutations



Gives a convenient basis: $|J, M, p\rangle$ = superpositions of spin configurations (Schur basis)

Tracing out spin degrees of freedom = ignoring M

But $M = \frac{1}{2}(N_\uparrow - N_\downarrow)$ is fixed anyway – so no information is lost!

For a given cell configuration, all the different $|J, M, p\rangle$ states can be distinguished by the quantum ignorant observer

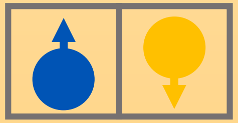
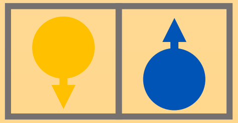
And there are as many of these as there are spin configurations, i.e. $\binom{2n}{n}$

WHY DOES IT WORK? EXAMPLES

For a given cell configuration, all the different $|J, M, p\rangle$ states can be distinguished by the ignorant observer

And there are as many of these as there are spin configurations, i.e. $\binom{2n}{n}$

Example: 2 particles

Classically: ignorant observer sees only the cell configuration:  and  look the same

$$|J = 1,\ M = 0,\ p = 0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad |J = 0,\ M = 0,\ p = 0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

Example: 3 particles

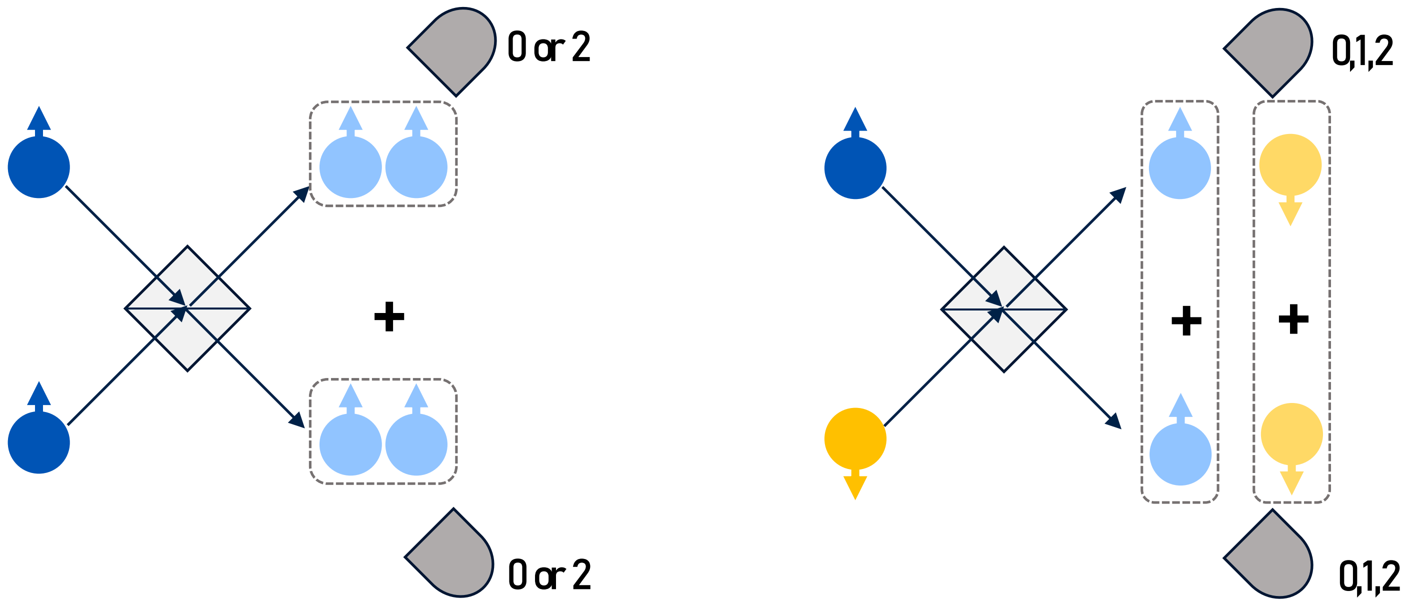
$$|J = 3/2,\ M = 1/2,\ p = 0\rangle = \frac{|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle}{\sqrt{3}}$$

$$|J = 1/2,\ M = 1/2,\ p = 0\rangle = \frac{|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle}{\sqrt{2}}$$

$$|J = 1/2,\ M = 1/2,\ p = 1\rangle = \frac{2\,|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle}{\sqrt{6}}$$

RELATED WORK

Recall the **Hong-Ou-Mandel effect** in quantum optics:

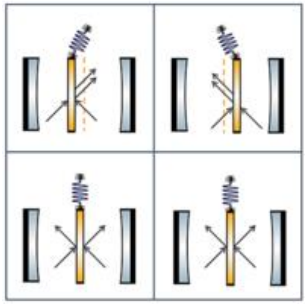


Non-polarising beam-splitter and photon number counting are able to tell if the polarisations are equal or opposite

These operations are polarisation-independent, so **accessible to an ignorant observer**

Recent / related works on thermodynamics with identical particles:

- Holmes et al., PRL 124, 210601 (2020); also NPJ 22, 113015 (2020)
- Watanabe et al., PRL 124, 210604 (2020)
- Myers and Deffner, PRE 101, 012110 (2020)
- Allahverdyan and Neuwenhuizen, PRE 73, 066119 (2006)



CONCLUSIONS

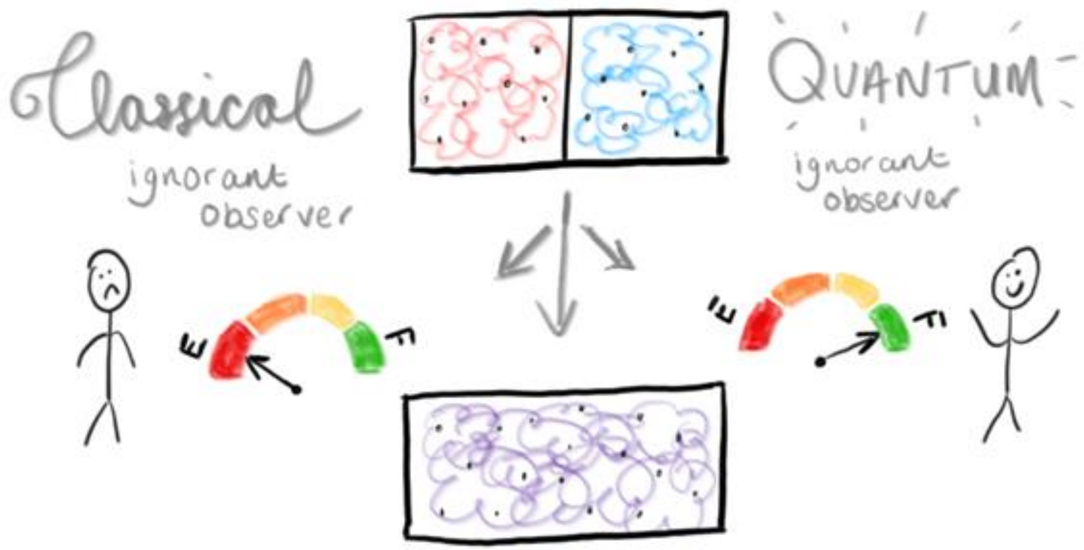
- With quantum particles, relational spin information is imprinted upon observable degrees of freedom
  Superpositions of classically indistinguishable configurations are distinguishable even by an ignorant observer

- In a (low density) macroscopic limit, **ignorance is bliss** – as much work can be extracted as if the particles were fully distinguishable: what we call a ***QUANTUM GIBBS PARADOX***

- Allowing fully quantum control, classical thermodynamics does **not** emerge in the macroscopic limit

OPEN QUESTIONS

- More realistic models, $H \neq 0$
- Understanding / approximating optimal operations
- Experimental proposals and practical implementations



Thank you

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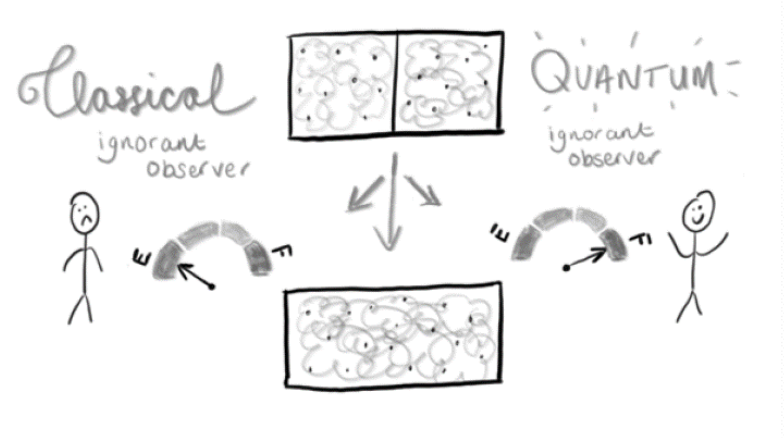
physicsworld

QUANTUM MECHANICS | RESEARCH UPDATE

Quantum mechanics gives new insights into the Gibbs paradox

20 Mar 2021 Lisa Tse



Ignorance is bliss A cartoon depicting the differences between 'ignorant' observers of mixing gases in the classical and quantum versions of the Gibbs paradox. (Courtesy: Bethan Morris, PhD student at University of Nottingham)

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