Asymptotic transport properties in open quantum spin chains

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Work with R. Floreanini and L. Memarzadeh

Outline I

1. **Open quantum spin chains**

2. **Master equation in the global approach**
   - Global Hamiltonian diagonalization
   - Global approach: master equation

3. **Stationary state**
   - Stationary Transport Properties
   - Bipartite entanglement

Open quantum systems: weak coupling limit

- Open quantum systems:

$$H = H_S + H_E + \lambda H_{int}$$

- Weak-coupling limit:

$$t \mapsto \tau := \lambda^2 t , \qquad \lambda \mapsto 0 \quad \& \quad t \mapsto +\infty$$

- Elimination of fast oscillating terms:

$$\exp\Big(i\, t\, (E_j - E_i) - (E_p - E_q)\Big), \quad E_\ell \quad \text{eigenvalues of} \quad H_S$$

Open quantum spin chains: nearest neighbour interactions

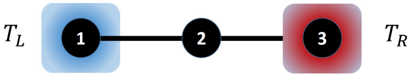

Figure: Open 3-spin chain

Global vs Local approach

$H_S = H_1 + H_2 + H_3 + g\Big(H_{12} + H_{23}\Big)$: if $g \ll 1$

- Local approach: E_ℓ eigenvalues of $H_{1,3}$ or
- Global approach: E_ℓ eigenvalues of H_S ?

Local approach: some literature

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Stationary transport properties: 3-qubit chain

F.B., R. Floreanini, L. Memarzadeh, PRA 102 (2020)

- Global approach: analytic stationary state
- Spin flux continuity equation: sinks and sources
- Local approach: stationary state up to first order perturbation in g: no sinks and sources
- The local stationary state does not emerge from the global stationary state by sending $g \to 0$

Spin chains of length N: nearest neighbour XX interactions

- XX Hamiltonian:

$$H = g \sum_{\ell=1}^{N-1} \left(\sigma_x^{(\ell)} \sigma_x^{(\ell+1)} + \sigma_y^{(\ell)} \sigma_y^{(\ell+1)} \right) + \Delta \sum_{\ell=1}^{N} \sigma_z^{(\ell)}$$

- $g > 0$: interspin coupling
- $\Delta > 0$: transverse constant magnetic field

Open quantum spin chains

- Left ($\ell = 1$) and right spin ($\ell = N$) coupled to independent, free Bosonic thermal baths
- Bath Hamiltonians: $\alpha = L, R$,

$$
\begin{aligned}
H_\alpha &= \int_0^{+\infty} \mathrm{d}\nu\, \nu\, \mathfrak{b}_\alpha^\dagger(\nu)\, \mathfrak{b}_\alpha(\nu) \\
\left[\mathfrak{b}_\alpha(\nu),\, \mathfrak{b}_\beta^\dagger(\nu')\right] &= \delta_{\alpha\beta}\, \delta(\nu - \nu')
\end{aligned}
$$

- Interaction Hamiltonian: $\lambda << 1$ dimensionless coupling constant

$$
\begin{aligned}
H' &= \lambda \sum_{\alpha=L,R} \left(\sigma_+^{(\alpha)} \mathfrak{B}_\alpha + \sigma_-^{(\alpha)} \mathfrak{B}_\alpha^\dagger\right) , \quad \sigma_\pm^{(\ell)} \equiv \frac{1}{2}\left(\sigma_x^{(\ell)} \pm i\sigma_y^{(\ell)}\right) \\
\mathfrak{B}_\alpha &= \int_0^\infty \mathrm{d}\nu\, h_\alpha(\nu)\, \mathfrak{b}_\alpha(\nu) , \quad [h_\alpha(\nu)]^* = h_\alpha(\nu) ,
\end{aligned}
$$

Thermal baths

- Bath Gibbs states at inverse temperatures $\beta_{L,R}$:

$$\rho_{env} = \frac{\mathrm{e}^{-\beta_L H_L}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta_L H_L}\right)} \otimes \frac{\mathrm{e}^{-\beta_R H_R}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta_R H_R}\right)}$$

- Thermal expectations:

$$\mathrm{Tr}_B\left(\rho_{env}\, b^\dagger_\alpha(\nu) b_{\alpha'}(\nu')\right) = \delta_{\alpha\alpha'}\delta(\nu-\nu')\, n_\alpha(\nu)$$

$$\mathrm{Tr}_B\left(\rho_{env}\, b_\alpha(\nu) b^\dagger_{\alpha'}(\nu')\right) = \delta_{\alpha\alpha'}\delta(\nu-\nu')\,(1+n_\alpha(\nu))$$

$$n_\alpha(\nu) = \frac{1}{\mathrm{e}^{\beta_\alpha \nu} - 1}\,, \qquad \nu \geq 0\,.$$

Weak-Coupling Limit

- Initial state: $\rho_{\mathrm{tot}}(0) = \rho(0) \otimes \rho_{env}$
- Weak-coupling limit conditions: $\lambda \|\mathfrak{B}_\alpha\| \ll \|H\|$
- Kraus operators:

$$A^\dagger_\alpha(\omega) = \sum_{E_i - E_j = \omega} |E_i\rangle\langle E_i| \, \sigma_+^{(\alpha)} \, |E_j\rangle\langle E_j|$$

- Global approach:

$$H|E_j\rangle = E_j \, |E_j\rangle \ , \quad H = g \sum_{\ell=1}^{N-1} \left(\sigma_x^{(\ell)} \sigma_x^{(\ell+1)} + \sigma_y^{(\ell)} \sigma_y^{(\ell+1)} \right) + \Delta \sum_{\ell=1}^{N} \sigma_z^{(\ell)}$$

- Local approach: $H_{loc} = \Delta \sum_{\ell=1}^{N} \sigma_z^{(\ell)}$.

Global Hamiltonian diagonalization

$$H = \Delta \sum_{\ell=1}^{N} \sigma_z^{(\ell)} + 2g \sum_{\ell=1}^{N-1} \left(\sigma_+^{(\ell)} \sigma_-^{(\ell+1)} + \sigma_-^{(\ell)} \sigma_+^{(\ell+1)} \right)$$

- Jordan-Wigner fermionization:

$$a_j := \prod_{k=1}^{j-1} (-\sigma_z^{(k)})\, \sigma_-^{(j)} , \quad a_j^\dagger = \prod_{k=1}^{j-1} (-\sigma_z^{(k)})\, \sigma_+^{(j)} , \quad \left\{ a_j \, , \, a_k^\dagger \right\} = \delta_{jk}$$

$$H = -N\Delta + 2g\, \widetilde{H}$$

$$\widetilde{H} = \gamma \sum_{j=1}^{N} a_j^\dagger a_j + \sum_{j=1}^{N-1} \left(a_j^\dagger\, a_{j+1} + a_{j+1}^\dagger a_j \right) , \; \gamma := \frac{\Delta}{g}$$

Global approach: global Hamiltonian diagonalization

$$\widetilde{H} = \gamma \sum_{j=1}^{N} a_j^\dagger a_j + \sum_{j=1}^{N-1} \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right)$$

- Bogoljubov transformation:

$$b_\ell := \sum_{j=1}^{N} u_{\ell j}\, a_j \,, \quad b_\ell^\dagger := \sum_{j=1}^{N} u_{\ell j}\, a_j^\dagger \,, \quad u_{\ell k} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\ell k \pi}{N+1}\right)$$

- Diagonal Fermionic Hamiltonian:

$$H = -N\,\Delta + \sum_{\ell=1}^{N} \left(2\,\Delta + 4g \cos\left(\frac{\ell\pi}{N+1}\right)\right) b_\ell^\dagger\, b_\ell$$

Global Hamiltonian: eigenvalues and eigenvectors

$$H = -N\,\Delta + \sum_{\ell=1}^{N}\left(2\,\Delta + 4g\cos\left(\frac{\ell\pi}{N+1}\right)\right) b_\ell^\dagger\, b_\ell$$

- Eigenvectors: $b_\ell^\dagger\, b_\ell|\mathbf{n}\rangle = n_\ell\,|\mathbf{n}\rangle$

$$|\mathbf{n}\rangle = (b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots(b_N^\dagger)^{n_N}\,|vac\rangle\ ,\quad \mathbf{n} = n_1, n_2, \cdots, n_N\ ,\ n_\ell = 0, 1$$

$$b_\ell|\mathbf{n}\rangle = (-1)^{\sum_{j=1}^{\ell-1} n_j}\sqrt{n_\ell}\,|\mathbf{n}_\ell^-\rangle\ ,\quad \mathbf{n}_\ell^- = n_1, \cdots, n_\ell - 1, \cdots n_N$$

$$b_\ell^\dagger|\mathbf{n}\rangle = (-1)^{\sum_{j=1}^{\ell-1} n_j}\sqrt{1 - n_\ell}\,|\mathbf{n}_\ell^+\rangle\ ,\quad \mathbf{n}_\ell^+ = n_1, \cdots, n_\ell + 1, \cdots n_N$$

- Eigenvalues: $H\,|\mathbf{n}\rangle = E_{\mathbf{n}}\,|\mathbf{n}\rangle$,

$$E_{\mathbf{n}} = \Delta\left(2\sum_{\ell=1}^{N} n_\ell\ -\ N\right) + 4g\sum_{\ell=1}^{N} n_\ell\ \cos\left(\frac{\ell\pi}{N+1}\right)$$

GKSL Master equation

$$\frac{\partial\rho(t)}{\partial t} = -i\left[H + \lambda^2 H_{LS}\,,\ \rho(t)\right] + \mathbb{D}[\rho(t)] = \mathbb{L}[\rho(t)]$$

- Lamb-shift correction: **all transition frequencies**

$$H_{LS} = \sum_{\alpha=L,R}\sum_{\omega}\left[S^{(\alpha)}_\omega A^\dagger_\alpha(\omega)A_\alpha(\omega) + \widetilde{S}^{(\alpha)}_\omega A_\alpha(\omega)A^\dagger_\alpha(\omega)\right]$$

$$S^{(\alpha)}_\omega = P\int_0^{+\infty} \mathrm{d}\nu\,[h_\alpha(\nu)]^2\,\frac{1+n_\alpha(\nu)}{\omega-\nu}$$

$$\widetilde{S}^{(\alpha)}_\omega = P\int_0^{+\infty} \mathrm{d}\nu\,[h_\alpha(\nu)]^2\,\frac{n_\alpha(\nu)}{\nu-\omega}$$

GKSL Master equation

$$\frac{\partial\rho(t)}{\partial t} = -i\big[H + \lambda^2\, H_{LS}\,,\, \rho(t)\big] + \mathbb{D}[\rho(t)] = \mathbb{L}[\rho(t)]$$

- Dissipator: **positive transition frequencies**

$$\mathbb{D}[\rho(t)] = \lambda^2 \sum_{\alpha=L,R} \sum_{\boldsymbol{\omega\geq 0}} \mathbb{D}^{(\alpha)}_{\omega}[\rho(t)]$$

GKSL Master equation

$$\frac{\partial \rho(t)}{\partial t} = -i\left[H + \lambda^2 H_{LS}\, ,\, \rho(t)\right] + \mathbb{D}[\rho(t)] = \mathbb{L}[\rho(t)]$$

- Dissipator: **positive transition frequencies**

$$\mathbb{D}[\rho(t)] = \lambda^2 \sum_{\alpha=L,R} \sum_{\omega \geq 0} \mathbb{D}_{\omega}^{(\alpha)}[\rho(t)]$$

Why $\omega \geq \mathbf{0}$?

GKSL Master equation

$$\frac{\partial\rho(t)}{\partial t} = -i\big[H + \lambda^2\, H_{LS}\,,\, \rho(t)\big] + \mathbb{D}[\rho(t)] = \mathbb{L}[\rho(t)]$$

- Dissipator: **positive transition frequencies**

$$\mathbb{D}[\rho(t)] = \lambda^2 \sum_{\alpha=L,R} \sum_{\boldsymbol{\omega\geq 0}} \mathbb{D}^{(\alpha)}_{\omega}[\rho(t)]$$

Why $\omega \geq \mathbf{0}$? Bath energies $\boldsymbol{\nu} \geq \mathbf{0}$; interaction terms $\boldsymbol{\sigma_+\, \mathfrak{b}_\alpha(\nu)}$

GKSL Master equation

$$\frac{\partial\rho(t)}{\partial t} = -i\big[H + \lambda^2\, H_{LS}\,,\, \rho(t)\big] + \mathbb{D}[\rho(t)] = \mathbb{L}[\rho(t)]$$

- Dissipator: $\mathbb{D}[\rho(t)] = \lambda^2 \sum_{\alpha=L,R}\ \sum_{\boldsymbol{\omega}\geq\mathbf{0}} \mathbb{D}^{(\alpha)}_{\omega}[\rho(t)]$

$$\begin{aligned}\mathbb{D}^{(\alpha)}_{\omega}[\rho(t)] &= C^{(\alpha)}_{\omega}\left[A_\alpha(\omega)\rho(t)A^\dagger_\alpha(\omega) - \frac{1}{2}\left\{A^\dagger_\alpha(\omega)A_\alpha(\omega),\rho(t)\right\}\right] \\ &+ \widetilde{C}^{(\alpha)}_{\omega}\left[A^\dagger_\alpha(\omega)\rho(t)A_\alpha(\omega) - \frac{1}{2}\left\{A_\alpha(\omega)A^\dagger_\alpha(\omega),\rho(t)\right\}\right]\end{aligned}$$

$$C^{(\alpha)}_{\omega} = 2\pi\,[h_\alpha(\omega)]^2\,\big(n_\alpha(\omega)+1\big)\ ,\qquad \widetilde{C}^{(\alpha)}_{\omega} = 2\pi\,[h_\alpha(\omega)]^2\, n_\alpha(\omega)$$

Kraus operators: $A_\alpha(\omega)$

- Ladder operators in the Fermionic representation:

$$\sigma_+^{(L)} = \sum_{\ell=1}^{N} u_{1\ell}\, b_\ell^\dagger\ , \quad \sigma_+^{(R)} = -\left(\mathrm{e}^{i\,\pi\,\sum_{\ell=1}^{N}\, b_\ell^\dagger\, b_\ell}\right) \sum_{\ell=1}^{N} u_{N\ell}\, b_\ell^\dagger$$

- Non-vanishing transition amplitudes:

$$\langle \mathbf{n}_{1_\ell} | \sigma_+^{(L)} | \mathbf{n}_{0_\ell} \rangle = (-1)^{\sum_{j=1}^{\ell-1} n_j}\, \sqrt{1-n_\ell}\, u_{1\ell}$$
$$\langle \mathbf{n}_{1_\ell} | \sigma_+^{(R)} | \mathbf{n}_{0_\ell} \rangle = (-1)^{\sum_{j=1}^{\ell+1} n_j}\, \sqrt{1-n_\ell}\, u_{N\ell}$$
$$\mathbf{n}_{1_\ell} = n_1, \cdots, n_\ell = 1, \cdots, n_N\ , \quad \mathbf{n}_{0_\ell} = n_1, \cdots, n_\ell = 0, \cdots, n_N$$

Kraus operators $A_\alpha(\omega)$

- Transition frequencies contributing to the dissipator:

$$\omega_\ell = E_{\mathbf{n}_{1_\ell}} - E_{\mathbf{n}_{0_\ell}} = 2\,\Delta + 4\,g\,\cos\left(\frac{\ell\pi}{N+1}\right)$$

- Kraus operators:

$$A_L^\dagger(\omega_\ell) = u_{1\ell}\sum_{\widehat{\mathbf{n}}_\ell}(-1)^{\sum_{j=1}^{\ell-1} n_j}\,|\mathbf{n}_{1_\ell}\rangle\langle\mathbf{n}_{0_\ell}|$$

$$A_R^\dagger(\omega_\ell) = u_{N\ell}\sum_{\widehat{\mathbf{n}}_\ell}(-1)^{\sum_{j=\ell+1}^{N} n_j}\,|\mathbf{n}_{1_\ell}\rangle\langle\mathbf{n}_{0_\ell}|$$

$\widehat{\mathbf{n}}_\ell$ binary n-tuples with n_ℓ fixed, $u_{\ell k} = \sqrt{\frac{2}{N+1}}\sin\left(\frac{\ell k\pi}{N+1}\right)$

Observations

- $A_\alpha(\omega_\ell)$ contribute to the dissipator only if

$$\omega_\ell = E_{\mathbf{n}_{1_\ell}} - E_{\mathbf{n}_{0_\ell}} = 2\,\Delta + 4\,g\,\cos\left(\frac{\ell\pi}{N+1}\right) \geq 0$$

- The sign of ω_ℓ depends on g and Δ:

$$\cos\left(\frac{\pi\ell}{N+1}\right) < 0 \quad \text{for} \quad N \geq \ell > \frac{N+1}{2}\ .$$

- Working assumption:

$$g \leq g^* := \frac{\Delta}{2\,\cos\left(\frac{\pi}{N+1}\right)} \Longrightarrow \omega_\ell \geq 0\ , \quad \ell = 1, 2, \ldots, N$$

Stationary state

Unique: the commutant of $\{A^\dagger_\alpha(\omega)\,,\,A_\alpha(\omega)\}_{\omega,\alpha}$ is trivial

- Diagonal Hamiltonian: $H+\lambda^2 H_{LS}=\sum_{\mathbf{n}}\widetilde{E}_{\mathbf{n}}\,|\mathbf{n}\rangle\langle\mathbf{n}|$
- Set $\rho_\infty=\sum_{\mathbf{n}}\Lambda_{\mathbf{n}}\,|\mathbf{n}\rangle\langle\mathbf{n}$ and ask $\mathbb{L}[\rho_\infty]=0$
- Solution: $\Lambda_{\mathbf{n}}=\prod_{\ell=1}^{N}\lambda^{(\ell)}_{n_\ell}, \quad \lambda^{(\ell)}_{n_\ell}=\frac{R^{(\ell)}_{n_\ell}}{R_\ell},$

$$R^{(\ell)}_{n_\ell}:=[h_L(\omega_\ell)]^2\Big(1-n_\ell+n_L(\omega_\ell)\Big)+[h_R(\omega_\ell)]^2\Big(1-n_\ell+n_R(\omega_\ell)\Big)$$
$$R_\ell:=[h_L(\omega_\ell)]^2\Big(1+2n_L(\omega_\ell)\Big)+[h_R(\omega_\ell)]^2\Big(1+2n_R(\omega_\ell)\Big)$$

Stationary state

- Identical baths: $h_{L,R}(\omega_\ell) = h$, $\beta_L = \beta_R = \beta$,

$$\lambda^{(\ell)}_{n_\ell} = \frac{e^{\beta(1-n_\ell)\omega_\ell}}{e^{\beta\omega_\ell}+1}$$

$$\rho_\infty = \sum_{\mathbf{n}} \prod_{\ell=1}^{N} \frac{e^{\beta(1-n_\ell)\omega_\ell}}{e^{\beta\omega_\ell}+1} |\mathbf{n}\rangle\langle\mathbf{n}| = \frac{e^{-\beta H}}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

- Even with equal temperatures stationary entanglement : no threshold temperature difference

S. Khandelwal et al., NJP **22** 2020

Tansport properties

- Time-dependence of averages of chain observables

$$\frac{d}{dt}\mathrm{Tr}\big[X\rho(t)\big] = \mathrm{Tr}\big[\mathbb{L}[\rho(t)]\,X\big] = \mathrm{Tr}\big[\widetilde{\mathbb{L}}[X]\,\rho(t)\big]$$

- Dual generator:

$$\widetilde{\mathbb{L}}[X] = i\big[H + \lambda^2 H_{LS}\,,\,X\big] + \widetilde{\mathbb{D}}[X]$$

$$\widetilde{\mathbb{D}}[X] = \lambda^2 \sum_{\alpha=L,R}\sum_{\omega_\ell\geq 0}^{N} \widetilde{\mathbb{D}}^{(\alpha)}_{\omega_\ell}[X]$$

$$\widetilde{\mathbb{D}}^{(\alpha)}_{\omega_\ell}[X] = C^{(\alpha)}_{\omega_\ell}\left[A^\dagger_\alpha(\omega_\ell)\,X\,A_\alpha(\omega_\ell) - \frac{1}{2}\left\{A^\dagger_\alpha(\omega_\ell)A_\alpha(\omega_\ell), X\right\}\right]$$
$$+\widetilde{C}^{(\alpha)}_{\omega_\ell}\left[A_\alpha(\omega_\ell)\,X\,A^\dagger_\alpha(\omega_\ell) - \frac{1}{2}\left\{A_\alpha(\omega_\ell)A^\dagger_\alpha(\omega_\ell), X\right\}\right]$$

Spin flow at site k: local X

$$X^{(k)} = \sigma_z^{(k)} , \qquad \frac{d}{dt}\mathrm{Tr}\left[\sigma^{(k)}\rho(t)\right]$$

- Lamb-shift Hamiltonian: current divergence,

$$i\left[H + \lambda^2 H_{LS}, \sigma_z^{(k)}\right] = (g + \kappa)\left(J^{(k-1,k)} - J^{(k,k+1)}\right)$$

$$J^{(k,k+1)} = 4i\left(\sigma_-^{(k)}\sigma_+^{(k+1)} - \sigma_+^{(k)}\sigma_-^{(k+1)}\right) = -4i\left(a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1}\right)$$

$$= -4i\sum_{j\,,\,\ell=1}^{N} u_{kj}u_{k+1\ell}(b_j b_\ell^\dagger + b_j^\dagger b_\ell)\ , \quad \kappa = \frac{i\lambda^2}{8\sqrt{2}}\sum_{\alpha=L,R}\sum_{\ell=1}^{N}\left(S_{\omega_\ell}^{(\alpha)} - \widetilde{S}_{\omega_\ell}^{(\alpha)}\right)$$

- Asymptotic current divergence: $\rho(t) = \rho_\infty$,

$$\langle J^{(k,k+1)}\rangle_\infty := \mathrm{Tr}\left(\rho_\infty\, J^{(k,k+1)}\right) = 0$$

Sinks and sources

$$\mathfrak{Q}_{\alpha}^{(k)} := \lambda^2 \sum_{\ell=1}^{N} \mathrm{Tr}\left(\rho_\infty\, \widetilde{\mathbb{D}}_{\omega_\ell}^{(\alpha)}[\sigma_z^{(k)}]\right) \,,$$

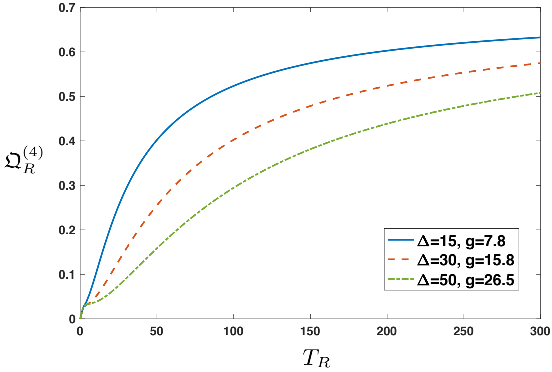
- Non-singly vanishing sink and source terms:

$$\begin{aligned} \mathfrak{Q}_{R}^{(k)} &= 2\pi\lambda^2 \sum_{\ell=1}^{N} u_{k\ell}^2\, u_{1\ell}^2\, [h_L(\omega_\ell)]^2\, [h_R(\omega_\ell)]^2\, \frac{n_R(\omega_\ell) - n_L(\omega_\ell)}{R_\ell} \\ &= -\mathfrak{Q}_{L}^{(k)} \simeq \frac{1}{N} \text{ when } N \to \infty \end{aligned}$$

- Stationarity: $\mathrm{Tr}\left(\rho_\infty\, \widetilde{\mathbb{D}}[\sigma^{(k)}]\right) = \mathfrak{Q}_{R}^{(k)} + \mathfrak{Q}_{L}^{(k)} = 0$
- Equal baths: $h_L(\omega) = h_R(\omega) = h$, $n_L(\omega_\ell) = n_R(\omega_\ell)$,

$$\mathfrak{Q}_{R}^{(k)} = \pi\lambda^2 \sum_{\ell=1}^{N} u_{k\ell}^2\, u_{1\ell}^2\, \frac{n_R(\omega_\ell) - n_L(\omega_\ell)}{1 + n_L(\omega_\ell) + n_R(\omega_\ell)} = 0$$

Source terms $\mathfrak{Q}_R^{(4)}$ vs T_R
$T_L = 0$, N=10, $\lambda = 1$, $\Delta = 15, 30, 50$, $g \simeq g^* = \frac{\Delta}{2\cos\left(\frac{\pi}{N+1}\right)}$



- Global approach: sinks and sources at $k \neq 1, N$ due to the non-local structure of Lindblad operators
- Sinks and sources decrease as $1/N$ due to $u_{k\ell}^2\, u_{1\ell}^2$
- Local approach: no sinks and sources for Kraus ops depend only on the leftmost and rightmost spins
- Global approach: sinks and sources when $g \to 0$ due to g in $n_{L,R}(\omega_\ell) \neq 0$

Heat Flow: non-local X

$$X = H\,,\qquad \mathfrak{H}(t) := \mathrm{Tr}\left(\frac{\mathrm{d}\rho(t)}{\mathrm{d}t}\,H\right) = \mathrm{Tr}\left(\mathbb{L}[\rho(t)]\,H\right)$$

- Stationary heat flow: $\rho(t) = \rho_\infty$, $\mathfrak{H}^{st}_L + \mathfrak{H}^{st}_R = 0$,

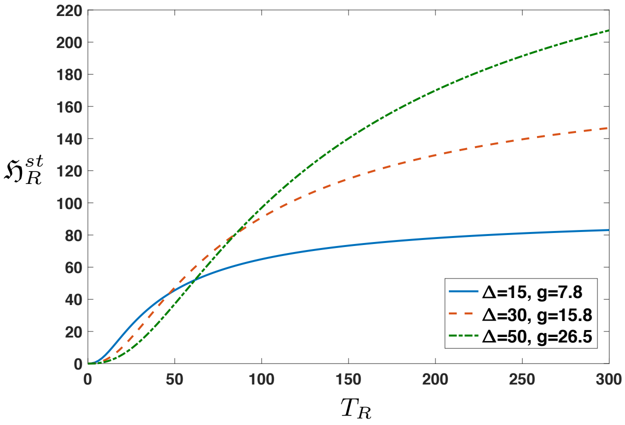
$$\mathfrak{H}^{st}_R = \sum_{\ell=1}^{N} \mathrm{Tr}\left(\mathbb{D}^{(R)}_{\omega_\ell}[\rho_\infty]\,H\right) \simeq 1 \text{ when } N \to \infty$$

$$= \lambda^2 \sum_{\ell=1}^{N} \omega_\ell\, u^2_{1\ell}\,[h_L(\omega_\ell)]^2[h_R(\omega_\ell)]^2\,\frac{n_R(\omega_\ell) - n_L(\omega_\ell)}{R_\ell}$$

- Equal baths: $h_L(\omega) = h_R(\omega) = h$, $n_L(\omega_\ell) = n_R(\omega_\ell)$,

$$\mathfrak{H}^{st}_R = \pi\,\lambda^2 \sum_{\ell=1}^{N} \omega_\ell\, u^2_{1\ell}\,\frac{n_R(\omega_\ell) - n_L(\omega_\ell)}{1 + n_L(\omega_\ell) + n_R(\omega_\ell)} = 0$$

Heat flow $\mathfrak{H}_R^{st}$ vs T_R: $T_L = 0$, $N = 8$, $\lambda = 1$, $\Delta = 15, 30, 50$, $g \simeq g^*$



Two-spin entanglement along the chain: concurrence

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$$

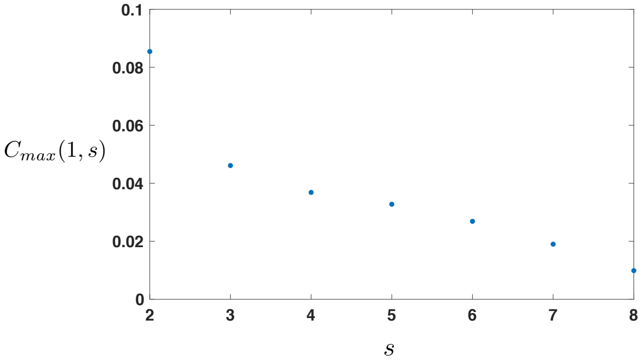
$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$: positive eigenvalues of $\rho\,(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$

- Structure of stationary two-spin reduced density matrices:

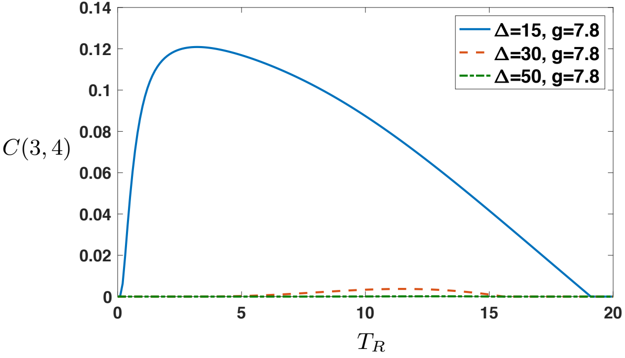
$$\rho_{(r,s)} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

- Concurrence: $C(r,s) = 2\,\max\left\{0\,,\,\left(|c| - \sqrt{a\,e}\right)\right\}$

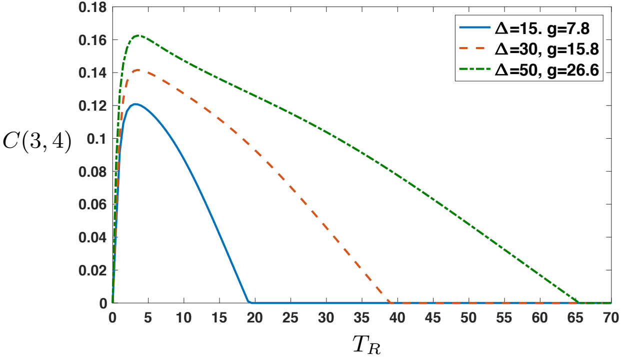
Maximum concurrence over T_R of spins 1 and $s = 2, 3, \cdots 8$ $N = 8, \lambda = 1, T_L = 0, \Delta = 15$ and $g = 7.8 \simeq g^*$



Concurrence of spins 3 and 4 vs T_R
$T_L = 0$, $N = 8$, $\Delta = 15, 30, 50$ and g close to g^* for $\Delta = 15$



Concurrence of spins 3 and 4 vs T_R
$T_L = 0$, $N = 8$, $\Delta = 15, 30, 50$ and g close to g^*



Conclusions

- N-spin chain with XX interactions
- End spins coupled to thermal Bosonic baths via energy preserving interactions
- Global approach: master equation
- Global approach: analytic stationary state
- Asymptotic transport: spin-flow sinks and sources
- Asymptotic transport: heat flow
- Bipartite concurrence as function of spin-distance, interaction strength and bath temperatures