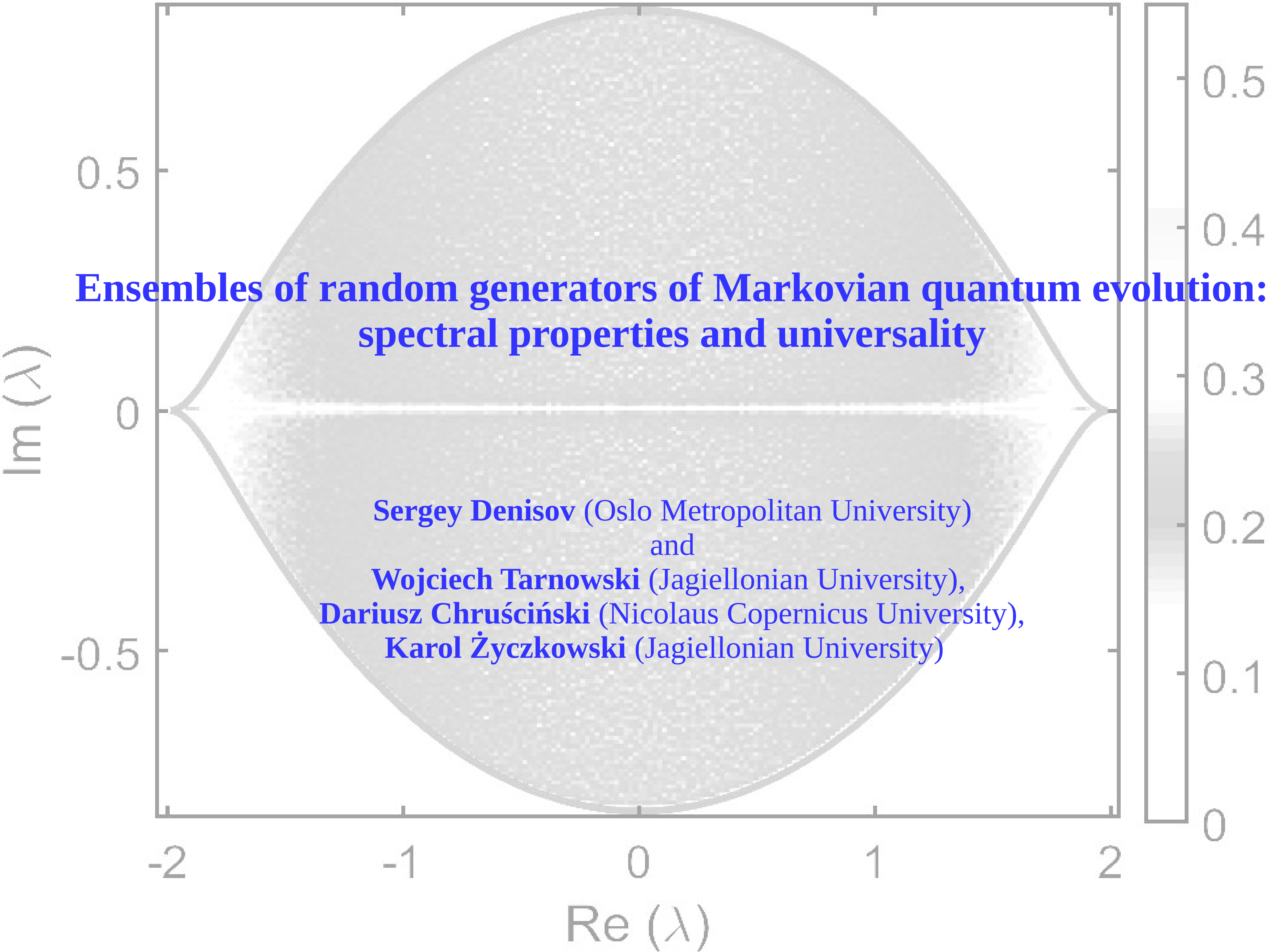

Ensembles of random generators of Markovian quantum evolution: spectral properties and universality

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and
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Dariusz Chruściński (Nicolaus Copernicus University),
Karol Życzkowski (Jagiellonian University)

Context

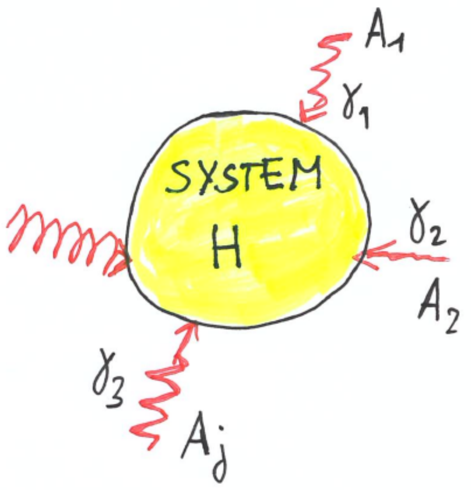
Gorini–Kossakowski–Sudarshan–Lindblad (GKS-L) equation

for a historical review see: *D. Chruściński and S. Pascazio, Open Sys. Inf. Dyn. 24, 1740001 (2017)*

$$\dot{\varrho} = \mathcal{L}(\rho) = \mathcal{L}_H \varrho + \mathcal{L}_d \varrho = -i[H, \varrho] + \sum_{m,n=1}^{N^2-1} k_{m,n}\left(F_n \rho F_m^\dagger - \frac{1}{2}\{F_m^\dagger F_n, \rho\}\right)$$

where *Kossakowski matrix* $K = \{k_{m,n}\}$ is positive, $K^\dagger = K \geq 0$

V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976)



$$\dot{\varrho} = -i[H, \varrho] + \sum_{j=1}^{N^2-1} \gamma_j \left(V_j \rho V_j^\dagger - \frac{1}{2}\{V_j^\dagger V_j, \rho\}\right)$$

G. Lindblad, Commun. Math. Phys. 48, 119 (1976)

Context

Lindblad evolution operator $\mathcal{L}$ is the only possible functional form of the generator with the following properties:

- propagator $\mathcal{R}_\tau = \exp(\mathcal{L}\tau)$ for any time τ is a positive map, i.e. it takes a positive operator into a positive operator, $\mathcal{R}_\tau A \to B$, $A, B > 0$;
- moreover, it is a *completely* positive: map $\mathbb{1}_M \otimes \mathcal{R}_\tau$ is positive for any M;
- propagator $\mathcal{R}_\tau$ preserves the trace; $\mathrm{Tr}[\mathcal{R}_\tau A] = \mathrm{Tr}[A]$;
- propagators for different times form a semi-group, $\mathcal{R}_{\tau+s} = \mathcal{R}_\tau \mathcal{R}_s$.

Any propagator is a completely positive and trace preserving (CPTP) map

Context

E. Kapit, P. Roushan, Ch. Neill, S. Boixo, and V. Smelyanskiy,
Entanglement and complexity of interacting qubits subject to asymmetric noise,
arxiv:1905.01792

$$\partial_t \rho = i\left[H\left(t\right), \rho\right] + \sum_{i=1}^{K} \left(O_i \rho O_i^\dagger + \frac{1}{2}\left\{ O_i^\dagger O_i, \rho \right\} \right)$$

- What are states to which the system evolves under the action of ‘randomly sampled’ generators? Are they non-trivial (with respect to some quantifiers)? Are they far from some trivial ‘classical’ states, f. e., the normalized identity?

- What is the speed of relaxation to the asymptotic state? Can we have a situation when the relaxation is fast and the corresponding asymptotic state is non-trivial & essentially quantum?

Dissipative Quantum Chaos: A notion of an ensemble of random operators of quantum Markovian evolution

Random GKS-L generators

$$\dot{\varrho} = \mathcal{L}(\rho) = \mathcal{L}_H\varrho + \mathcal{L}_d\varrho = -i[H,\varrho] + \sum_{m,n=1}^{N^2-1} k_{m,n}\Big(F_n\rho F_m^\dagger - \frac{1}{2}\{F_m^\dagger F_n,\rho\}\Big)$$

$$\mathcal{L}(\cdot) = -i(H\otimes\mathbb{1} + \mathbb{1}\otimes\overline{H}) + \\ + \sum_j \gamma_j V_j \otimes \overline{V} - \frac{1}{2}\sum_j \gamma_j\big(V_j^\dagger V_j\otimes\mathbb{1} + \mathbb{1}\otimes V_j^\intercal\overline{V}_j\big)$$

Since $\sum_j \gamma_j V_j \rho V_j^\dagger = \Phi(\rho)$ defines a map Φ, and $\sum_j \gamma_j V_j^\dagger V_j = \Phi^\dagger(\mathbb{I})$ where $\Phi^\dagger$ is the dual map, the **dissipative Lindblad generator** can be represented by an (auxiliary) discrete map Φ,

$$\mathcal{L}(\varrho) = -i[H,\varrho] + \Phi(\varrho) - \frac{1}{2}\Big(\Phi^\dagger(\mathbb{1})\varrho - \varrho\Phi^\dagger(\mathbb{1})\Big)$$

Random GKS-L generators: $H = 0$

$$\mathcal{L}(\rho) \;=\; \Phi(\rho) - \frac{1}{2}\Big(\Phi^\dagger(\mathbb{I})\rho + \rho\Phi^\dagger(\mathbb{I})\Big)$$

If Φ is trace preserving, than $\Phi^\dagger$ is **unital**, hence $\Phi^\dagger(\mathbb{I}) = \mathbb{I}$ which implies $\mathcal{L}(\rho) \;=\; \Phi(\rho) - \rho\,.$

In the general case the dual map $\Phi^\dagger$ is **not unital**,
so $\Phi^\dagger(\mathbb{I}) = \mathbb{I} + X$ where a Hermitian matrix $X = X^\dagger \neq 0$.

Lindblad operator associated with a map Φ

Then the generator reads

$$\mathcal{L} \;=\; \Phi - \mathbb{1}\otimes\mathbb{1} - \frac{1}{2}\Big(X\otimes\mathbb{1} + \mathbb{1}\otimes\bar{X}\Big) \qquad (**)$$

Recall: spectra of random CPTP maps

Random CPTP maps (or *channels*)

W. Bruzda, V. Cappellini, H.-J. Sommers, and K. Życzkowski (2009)

Spectral density in the unit disk

The spectrum of Φ consists of:

i) the **Frobenius-Perron** **leading eigenvalue** $z_1 = 1$,

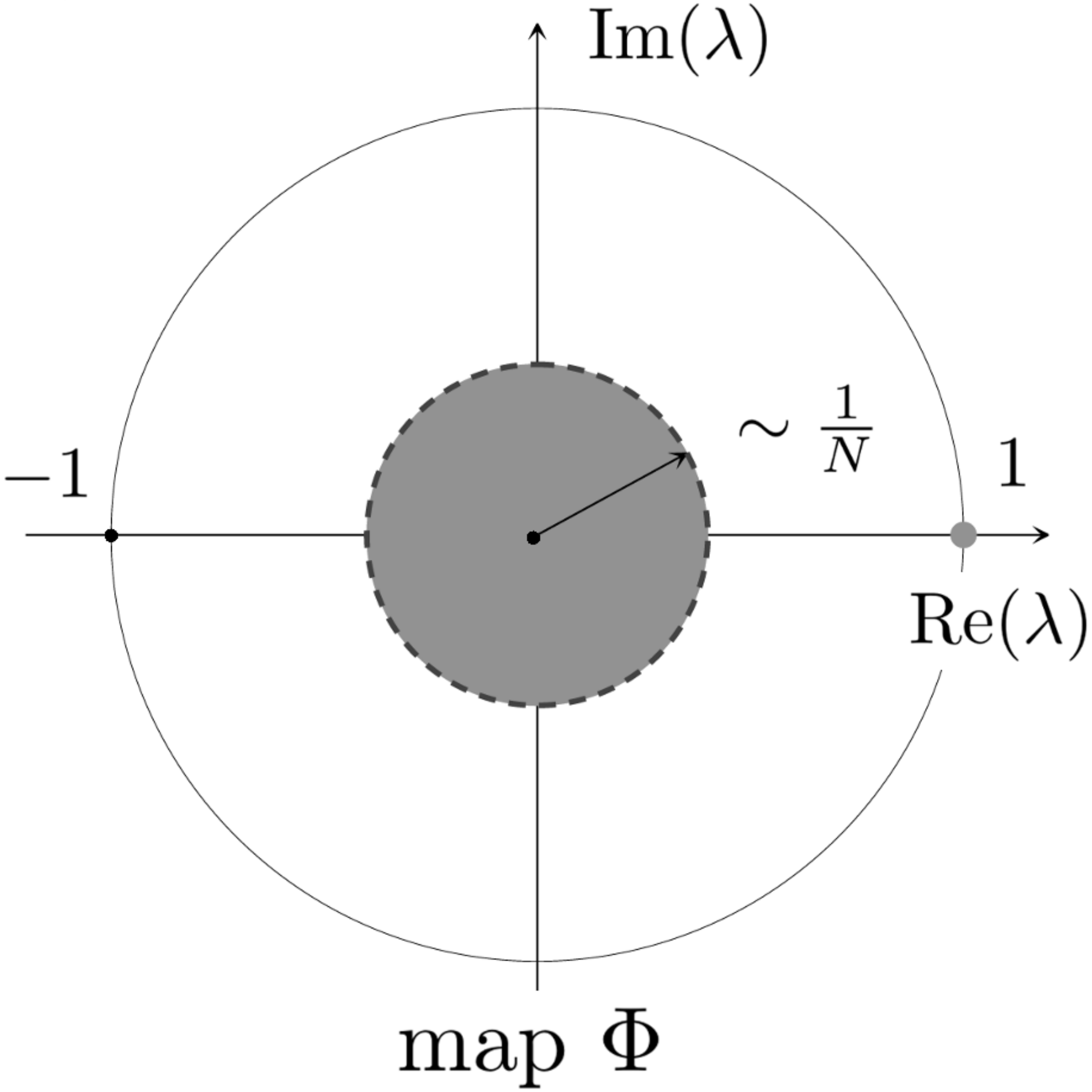
ii) the component at the real axis, the distribution of which is asymptotically given by the step function $P(x) = \frac{1}{2}\Theta(x-1)\Theta(1-x)$,

iii) complex eigenvalues, which cover the disk of radius $r = |z_2| \leq 1$ uniformly according to the **Girko distribution**.

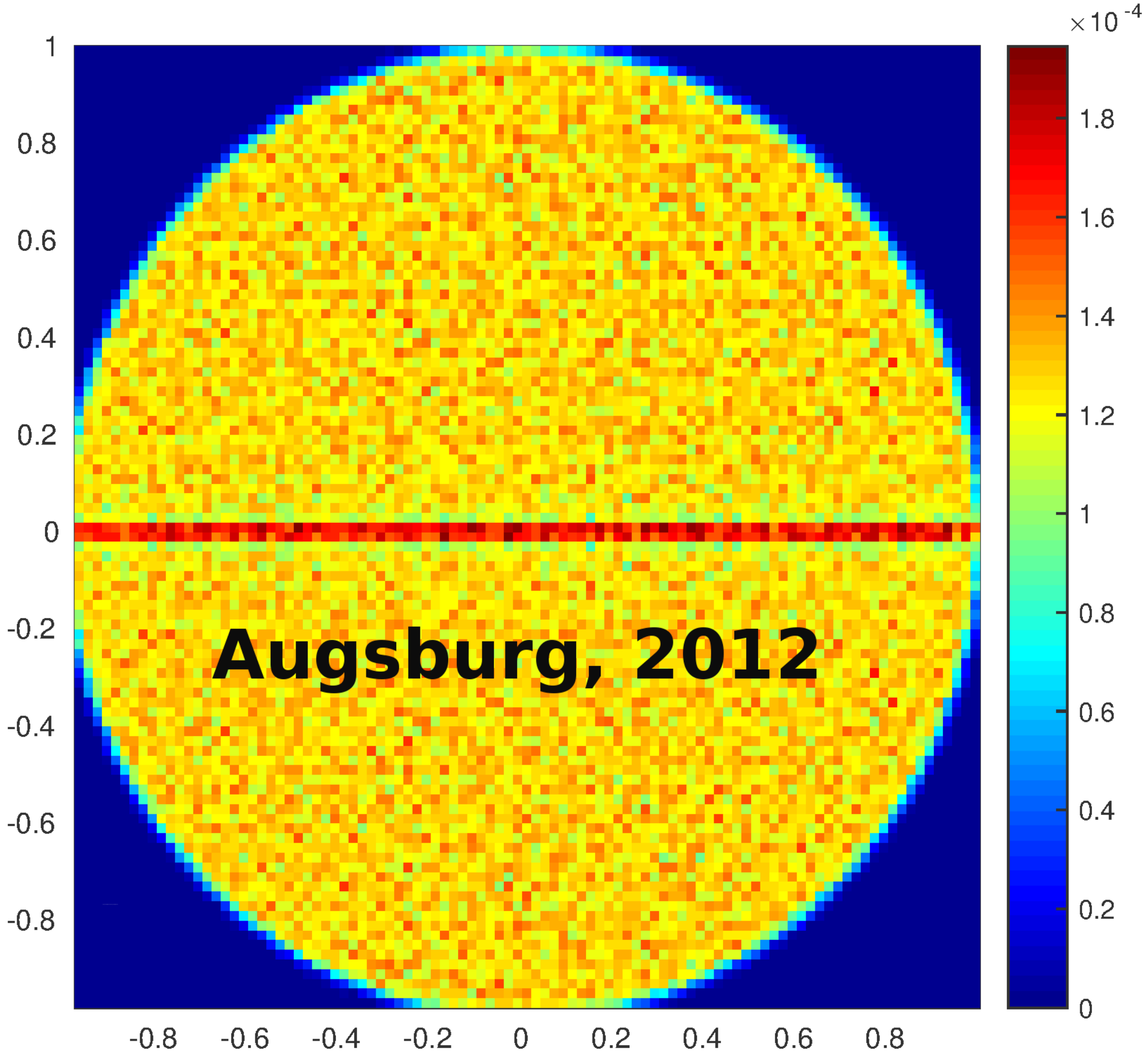
Subleading eigenvalue $r = |z_2|$ and **spectral gap** $1 - r$

The radius r is determined by the trace condition: Since the avearge $\langle \mathrm{Tr} D^2 \rangle = \langle \mathrm{Tr} \Phi\Phi^\dagger \rangle \approx \mathrm{const}$ then $r \sim 1/N$ so the spectrum of the rescaled matrix $\Phi' := N\Phi$ covers the entire unit disk.

Random CPTP maps (or *channels*)

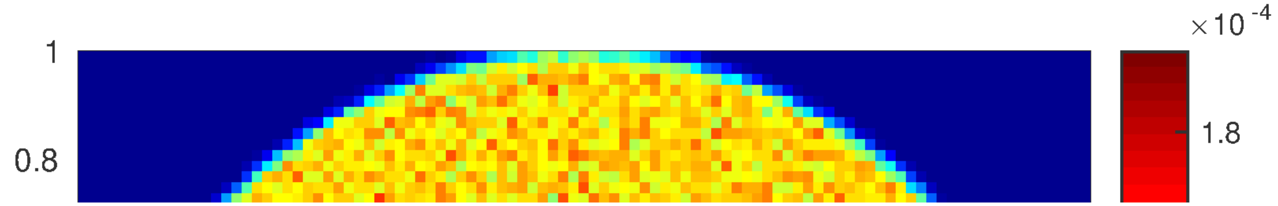


Random CPTP maps (or *channels*)



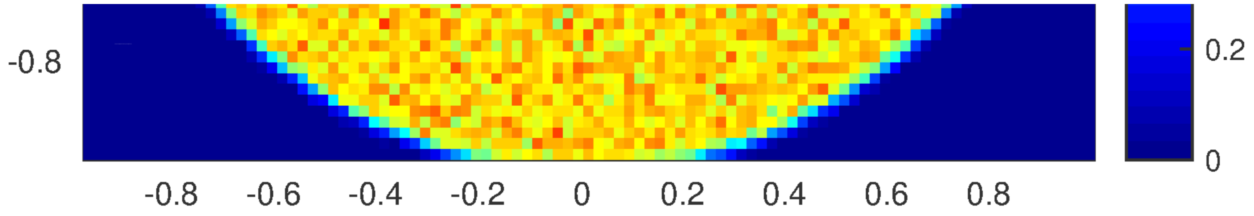
Augsburg, 2012

Random CPTP maps (or *channels*)



RESULTS OF THE SAMPLING

Results of the stochastic sampling confirm the key observation made in Ref. [1]: upon the increase N probability density functions (pdf's) of the eigenvalues approach the uniform distirbution over the unit circle, so-called *Girko distribution* [2].

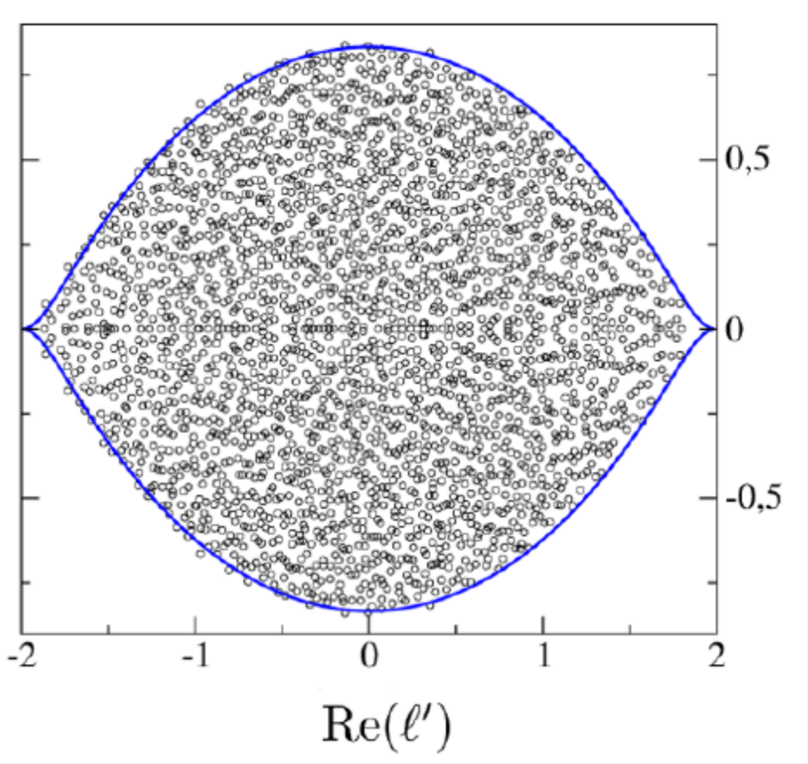


Random GKS-L generator

is defined by an assumption that $K = GG^\dagger$ is a **random Wishart matrix** which describes dissipative part of the dynamics.

(for simplicity we assume the Hamiltonian part vanishes, $H = 0$, and dicuss here only **disipative** part L of the generator.)

Spectrum of a random Lindblad operator



Numerical results for spectrum of the rescaled **Lindblad operator** $\mathcal{L}' = N(\mathcal{L} + 1)$

obtained for $N = 50$.

How to get the **blue** boundary?

RM-model of random GKS-L generators

Take the Lindblad operator associated with a map Φ

$$\mathcal{L} \;=\; \Phi - \mathbb{1}\otimes\mathbb{1} - \frac{1}{2}\Big(X\otimes\mathbb{1} + \mathbb{1}\otimes\bar{X}\Big) \qquad (**)$$

and replace it by random matrix model

$$\mathcal{L}_{RMT} \;=\; G_R - \mathbb{1}\otimes\mathbb{1} - \frac{1}{2}\Big(C\otimes\mathbb{1} + \mathbb{1}\otimes\bar{C}\Big) \qquad (***)$$

where G_R is a **real** **Ginibre** matrix, while the correction term C is a random hermitian matrix from **Gaussian orthogonal** ensemble (GOE).

RM-model of random GKS-L generators

Take the Lindblad operator associated with a map Φ

$$\mathcal{L} \;=\; \Phi - \mathbb{1}\otimes\mathbb{1} - \frac{1}{2}\Big(X\otimes\mathbb{1} + \mathbb{1}\otimes\bar{X}\Big) \qquad (**)$$

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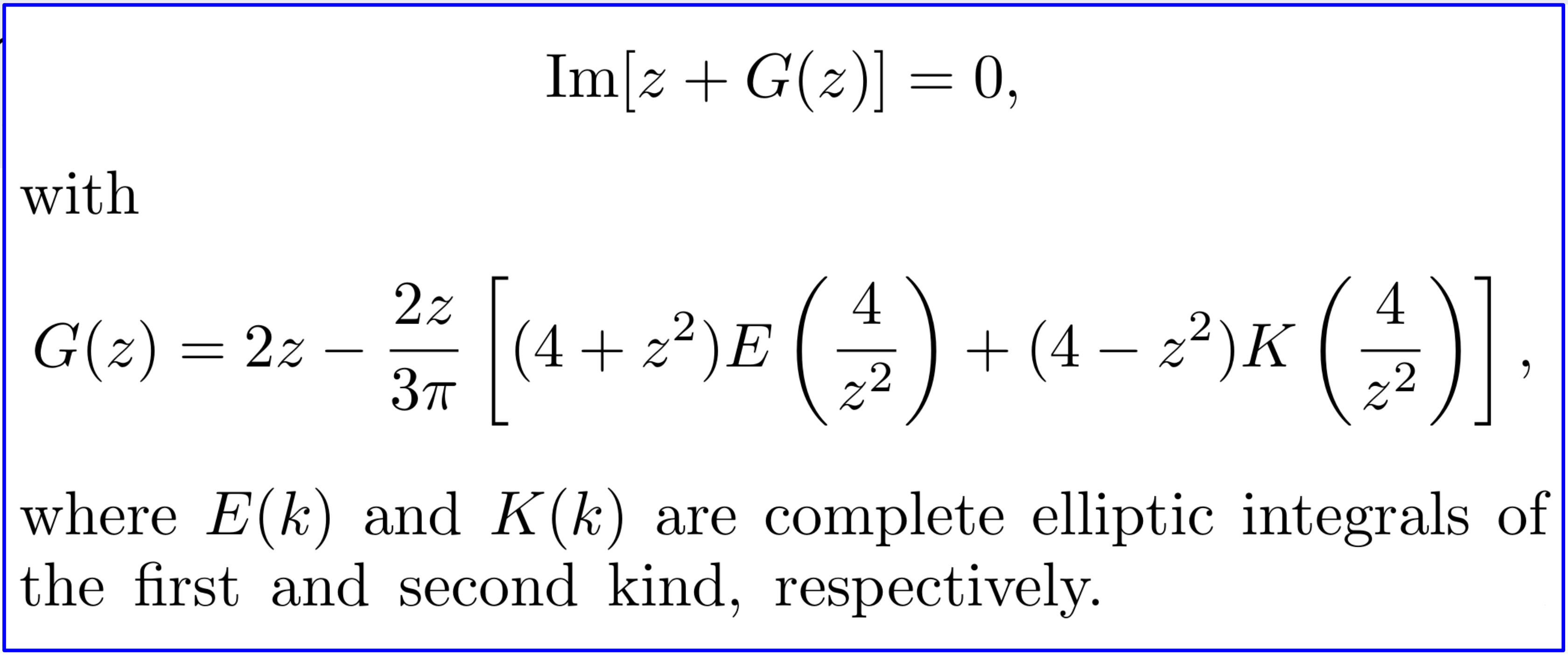

$$\mathrm{Im}[z + G(z)] = 0,$$

with

$$G(z) = 2z - \frac{2z}{3\pi}\left[(4+z^2)E\left(\frac{4}{z^2}\right) + (4-z^2)K\left(\frac{4}{z^2}\right)\right],$$

where $E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kind, respectively.

Take the Lindblad operator associated with a map Φ

$$\mathcal{L} \; = \; \Phi - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}\Big(X \otimes \mathbb{1} + \mathbb{1} \otimes \bar{X} \Big) \qquad (**)$$

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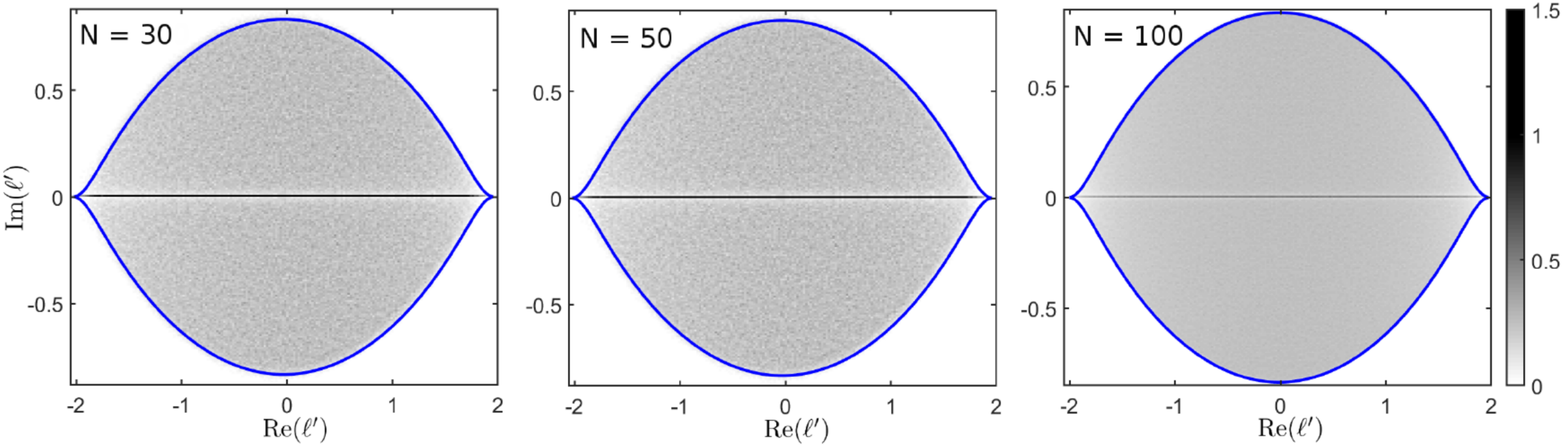
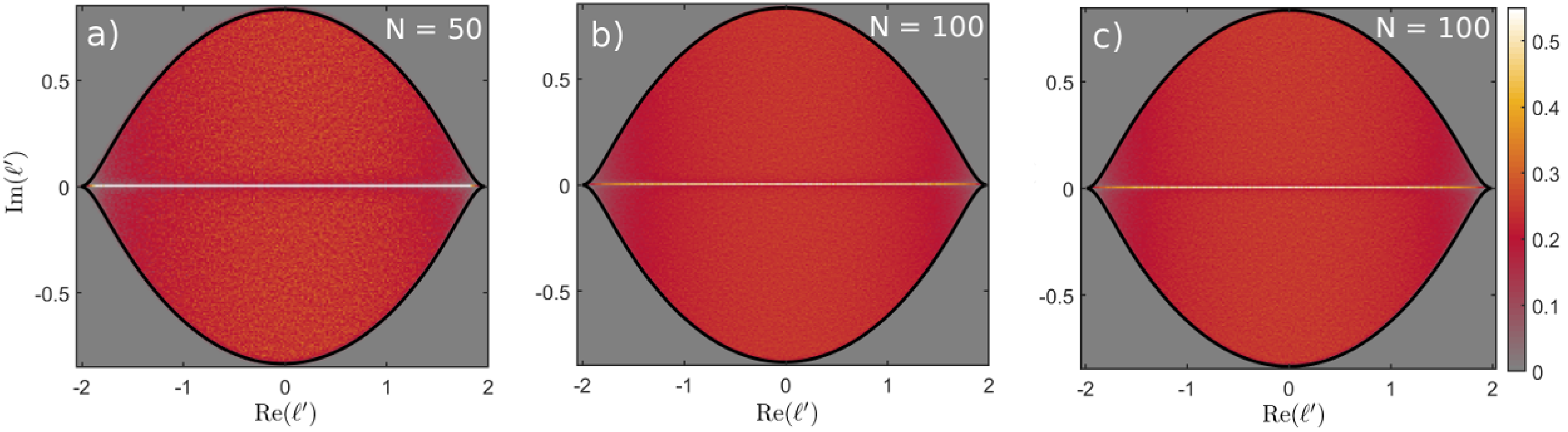


Spectra of rescaled Linblad operators $\mathcal{L}'$ for $N = 30, \; 50$ and 100 compared with the (**asymptotic**) RMT **boundary** derived from $(***)$.

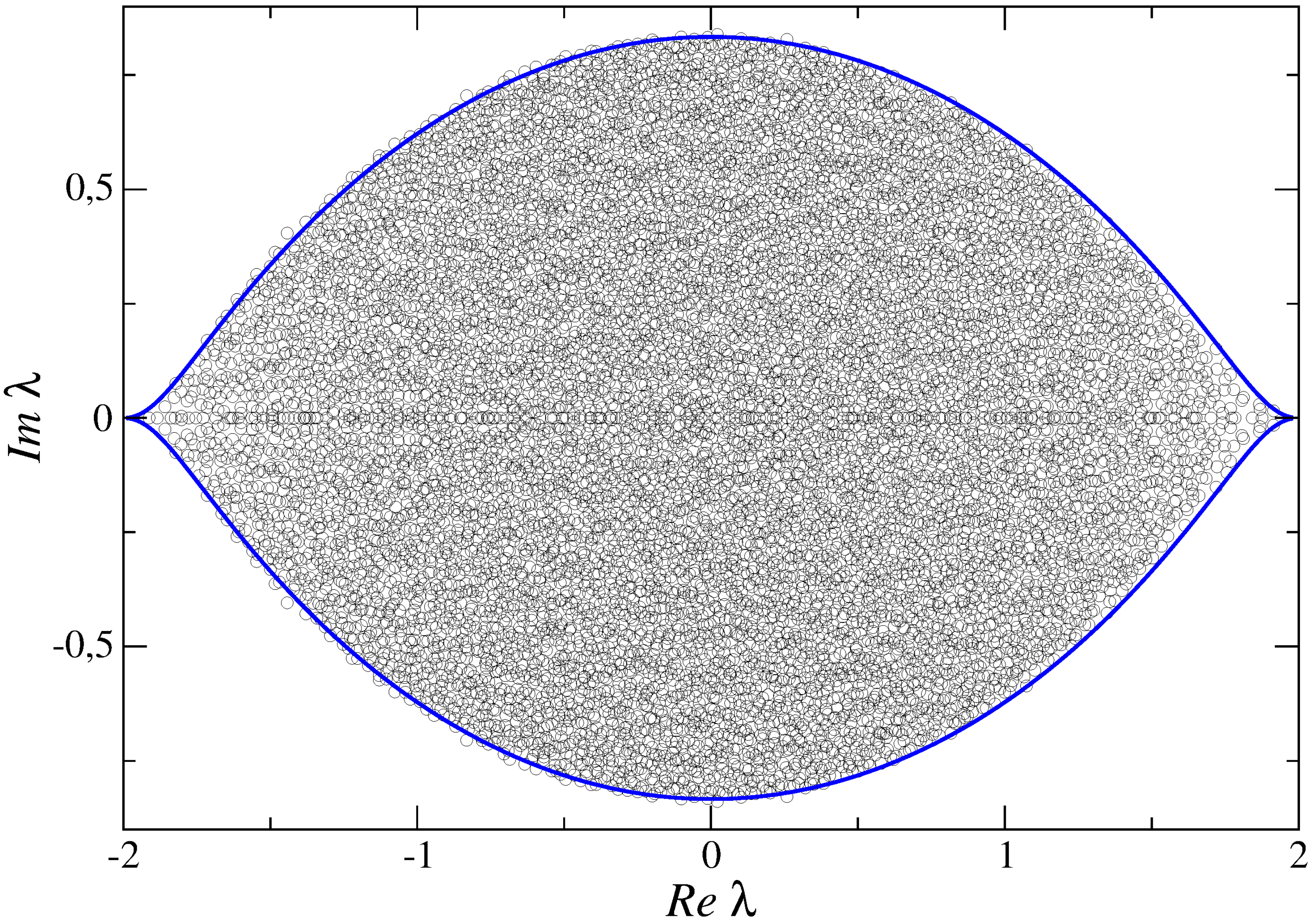
Universality of the Lemon

Kossakowski matrix can be sampled in infinitely many ways
[see, e.g., K. Życzkowski, K. A. Penson, I. Nechita, B. Collins, *Generating random density matrices* (2011)]



The spectral density is independent of the sampling procedure used (if the sampling is not 'pathological')

Typicality of the Lemon



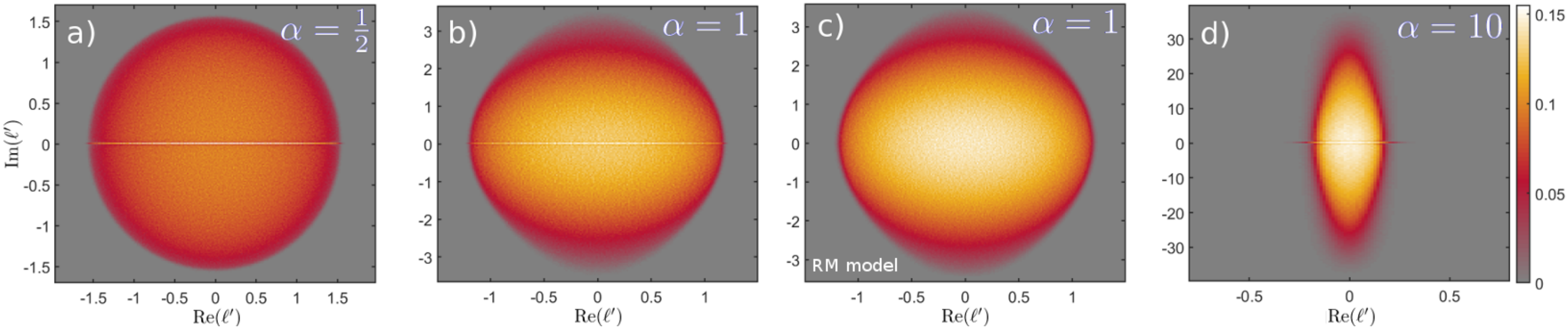
Eigenvalues of a single realization of a purely dissipative GKS-L generator for *N*=100

Random GKS-L generator: general case

$$\mathcal{L}(\rho) = -\frac{i\alpha}{\hbar}(H\rho - \rho H) + \Phi(\rho) - \frac{1}{2}\left(\Phi^\dagger(\mathbb{1})\rho + \rho\,\Phi^\dagger(\mathbb{1})\right)$$

where H is sampled from the GUE ensemble and normalized, Tr $H^2 = 1/N$.

The parameter $\alpha \geq 0$ weights contribution of the unitary component.



Spectra of the rescaled GKS-L generators $\mathcal{L}' = N(\mathcal{L} + 1)$

Random GKS-L generator: general case

RM-model

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1} - \left(\frac{1}{2} X + i\alpha H \right) \otimes \mathbb{1} + \mathbb{1} \otimes \left(\frac{1}{2} \overline{X} - i\alpha \overline{H} \right)$$

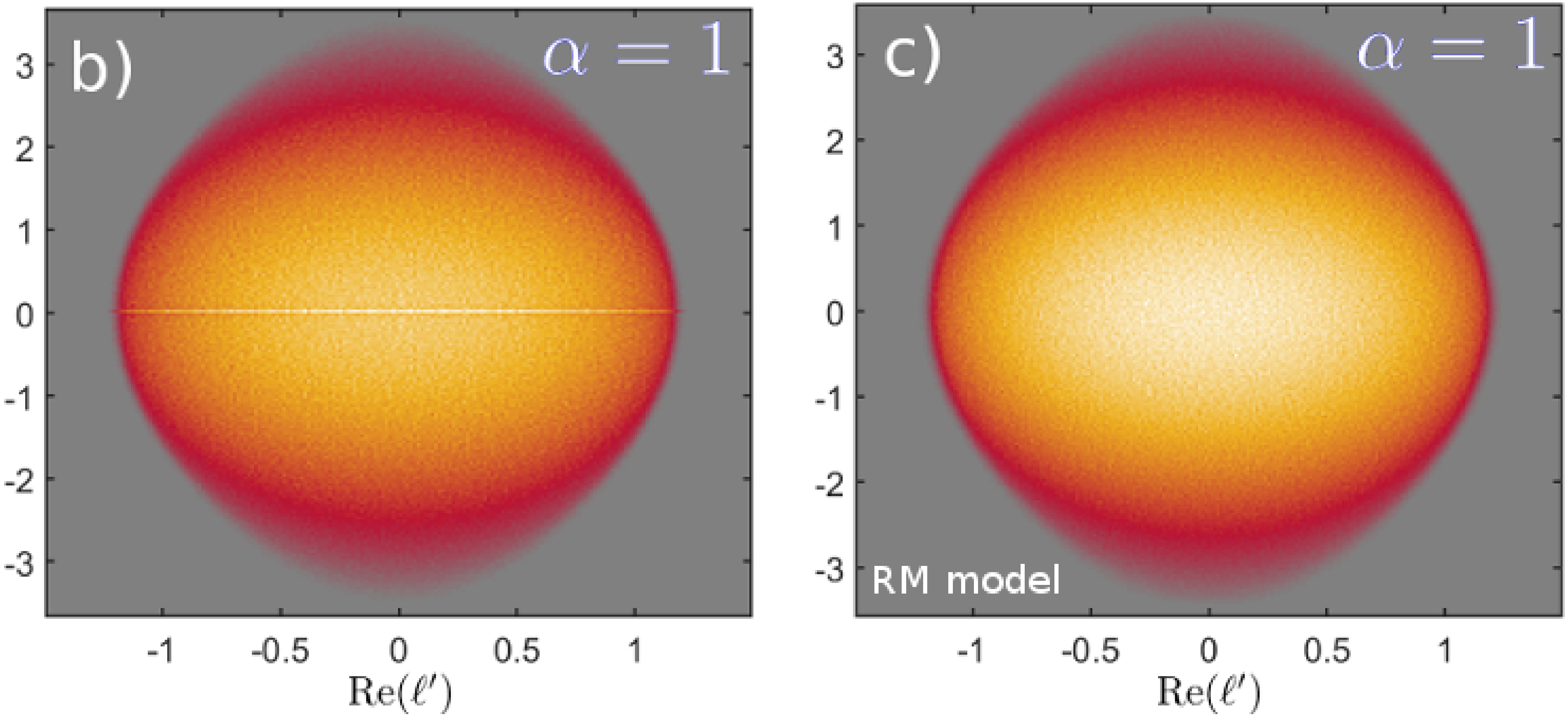
⬇

$$\widehat{\mathcal{L}}' \approx G_R - (W \otimes \mathbb{1} + \mathbb{1} \otimes \overline{W})$$

where $W = C + i\alpha H'$ and *C* (*H*') is a random GOE (GUE) matrix.

Random GKS-L generator: general case

RM-model



Random GKS-L generator: general case

RM-model

$$\widehat{\mathcal{L}}' \approx G_R - (W \otimes \mathbb{1} + \mathbb{1} \otimes \overline{W})$$

where $W = C + i\alpha H'$ and *C* (*H'*) is a random GOE (GUE) matrix.

The eigenvalues of W uniformly cover an ellipse with semi-axes $\frac{1}{\sqrt{1+4\alpha^2}}$ and $\frac{4\alpha^2}{\sqrt{1+4\alpha^2}}$. Spectral density of $\widehat{\mathcal{L}}'$ is therefore a (classical) convolution of two uniform densities supported on these ellipses followed by free convolution with the Girko disk of unit radius

Random GKS-L generator: general case

RM-model

where $W =$ … matrix.

The e … se with
semi-axe … of $\widehat{\mathcal{L}}'$ is
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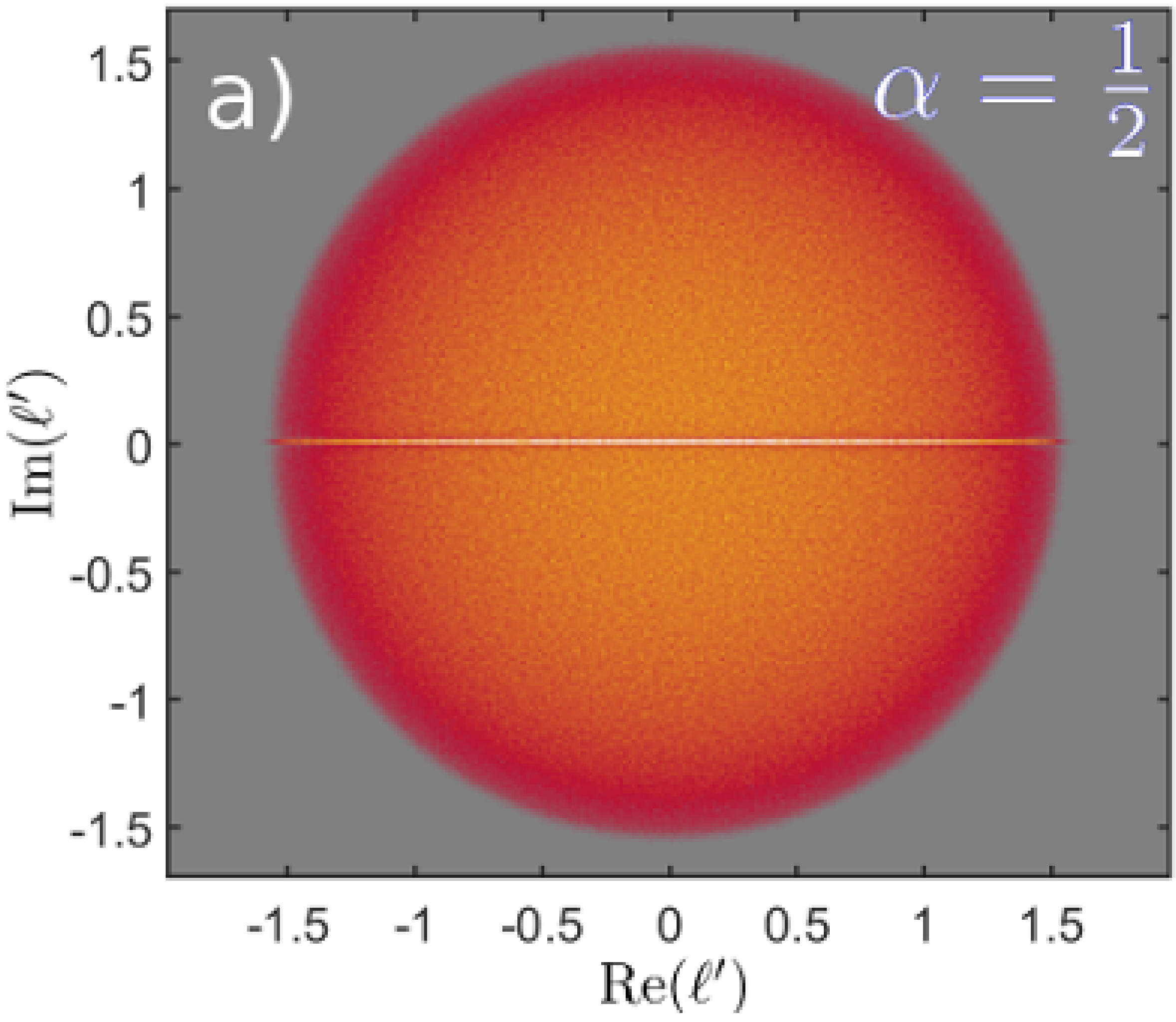
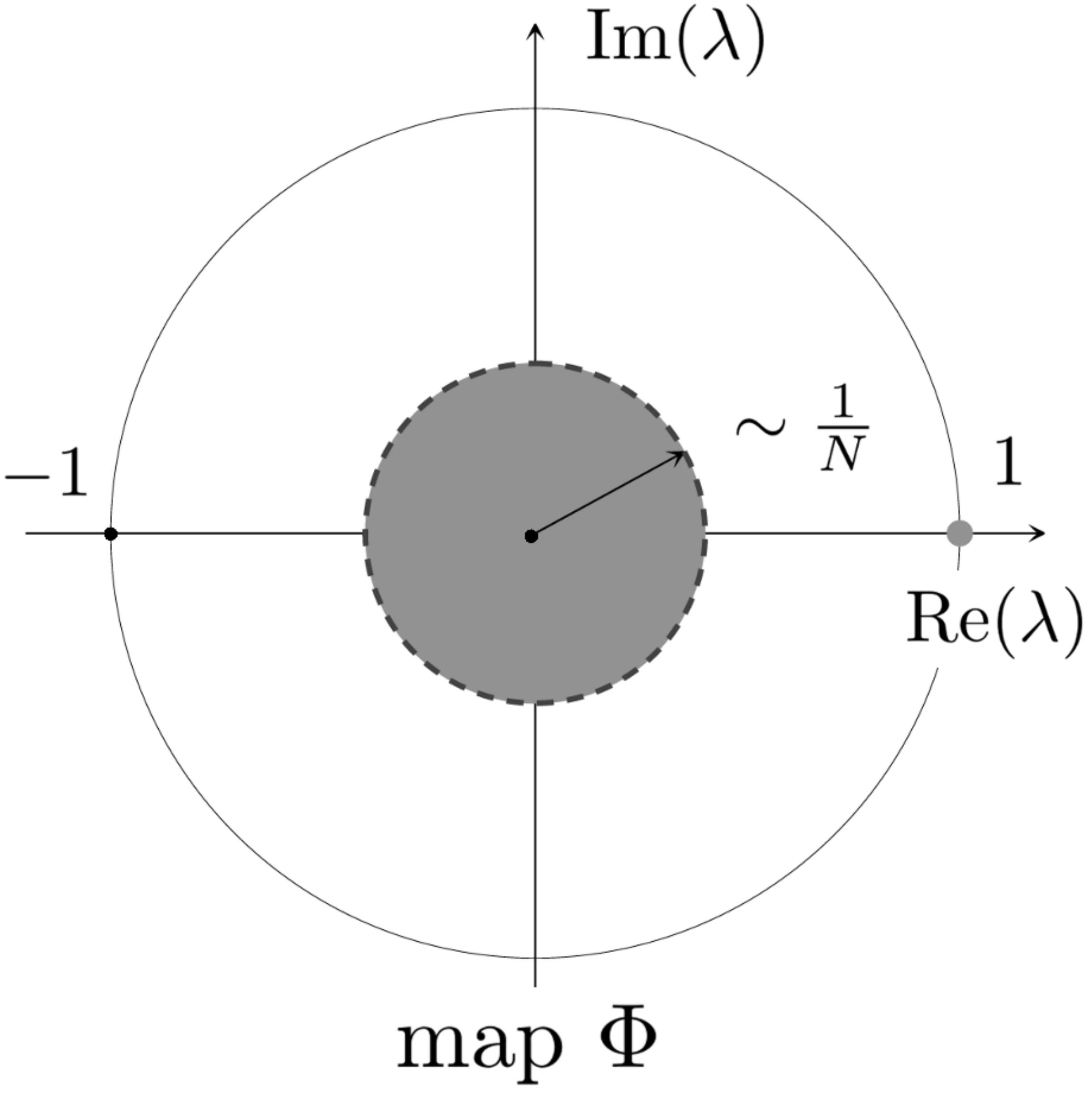
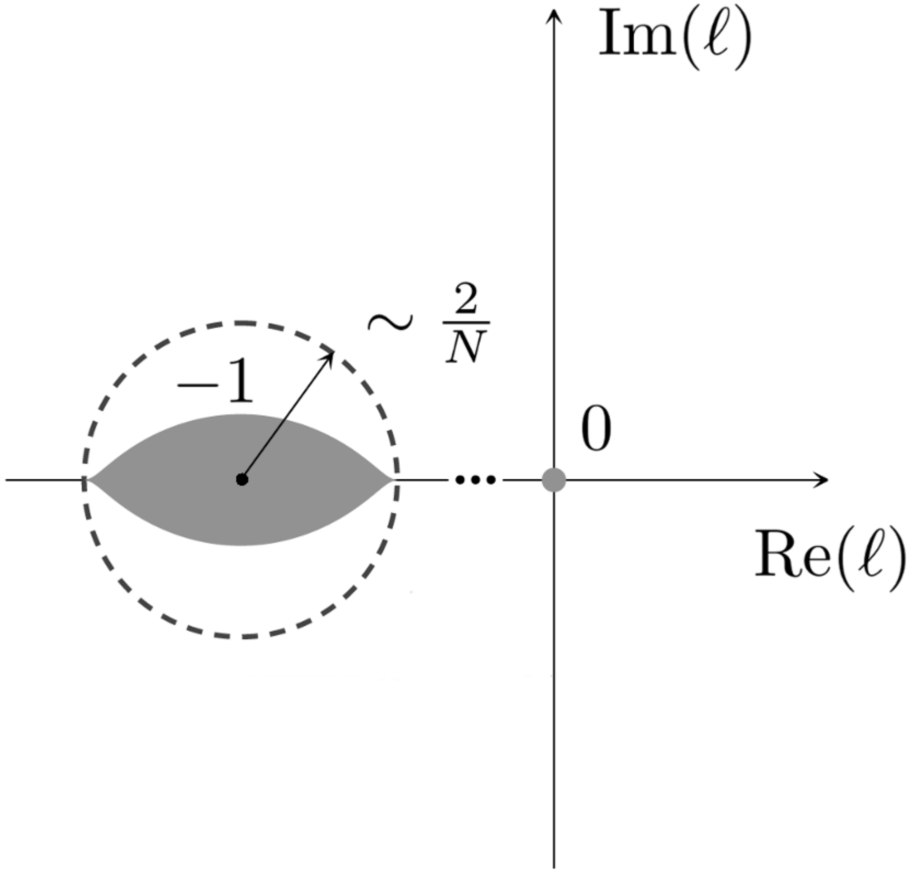


Spectra of a random map and GKS-L generator: a comparison



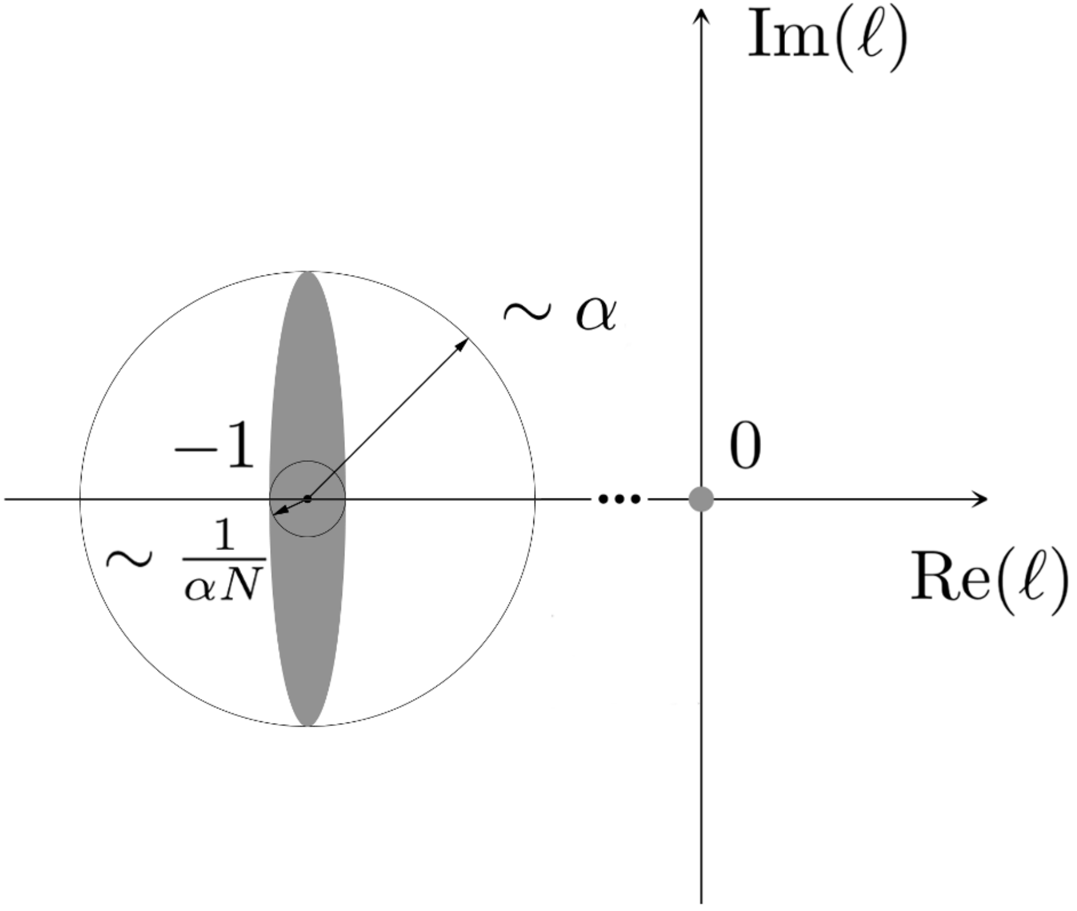
Spectra of a random map and GKS-L generator: a comparison



purely dissipative Lindblad operator

$\mathcal{L}_D$

Spectra of a random map and GKS-L generator: a comparison



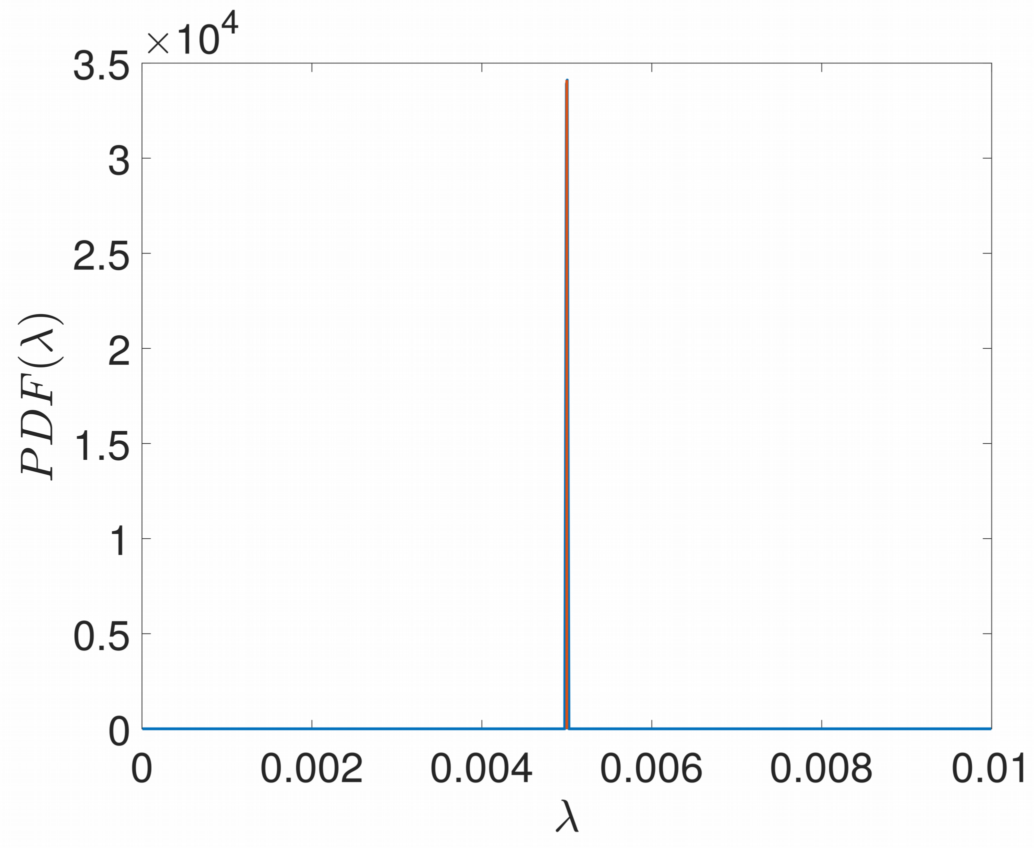
generic Lindblad operator

$$\mathcal{L} = \mathcal{L}_U + \mathcal{L}_D$$

Steady states of random GKS-L generators $\mathcal{L}\varrho_\infty = 0$

They are the steady states of random CPTP maps!

I.e., normalized identities ‘coated’ by GUE “fluctuations”

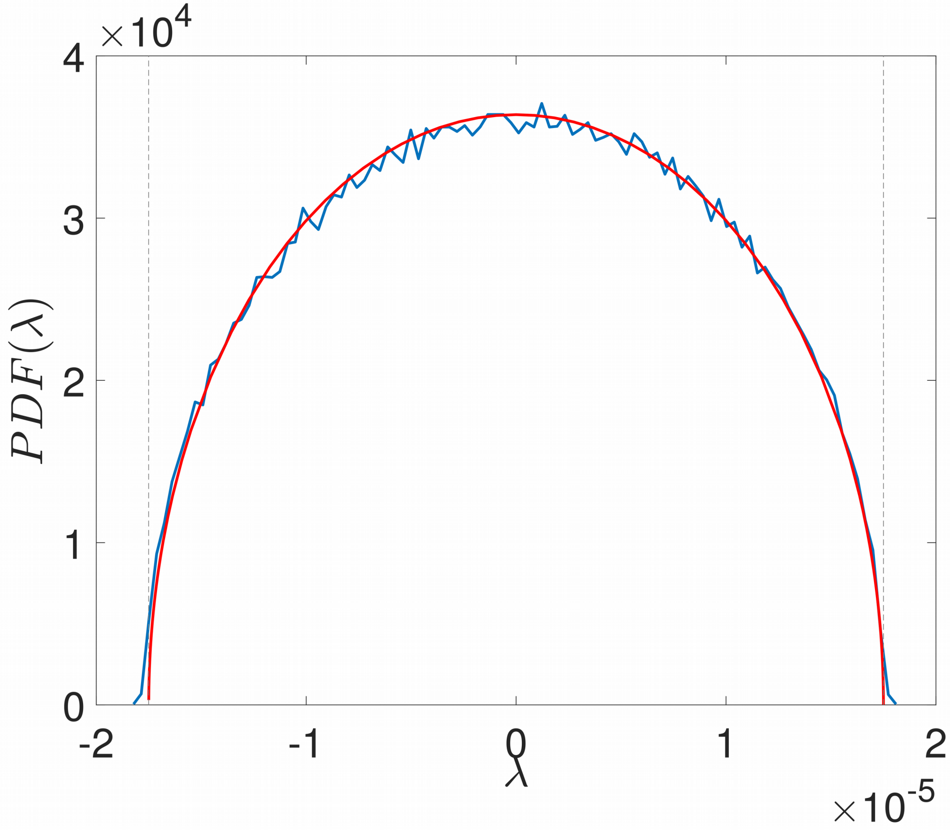


Spectral density of ϱ_∞ for *N* = 200

Steady states of random GKS-L generators $\mathcal{L}\varrho_\infty = 0$

They are the steady states of random CPTP maps!

I.e., <u>normalized identities ‘coated’ by GUE “fluctuations”</u>

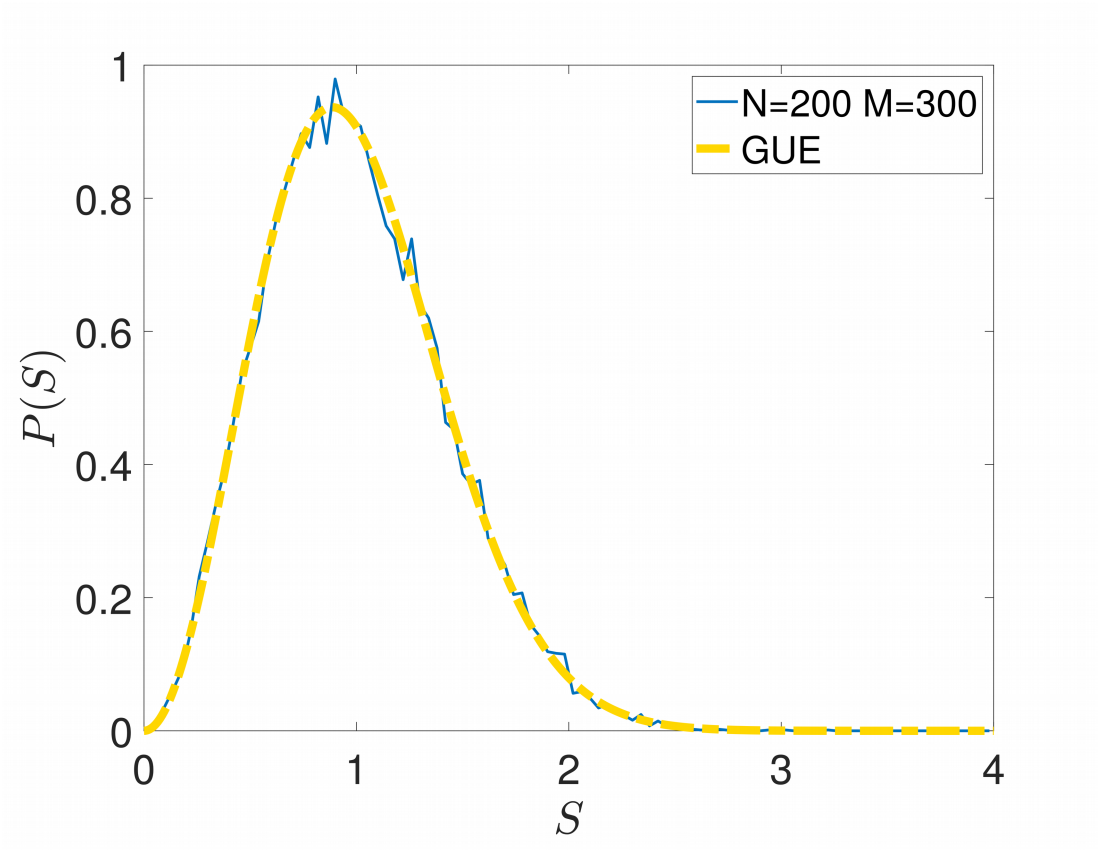


Spectral density of ϱ_∞ for *N* = 200 (minus the peak at 1/200)

Steady states of random GKS-L generators $\mathcal{L}\varrho_\infty = 0$

They are the steady states of random CPTP maps!

I.e., normalized identities ‘coated’ by GUE “fluctuations”



Level spacing distribution of ϱ_∞ for N = 200

Conclusions

- Relaxation with a randomly sampled GKS-L generator **is exponentially fast**

- The corresponding **asymptotic state is not ‘complex’**
(close to a trivial state, the normalized identity)

  - Is it possible to design sampling procedure which yields non-exponential relaxation (cut-off relaxation, power-law decay, etc)?

  - Is it possible to design a sampling procedure which gives ‘*complex*’ states (of high purity, high entanglement, etc)?

  - Is there an ensemble of random GKS-L generators with exponentially fast relaxation *and* non-trivial asymptotic states?