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The quantum mechanics canonically associated to free probability (joint work with Tarek Hamdi and Yun Gang Lu)

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First of all I want to dedicate my talk **to the memory of Andrzej Kossakowski**.

I had the pleasure and honor to write a paper with him, namely:

L. Accardi, D. Chruściński, A. Kossakowski, T. Matsuoka, M. Ohya:

On classical and quantum liftings,

Open systems and Information Dynamics (17) (4) (2010) 361-387

Plan of the talk.

1) Short review of the **quantum mechanics canonically associated with a classical random variable** with all moments.

2) The canonical quantum decomposition in the **semi-circle case**.

3) The $*$**–Lie–algebra** associated to the standard semi–circle distribution.

4) Free **momentum** and free **kinetic energy**.

5) The free **harmonic oscillator**.

6) The free **momentum evolution**.

7) The **inverse normal order problem**.

8) The free **kinetic energy evolution**.

Short review of the quantum mechanics canonically associated with a classical random variable with all moments

Let X be a classical real valued random variable with all moments and probability distribution μ. Denote $(\Phi_n)_{n\in\mathbb{N}}$ the orthogonal polynomials of X, (ω_n), (α_n) its Jacobi sequences and

$$\Gamma_X \equiv \Gamma\left(\mathbb{C}, \{\omega_n\}_{n=1}^{\infty}\right) \subseteq L^2(\mathbb{R}, \mu) \qquad (1)$$

the closure, in $L^2(\mathbb{R}, \mu)$, of the linear span of the Φ_n's. It is known from the paper:
Accardi L., Bozejko M.:
Interacting Fock spaces and Gaussianization of probability measures,
Infin. Dimens. Anal. Quantum Probab. Relat. Top. (IDA-QP) 1 (4) (1998) 663-670
Volterra Preprint N. 321 (1998)
that, identifying X with the multiplication operator by X in Γ_X, the orthogonal gradation

$$\Gamma_X \equiv \bigoplus_{n\in\mathbb{N}} \mathbb{C}\cdot\Phi_n \qquad (2)$$

and writing the **Jacobi tri–diagonal relation**,

$$X\Phi_n = \Phi_{n+1} + \alpha_n\Phi_n + \omega_n\Phi_{n-1} \quad ; \quad \Phi_{-1} := 0$$

in operator form, i.e.

$$X = a^+ + a^0 + a^- \tag{3}$$

one obtains a decomposition of X as a sum of 3 linear operators

$$a^+X\Phi_n = \Phi_{n+1} \,;\, a^0\Phi_n = \alpha_n\Phi_n \,;\, a^-\Phi_n = \omega_n\Phi_{n-1}$$

called respectively **creation, annihilation and preservation (CAP)** operators.
The CAP operators leave invariant the linear span of the Φ_n's.
The identity (3) **canonical quantum decomposition** of X.
It is **unique** up to isomorphisms.

Moreover, define Λ by

$$\Lambda\Phi_n = n\Phi_n$$

(**number operator**) and similarly, for any sequence (F_n) of complex numbers,

$$F_\Lambda\Phi_n := F_n\Phi_n$$

the operators a^+, a^- and the operator Λ satisfy the following **multiplication table**

$$a^+a^- = \omega_\Lambda \quad ; \quad a^-a^+ = \omega_{\Lambda+1} \tag{4}$$

$$F_\Lambda a^+ = a^+F_{\Lambda+1} \quad ; \quad a^-F_\Lambda = F_{\Lambda+1}a^- \tag{5}$$

which implies the **commutation relations**

$$[a^-, a^+] = \omega_{\Lambda+1} - \omega_\Lambda =: \partial\omega_\Lambda \tag{6}$$

$$[a^+, a^+] = [a^-, a^-] = 0$$

$$[a^+, F_\Lambda] = -a^+(F_{\Lambda+1} - F_\Lambda) =: -a^+\partial F_\Lambda \tag{7}$$

Look at the right hand side of the commutation relations

$$[a^-, a^+] = \omega_{\Lambda+1} - \omega_\Lambda =: \partial\omega_\Lambda \tag{8}$$

and notice that, if $\partial\omega_\Lambda = \hbar$ is a constant, they become the **Heisenberg commutation relations** (CCR) for quantum systems with one
degree of freedom:

$$[a^-, a^+] = \hbar \quad ; \quad [a^+, a^+] = [a^-, a^-] = 0 \tag{9}$$

Theorem

The **unique** symmetric (all odd moments vanish) classical random variable X whose associated canonical commutation relation are the Heisenberg commutation relations (10) for a fixed $\hbar > 0$ is the **standard gaussian with variance** $\hbar$.

Proof.

$$\partial\omega_\Lambda = \hbar \iff \omega_{n+1} - \omega_n = \hbar$$

$$\iff \omega_n = \hbar n + c \quad , \hbar, c \in \mathbb{N} \tag{10}$$

One proves that, for symmetric classical random variables, $c = 0$ and it is known, from the theory of orthogonal polynomials, that **the only measure** with principal Jacobi sequence given by $\hbar n$ is the **standard gaussian with variance** $\hbar$. $\square$

Remark. The case $c \neq 0$ corresponds to the **translates of the Gaussian**.

All the **Poisson random variables** with jump intensity $\hbar$ have **the same** principal Jacobi sequence as the standard gaussian with variance $\hbar$.

This explains why the usual Boson Fock space can be represented as L^2 of a gaussian measure, **but also** as L^2 of a Poisson measure (with respect to a different orthogonal gradation) and why both distributions appear so frequently in boson quantum physics.

The quantum algebra generated by a classical random variable

The $*$–algebra

$$\mathcal{A}_X^0 := \text{algebraic span of } \ a^+ \ , \ a^- \ , \ a^0 \ , \ \partial\omega_\Lambda$$

is called the **reduced quantum algebra canonically associated to the classical random variable** $X = a^+ + a^0 + a^-$.

$\mathcal{A}_X^0$ is **commutative** iff the probability distribution of X is a δ–measure on some point.

The vector Φ_0 (vacuum vector) induces on $\mathcal{A}_X^0$ the state

$$\varphi := \langle \cdot , \cdot \rangle$$

Recall that a pair $(\mathcal{B}, \varphi)$, where $\mathcal{B}$ is a $*$–algebra and φ a state on $\mathcal{B}$ is called an **algebraic probability space**:

quantum if $\mathcal{B}$ is non–commutative,

classical if $\mathcal{B}$ is commutative.

Therefore the pair

$$(\mathcal{A}_X^0, \varphi)$$

is an example of **quantum probability space**.
Summarizing:
Every **classical random variable** X is canonically associated with a **quantum probability space**.
Quantum probability spaces are the main object of study in quantum probability.

Remark. When $\partial\omega_\Lambda$ is not a multiple of the identity (i.e. in all cases except the Gauss–Poisson case) it is convenient to enlarge the $*-$algebra $\mathcal{A}_X^0$ by including in it arbitrary functions of Λ. The $*-$algebra

$$\mathcal{A}_X := \text{algebraic span of } \ a^+ \ , \ a^- \ , \ a^0 \ , \ F_\Lambda$$

is called the **quantum algebra canonically associated to the classical random variable** X.

The term **normal order** in the quantum algebra $\mathcal{A}_X$ of X is similar to that in usual quantum mechanics with the only **difference** that, in this case, the normally ordered expressions are sum of terms of the form

$$(a^+)^m a^n F_\Lambda$$

where F_Λ is a function of Λ.
The term normal order procedure is **more subtle** than the usual one because one must keep into account the commutation relations

$$F_\Lambda a^+ = a^+ F_{\Lambda+1} \quad ; \quad a^- F_\Lambda = F_{\Lambda+1} a^- \quad (11)$$

The canonical momentum operator associated with a classical random variable X.

It is known that the classical random variable X is (polynomially) **symmetric** (i.e. all its odd moments vanish) **if and only if** in the canonical quantum decomposition of X,

$$X = a^{+} + a^{0} + a^{-} \tag{12}$$

$a^0 = 0$.
For a **symmetric** classical random variable X, the Hermitean operator

$$P_X := i(a^{+} - a) \tag{13}$$

called the **momentum operator** canonically associated to X, satisfies the following commutation relation

$$[X, P_X] = i2\partial\omega_{\Lambda} \tag{14}$$

For mean zero Gaussian random variables (i.e. $\partial\omega_{\Lambda} = \hbar$), (14) reduces to the original Heisenberg commutation relation between position and its canonical conjugate momentum.

The quantum mechanics associated with a classical symmetric random variable X

Once the operators position X and momentum P_X are available, one can introduce all operators of physical interest, like kinetic energy $P_X^2/2$, potential energy ....
Hence **any classical symmetric random variable with all moments uniquely determines its own quantum mechanics** and usual quantum mechanics corresponds to mean zero Gaussian random variables whose covariance plays the role of Planck's constant.
Thus **every classical random variable determines its own quantum mechanics**.
The main known examples are:
– Boson QM corresponding to the Gauss–Poisson class.
– Fermion QM corresponding to the Bernoulli class.
– Quadratic QM corresponding to the 3 non–standard classes of Meixner measures.

The quantum mechanics associated with the semi–circle (Wigner) random variable X

One knows that the centered Gaussian measure with variance c, is characterized by $\omega_{n+1} - \omega_n = \hbar$ for each $n \geq 0$, where $\hbar$ is an arbitrary **strictly positive** constant.
The case

$$\omega_{n+1} - \omega_n = 0 \qquad , \quad \forall n \geq 1$$

(later on, it will be clear why $n \geq 1$ and not $n \geq 0$), is mathematically and physically intriguing because this case characterizes the principal Jacobi sequences of the form

$$\omega_n = \omega \geq 0 \quad ; \quad \forall n \in \mathbb{N}^* \tag{15}$$

and it is known from classical probability that, if ω is a strictly positive constant, the principal Jacobi sequences (15) **characterize the symmetric semi–circle laws** with variance ω.

Since these laws play the role of the Gaussian for free probability, the following problem naturally arises:

which is the quantum mechanics canonically associated to free probability?

The canonical quantum decomposition in the semi-circle case

The unique symmetric classical real valued random variable with principal Jacobi sequence $(\omega_n)_{n\geq 1}$ satisfying

$$\omega_n = \omega > 0 \quad , \quad \forall n \in \mathbb{N}^* := \mathbb{N}\setminus\{0\} \quad ; \quad \omega_0 := 0 \tag{16}$$

is called the **semi–circle random variable with parameter** ω.
Its law is the semi–circle distribution with parameter ω and its canonical quantum decomposition is

$$X := a + a^+$$

where a, a^+ are the creation–annihilation operators acting on the 1MIFS (1), with (ω_n) given by (16), and denoted

$$\Gamma_X := \bigoplus_{n\in\mathbb{N}} \mathbb{C}\cdot\Phi_n = L^2(\mathbb{R},\mu).$$

The monic basis of Γ_X is

$$\left\{\Phi_n := a^{+n}\Phi_0 \ : \ n \in \mathbb{N}\right\} \tag{17}$$

where Φ_0 is the vacuum vector and we use the convention that for any linear operator Y, $Y^0 = id$.

For simplicity of notations we **normalize the semi–circle principal Jacobi sequence** (16) so that

$$\omega = \omega_n = 1 \qquad , \qquad \forall n \geq 1 \tag{18}$$

(see discussion in section). With this normalization $\|\Phi_n\| = \omega_n! = 1$, i.e. **the monic polynomials coincide with the normalized polynomials** and are an ortho–normal basis of Γ_X. Recall that we use the convention

$$\Phi_{-n} = 0 \qquad ; \qquad \forall n \in \mathbb{N}^* \tag{19}$$

and, because of (18),

$$a\Phi_n = \Phi_{n-1} \quad ; \quad a^+\Phi_n = \Phi_{n+1} \quad ; \quad [a, a^+] = \partial\omega_\Lambda \tag{20}$$

The following Lemma recalls some properties of the canonical quantum decomposition of the semi–circle random variable.

Lemma 1 In the canonical representation of the semi–circle random variable with $\omega = 1$, the following multiplication table holds:

$$\omega_{\Lambda+1} = aa^{+} = 1 \quad ; \tag{21}$$

$$\omega_{\Lambda}\Big|_{\{\Phi_0\}^{\perp}} = a^{+}a\Big|_{\{\Phi_0\}^{\perp}} = 1 \quad ; \quad a^{+}a\Phi_0 = 0 \tag{22}$$

In particular

$$\omega_{\Lambda} = a^{+}a = 1 - \Phi_0\Phi_0^{*} \tag{23}$$

where, for any $\xi \in \Gamma_X$, ξ^{*} denotes the linear functional $\xi^{*} : \eta \in \Gamma_X \to \xi^{*}(\eta) := \langle \xi, \eta \rangle$,

$$\partial\omega_{\Lambda} = \Phi_0\Phi_0^{*} = \delta_{0,\Lambda} \tag{24}$$

$$[a, a^{+}] = \partial\omega_{\Lambda} = \Phi_0\Phi_0^{*} \tag{25}$$

Remark. Note that (24) implies that, for $m \geq 1$

$$[a, a^+]\Phi_n = \partial\omega_\Lambda \Phi_n = 0 \quad , \quad \forall n \geq 1$$

Thus, in the canonical representation of the semi–circle law, **the non–commutativity of a and a^+ is restricted to the vacuum space**.

The $*$–Lie–algebra associated to the standard semi–circle distribution

Denote

$$P_{\Phi_m} := \text{ the projection onto } \mathbb{C} \cdot \Phi_m \tag{26}$$

$$\left(P_{\Phi_m} := \delta_{m,\Lambda} = \Phi_m \Phi_m^*\right)$$

Theorem 1 The $*$–Lie algebra generated by a and a^+ is:

$$\mathcal{L}_0 = (\mathbb{C} \cdot a) \oplus (\mathbb{C} \cdot a^+) \oplus \mathcal{L}_{rank\,1}(\Gamma_X) \tag{27}$$

where $\mathcal{L}_{rank\,1}(\Gamma_X)$ denotes the $*$ algebra of rank 1 operators on Γ_X generated by the (Φ_n).

Analytic forms of free momentum and free kinetic energy operator

The μ–Hilbert transform

Let μ_a be the semi-circle measure supported on $[-a, a]$ and λ be the Lebesgue measure on $\mathbb{R}$. Recall that the Hilbert transform of a function on the interval $I_a = [-a, a] \subseteq \mathbb{R}$ is defined by

$$H_{I_a} := \chi_{I_a}(x) f(x) = \frac{1}{\pi} \text{p.v.} \int_{I_a} \frac{f(y)}{x - y} dy \qquad (28)$$

where $\chi_{I_a}(x) = 1$ if $x \in I_a$, $= 0$ if $x \notin I_a$.
In this paper we are interested in the weighed Hilbert transform over the interval $I_2 = [-2, 2]$:

$$H_\mu f(x) := H_{I_2} = \chi_{I_2}(x) \left[f(y) \sqrt{4 - y^2} \right] (x) \qquad (29)$$

i.e. the Hilbert transform with respect to the semi–circle measure μ over the interval I_2. H_μ is a **skew-adjoint** operator on $L^2([-2, 2], \mu)$.

Representation on $L^2([-2,2],\mu)$

Since μ has bounded support, the polynomials $(\Phi_n)_{n\geq 0}$ form a **complete orthogonal system** in $L^2([-2,2],\mu)$.

$$(\omega_n)_{n\geq 1} \equiv 1 \quad ; \quad (\alpha_n)_{n\geq 0} \equiv 0$$

the polynomials $(\Phi_n)_{n\geq 0}$ are explicitly given by:

$$\Phi_n(x) = \frac{\sin((n+1)\arccos(x/2))}{\sin(\arccos(x/2))}, \quad n \geq 0 \tag{30}$$

and satisfy the following **monic Jacobi relation**

$$x\Phi_n(x) = \Phi_{n+1}(x) + \Phi_{n-1}(x), \quad \forall x \in [-2,2] \tag{31}$$

The monic Chebyshev polynomial of first kind T_n given by

$$T_n(x) = 2\cos(n \arccos(x/2)) \tag{32}$$

are related with the Φ_n through the following relations

$$T_{n+1}(x) = \Phi_{n+1}(x) - \Phi_{n-1}(x) \tag{33}$$

$$2T_{n+1}(x) = xT_n(x) - \left(4 - x^2\right) \Phi_{n-1}(x) \tag{34}$$

$$T_{n+1}(x) = 2\Phi_{n+1}(x) - x\Phi_n(x) \tag{35}$$

and the two classes of polynomials are connected via the μ–Hilbert transform H_μ as follows

$$H_\mu \Phi_n = T_{n+1} \quad , \quad n \geq 0. \tag{36}$$

Proposition 1 The operators P_X and iH_μ coincide on $L^2([-2,2],\mu)$, i.e. for any $f \in L^2([-2,2],\mu)$ one has

$$P_X f(x) = iH_\mu f(x) = 2i\text{p.v.} \int_{-2}^{2} \frac{f(y)}{x-y}\mu(dy). \tag{37}$$

In particular, the free kinetic energy operator is given by

$$E_X := \frac{1}{2}P_X^2 = -\frac{1}{2}H_\mu^2 \tag{38}$$

Generalized Schrödinger representation

Let Q denote the operator of multiplication by the coordinate in $L^2([-2,2],\mu)$ and V the isometry from $L^2([-2,2],\mu)$ into $L^2(\mathbb{R},\lambda)$ given by

$$f \in L^2([-2,2],\mu) \mapsto f(Q)\rho \in L^2(\mathbb{R},\lambda) \tag{39}$$

where $\rho \in L^2(\mathbb{R},\lambda)$ is defined by

$$\rho(x) := \frac{1}{\sqrt{2\pi}}(4-x^2)^{1/4}\chi_{[-2,2]}(x), \quad x \in \mathbb{R} \tag{40}$$

Notice that, with 1 denoting the constant function $= 1$,

$$V\Phi_0 = V1 = \rho \ \in L^2(\mathbb{R}, \lambda) \tag{41}$$

Define a^ε ($\varepsilon \in \{+, 0, -\}$) the free CAP operators and set

$$A^\varepsilon := V a^\varepsilon V^* \tag{42}$$

Since V is isometric, the A^ε satisfy the free commutation relations:

$$[A^-, A^+] = (V\Phi_0)(V\Phi_0)^* = \rho\rho^* \tag{43}$$

Theorem 2 1. The free position operator X is mapped into the usual position operator $Q = VXV^*$ in $L^2(\mathbb{R}, \lambda)$:

$$Qf(x) = xf(x) \quad , \quad x \in \mathbb{R} \tag{44}$$

2. The free momentum operator P_X is mapped into the operator $P = VP_XV^*$ in $L^2(\mathbb{R}, \lambda)$ given by:

$$P = i\rho H_\mu \rho^{-1} \tag{45}$$

where, here and in the following, $\rho^{-1} := 1/\rho$ is understood on the support of ρ and we use the same symbol for ρ and the multiplication operator by ρ.

3. The free CAP operators a^ε ($\varepsilon \in \{+, 0, -\}$) are mapped by $V(\,\cdot\,)V^*$ into the following CAP operators

$$A^+ = \frac{1}{2}\Big(X + \rho H_\mu \rho^{-1}\Big) A^- = \frac{1}{2}\Big(X - \rho H_\mu \rho^{-1}\Big)$$

4. The free kinetic energy operator E_X is mapped into the operator
$E := VE_XV^*$ in $L^2(\mathbb{R}, \lambda)$ given by:

$$E = \frac{1}{2}P^2 = -\frac{1}{2}\rho H_\mu^2 \rho^{-1} \tag{46}$$

Harmonic oscillators in generalized quantum mechanics

The results in this slide are true
for any principal Jacobi sequence (ω_n).
The Hamiltonian of the
generalized harmonic oscillator
with frequency $\widehat{\omega}$ is defined by:

$$H_{\widehat{\omega}} := \frac{1}{2}(p^2 + \widehat{\omega}^2 q^2) \tag{47}$$

In generalized quantum mechanics this becomes

$$H_{\widehat{\omega}} \tag{48}$$

$$= \frac{1}{2}((\widehat{\omega}^2 - 1)(a^{+2} + a^2) + (\widehat{\omega}^2 + 1)(\omega_\Lambda + \omega_{\Lambda+1}))$$

Putting $\widehat{\omega}^2 = 1$ in (47), one finds

$$H_1 = \omega_\Lambda + \omega_{\Lambda+1} \tag{49}$$

Semi–circle harmonic oscillators

In the semi–circle case with $\omega_n = 1$ for each n, this implies

$$e^{itH_1} + e^{it(\omega_\Lambda+\omega_{\Lambda+1})}P_{\Phi_0}^{\perp}$$

$$= e^{it\omega_1}P_{\Phi_0} + e^{it2\omega_1}P_{\Phi_0}^{\perp}. \qquad (50)$$

Therefore, **for any unit vector**

$$\xi = P_{\Phi_0}\xi + P_{\Phi_0}^{\perp}\xi =: \xi_0 + \xi^{\perp} \in \Gamma_X$$

one has

$$\langle \xi, e^{itH_1}\xi \rangle = \|\xi_0\|^2 e^{it\omega_1} + \|\xi^{\perp}\|^2 e^{it2\omega_1}.$$

Thus, denoting

$$p_\xi := |\xi_0|^2 \in [0,1] \quad ; \quad 1 - p_\xi := \|\xi^{\perp}\|^2$$

one obtains

$$\langle \xi, e^{it(\omega_\Lambda+\omega_{\Lambda+1})}\xi \rangle = p_\xi e^{it\omega_1} + (1-p_\xi)e^{it2\omega_1}$$

i.e. the characteristic function of the **Bernoulli random variable** with values $\{\omega_1, 2\omega_1\}$ and distribution $(p_\xi, 1-p_\xi)$.

The case $\widehat{\omega}^2 \neq 1$

In this case:

the generalized harmonic oscillator Hamiltonian is

$$= \frac{1}{2}((\widehat{\omega}^2 - 1)(a^{+2} + a^2) + (\widehat{\omega}^2 + 1)(\omega_\Lambda + \omega_{\Lambda+1}))$$

so that the **kinetic energy (free) Hamiltonian** is

$$\frac{1}{2}p^2 = \frac{1}{2}\left(a^{+2} + a^2\right) + \frac{1}{2}(\omega_\Lambda + \omega_{\Lambda+1}))$$

The semi–circle coherent vectors are

$$\psi_z := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\omega_n!}} \Phi_n = \sum_{n=0}^{\infty} z^n \Phi_n \qquad , \qquad |z| < 1$$

so that the associated kernel is given by the geometric series.

$$\langle \psi_u, \psi_v \rangle = \sum_{n=0}^{\infty} (\bar{u}v)^n = \frac{1}{1 - \bar{u}v} \qquad , \qquad |u|, |v| < 1$$

Evolutions generated by the free momentum

Recall that the **momentum operator** canonically associated to the semi–circle random variable $X = a^+ + a$ is

$$P := i(a^+ - a)$$

Both X and P are bounded operators.
We study the Schrödinger evolution e^{itP} and the associated Heisenberg evolution $e^{itP}(\,\cdot\,)e^{-itP}$

Lemma 2

$$a_t^+ := e^{itP} a^+ e^{-itP}$$

$$= a^+ + \int_0^t ds (e^{isP_X}\Phi_0)(e^{isP_X}\Phi_0)^* \qquad (51)$$

hence, for any $n \in \mathbb{N}$,

$$a_t^+ \Phi_n = \Phi_{n+1} + \int_0^t ds \langle e^{isP_X}\Phi_0, \Phi_n\rangle e^{isP_X}\Phi_0 \qquad (52)$$

In other words, $a_t^+ = e^{itP_X} a^+ e^{-itP_X}$ is **completely determined by** $e^{itP_X}\Phi_0$.
But then also the action of $e^{itP_X}(\,\cdot\,)e^{-itP_X}$ on the whole quantum algebra $\mathcal{A}_X$ is completely determined by $e^{itP_X}\Phi_0$.

Action on the number vectors of the Schrödinger evolution generated by P_X

We want to compute

$$e^{itP}\Phi_n = \sum_{k\geq 0} \frac{(it)^k}{k!} P^k \Phi_n = \sum_{k\geq 0} \frac{(it)^k}{k!} (i)^k (a^+ - a)^k \Phi_n$$

$$= \sum_{k\geq 0} \frac{(-t)^k}{k!} \sum_{\varepsilon\in\{+,-\}^k} (-1)^{|\{j:\varepsilon(j)=-1\}|} a^{\varepsilon(k)} \dots a^{\varepsilon(1)} \Phi_n \tag{53}$$

Thus the problem is to evaluate the products

$$a^{\varepsilon(k)} \dots a^{\varepsilon(1)} \Phi_n \tag{54}$$

Usual approach: reduce the products

$$a^{\varepsilon(k)} \dots a^{\varepsilon(1)} \tag{55}$$

to their **normally ordered form** using the commutation relation

$$\omega_{\Lambda+1} = aa^+ = 1$$

Any product (55) can be reduced to the form

$$a^{\varepsilon(k)} \cdots a^{\varepsilon(1)} = (a^{+})^{m_{+}} a^{m_{-}} \tag{56}$$

where the numbers $m_{+}, m_{-} \in \mathbb{N}$ are uniquely determined by ε and, when ε varies in $\{+,-\}^{k}$, are all possible pairs satisfying

$$m_{+} + m_{-} \in \{0, 1, \ldots, k\} \tag{57}$$

So we know that e^{itP} can be written in the form

$$e^{itP} = \sum_{m,n \geq 0} I_{m,n}(t)(a^{+})^{m}(a^{-})^{n} \tag{58}$$

for some numerical coefficients $I_{m,n}(t)$.
The problem is to
calculate these coefficients.
The solution of this problem requires the solution of the **inverse normal order problem**, namely:
parametrize the set of $\varepsilon \in \{+,-\}^{k}$ that satisfy the identity

$$a^{\varepsilon(k)} \cdots a^{\varepsilon(1)} = (a^{+})^{m_{+}} a^{m_{-}} \tag{59}$$

when k varies in $\mathbb{N}$.

The vacuum distribution of P and X

Lemma 3 X and $P = P_X$ have the same spectrum and the vacuum distribution of X and of the X–momentum operator

$$P := -i(a^- - a^+) \tag{60}$$

coincide (semi–circle distribution on the interval $[-2, 2]$).

The evolutions e^{itP}, e^{itX}

Theorem 3 If $\omega_n = 1$ for any $n \geq 1$ then, for any $t \in \mathbb{C}$,

$$e^{itP} = \sum_{m,n \geq 0} I_{m,n}(t)(a^+)^m(a^-)^n \tag{61}$$

where both series converges strongly on $\Gamma\Big(\mathbb{C}, \{\omega_n\}_{n\geq 1}\Big)$ and

$$e^{itP} = \sum_{m,n \geq 0} I_{m,n}(t)(a^+)^m(a^-)^n \tag{62}$$

$$I_{m,n}(t) := \sum_{p \geq 0} \frac{t^{m+n+2p}}{(m+n+2p)!} \tag{63}$$

$$(-1)^{p+m}|\Theta_{m+n+2p}(m,n)| = \langle \Phi_m, e^{itP}\Phi_n \rangle$$

where, for any $m_+, m_-, p \geq 0$,

$$|\Theta_{m_+ + m_- + 2p}(m_+, m_-)| \tag{64}$$

$$= \frac{m_+ + m_- + 1}{2p + m_+ + m_- + 1} \binom{2p + m_+ + m_- + 1}{p}$$

The expression (63) looks complicated, but it can be written in terms of **familiar functions in mathematical physics**.

Lemma 4 Let, for any $m, n \in \mathbb{N}$,

$$J_n(t) := \sum_{p \geq 0} \frac{(-1)^p}{p!(n+p)!} \left(\frac{t}{2}\right)^{n+2p} \qquad (65)$$

denote the **Bessel function of first kind**. Then, for any $t \in \mathbb{C}^*$

$$I_{0,n}(t) = (n+1)\frac{J_{n+1}(2t)}{t} = J_{n+2}(2t) + J_n(2t)$$

$$I_{m,n}(t) = (-1)^m I_{0,m+n}(t) = (-1)^n I_{m+n,0}(t)$$

$$\frac{(-1)^m}{t}(m+n+1)J_{m+n+1}(2t)$$

$$I_{m,n}(0) = \delta_{m+n,0}$$

From the above result we deduce the explicit action of e^{itP} on the orthogonal polynomials $\Phi_k(x)$ (i.e. the non–normalized number vectors).

Proposition 2 For any $t \in \mathbb{C}$ and any $k \in \mathbb{N}$,

$$e^{itP}\Phi_k(x) = \sum_{l=0}^{\infty} \langle \Phi_l, e^{itP}\Phi_k \rangle \Phi_l(x)$$

where

$$\langle \Phi_l, e^{itP}\Phi_k \rangle = \sum_{m=0}^{l \wedge k} I_{l-m,k-m}(t) \tag{66}$$

In particular,

$$e^{itP}\Phi_0(x)$$

$$= \frac{1}{t} \sum_{l=0}^{\infty} (-1)^l (l+1) J_{l+1}(2t) \Phi_l(x) \tag{67}$$

Remark 1 Notice that the **characteristic function** of P with respect to the state $\langle \Phi_l, \cdot \Phi_l \rangle$ on $\mathcal{B}(\Gamma\Big(\mathbb{C}, \{\omega_n\}_{n\geq 1}))$ is

$$\langle \Phi_l, e^{itP} \Phi_l \rangle = I_{l,l}(t) = \sum_{m=0}^{l} I_{m,m}(t) \qquad (68)$$

In particular, the vacuum expectations of e^{itP} is given by

$$\langle \Phi_0, e^{itP} \Phi_0 \rangle = I_{0,0}(t) = \sum_{p=0}^{\infty} \frac{(-t^2)^p}{(2p)!} C_p = \frac{J_1(2t)}{t} \qquad (69)$$

Formula (69) was known in the literature (the characteristic function of the semi–circle law). Formula (68) **seems to be new**.
At the moment we do not know the explicit form of the probability measure corresponding to this characteristic function.

Lemma 5 *We have*

$$\sum_{m\geq 1}(-1)^m J_m(2t)\sin(m\theta)$$

$$= -\frac{\sin(2t\sin\theta)}{2} - \frac{\sin(\theta)}{2\pi}\int_0^\pi \frac{\cos(2t\sin\varphi)}{\cos(\theta)-\cos(\varphi)}d\varphi \qquad (70)$$

and

$$\sum_{m\geq 1}(-1)^m J_m(2t)\cos(m\theta)$$

$$= \sum_{m\geq 1}(-1)^m J_m(2t)\cos(m\theta) = \frac{\cos(2t\sin\theta) - J_0(2t)}{2}$$

Proposition 3 *For any $t \in \mathbb{C}$, one has*

$$e^{itX}\Phi_0(x) = e^{itx} - ixJ_1(2t) \qquad (71)$$

and for any $k \geq 1$,

$$e^{itX}\Phi_k(x) = J_0(2t)\Phi_k(x) + x\sum_{n=1}^{k} i^n J_n(2t)\phi_{k-n+1}(x) + \qquad (72)$$

Remark 2 *Since the vacuum distributions of X and P coincide with the semi-circle distribution μ, the vacuum expectations of e^{itP} and e^{itX} coincide with the* ***the characteristic function*** *of the measure μ which can be computed directly by expending e^{ixt} into a power series then using the fact that odd-moments are zero and even-moments are the Catalan numbers interchanging integral and sum, one obtains:*

$$\varphi_\mu(t) = \sum_{n \geq 0} \frac{(it)^n}{n!} \int_{\mathbb{R}} x^n d\mu(x)$$

$$= \sum_{n\geq 0} \frac{(-1)^n}{n+1}\binom{2n}{n}\frac{t^{2n}}{(2n)!} = \frac{J_1(2t)}{t} \qquad (73)$$

The 1–parameter $*$–automorphism groups associated to P

In this section, we determine the action of $e^{itP}(\ \cdot\ )e^{-itP}$ on the algebra generated by creation and annihilation operators. For this it is sufficient to determine this action on a^+.

Proposition 4 One has

$$\partial\omega_{\Lambda,t} := e^{itP}\partial\omega_\Lambda e^{-itP} = (e^{itP}\Phi_0)(e^{itP}\Phi_0)^* \qquad (74)$$

$$a_t^+ := e^{itP}a^+e^{-itP} = a^+ + \omega\int_0^t ds(e^{isP}\Phi_0)(e^{isP}\Phi_0)^* \qquad (75)$$

$$X_t := e^{itP}Xe^{-itP} = X + 2\omega\int_0^t ds(e^{isP}\Phi_0)(e^{isP}\Phi_0)^* \qquad (76)$$

Remark. From formula (76) one deduced an elegant extension of the action of the usual quantum mechanical formula

$$e^{itP}Xe^{-itP} = X + t$$

The main difference is that **translation by a number is here replaced by translation by translation by an operator**.

In order to give a more explicit form to (75), one needs the following estimates.

Lemma 6 *The following inequalities hold*

$$|I_{0,n}(s)| \leq \frac{|s|^n}{n!} e^{|s|^2} \left(1 + \frac{|s|^2}{2}\right) \qquad (77)$$

and

$$|I_{m,n}(s)| \leq \frac{|s|^{m+n}}{m!n!} e^{|s|^2} \left(1 + \frac{|s|^2}{2}\right) \qquad (78)$$

Theorem 4 One has, for each $t \in \mathbb{R}$,

$$a_t^+ = a^+ + \omega \sum_{m,n,p,q \geq 0} \frac{(-1)^{m+n+p+q}(m+1)(n+1)}{p!q!(p+m+1)!(q+n+1)!} \frac{\quad}{m \cdots} \tag{79}$$

Remark 3 Using the expression (4), one can also rewrite (79) as

$$a_t^+ = a^+ + \omega \sum_{m,n \geq 0} (-1)^{m+n}(m+1)(n+1) \cdot$$

$$\cdot \int_0^t \frac{ds}{s^2} J_{m+1}(2s) J_{n+1}(2s) \Phi_m \Phi_n^* \tag{80}$$

Evolutions associated to the kinetic energy operator

Proposition 5 If $\omega_n = 1$ for any $n \geq 1$, for any t,

$$e^{itP^2} = \sum_{m,n\geq 0} I^{(2)}_{m,n}(t)(a^+)^m a^n \tag{81}$$

where for any $m, n \in \mathbb{N}$,

$$I^{(2)}_{m,n}(t) = \chi_{2\mathbb{N}}\,(m+n)\,(-1)^{\frac{3m+n}{2}} \cdot$$

$$\cdot \sum_{p\geq 0} \frac{(it)^{\frac{m+n}{2}+p}}{\left(\frac{m+n}{2}+p\right)!} |\Theta_{m+n+2p}(m,n)| \tag{82}$$

Proposition 6 For any $m, n \in \mathbb{N}$,

$$I^{(2)}_{m,n}(t) = (-1)^m I^{(2)}_{0,m+n}(t) = (-1)^n I^{(2)}_{m+n,0}(t) \tag{83}$$

with

$$I^{(2)}_{0,n}(t) = \chi_{2\mathbb{N}}(n) \frac{(-it)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!} {}_1F_1\left(\frac{n+1}{2}; n+2; 4it\right) \tag{84}$$

where ${}_1F_1$ is the confluent hypergeometric function

$${}_1F_1(a,b,z) \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a;b;z)$$

where:

$$a^{(0)} = 1 \quad , \quad a^{(n)} = a(a+1)(a+2)\cdots(a+n-1)$$

The $1-$parameter $*-$automorphism groups associated to P^2

With the notation $\partial\omega_\Lambda := \omega_{\Lambda+1} - \omega_\Lambda$, we now study the Heisenberg evolutions:

$$\partial\omega_{\Lambda,t} := e^{itP^2}\partial\omega_\Lambda e^{-itP^2} = (e^{itP^2}\Phi_0)(e^{itP^2}\Phi_0)^* \tag{85}$$

$$a_t^+ := e^{itP^2}a^+e^{-itP^2}$$

Recall that, once one knows a_t^+ and $\partial\omega_{\Lambda,t}$, one has the action of the Heisenberg evolution **on the whole reduced quantum algebra of** X.

Theorem 5 One has

$$a_t^+ = a^+ + \tag{86}$$

$$\sum_{m,p,n,q \geq 0} \frac{(-1)^{m+q} |\Theta_{2m+2p}(2m,0)| |\Theta_{2n+2q}(2n,0)|}{(m+p)!\,(n+q)!}$$

$$\frac{(it)^{m+n+p+q+1}}{m+n+p+q+1} \left(T_{2m-1} \Phi_{2n}^* - \Phi_{2n} T_{2m-1}^* \right)$$

where

$$|\Theta_{2m+2p}(2n,0)| = \frac{(2m+1)(2m+2p)!}{p!(2m+p+1)!}$$

and

$$T_n(x) = 2\cos(n \arccos(x/2)) \tag{87}$$

are the monic Chebyshev polynomial of first kind T_n, related with the Φ_n through the identities

$$T_{n+1}(x) = \Phi_{n+1}(x) - \Phi_{n-1}(x) \tag{88}$$

Theorem 6 One has

$$a_t^+ = a^+ + \tag{89}$$

$$\sum_{m,p,n,q\geq 0} \frac{(-1)^{m+q}|\Theta_{2m+2p}(2m,0)||\Theta_{2n+2q}(2n,0)|}{(m+p)!\,(n+q)!}$$

$$\frac{(it)^{m+n+p+q+1}}{m+n+p+q+1}\left(T_{2m-1}\Phi_{2n}^* - \Phi_{2n}T_{2m-1}^*\right)$$

where

$$|\Theta_{2m+2p}(2n,0)| = \frac{(2m+1)(2m+2p)!}{p!(2m+p+1)!}$$

are again the monic Chebyshev polynomial of first kind T_n

Abstract It is now known that each classical random variable has a canonical quantum decomposition in terms of creation, annihilation and preservation (CAP) operators satisfying commutation relations uniquely determined by their Jacobi coefficients or their multi–dimensional extensions.
Symmetric classical random variables also have a canonically conjugated moment and the two are intertwined by a generalization, to arbitrary random variables, of the Gauss–Fourier transform.
Thus every classical random variable determines its own quantum mechanics. Usual QM corresponds to the Gaussian–Poisson class.
Quadratic QM corresponds to the 3 non–standard classes of Meixner measures.
According to the information complexity index for probability measures on $\mathbb{R}$, the semi–circle–arcsine class has a lower complexity index than

the above mentioned 5 classes. Thus it is interesting to investigate how does the QM associated to these probability measures look like. I will discuss the solution of this problem in the case of the semi–circle law. It turns out that in this case the canonically conjugate moment is given by the Hilbert transform with respect to the semi–circle measure. This allows to express the momentum evolution and the free evolution (generated by kinetic energy) in terms respectively of Bessel functions and of confluent hypergeometric series.