jan.kolodynski@cent.uw.edu.pl

# Quantum frequency estimation with local decoherence of arbitrary type

**Jan (Janek) Kołodyński** (QOT, CeNT) and **Rafał Demkowicz-Dobrzański** (FUW)

*University of Warsaw, Poland*

**Andrea Smirne**, **Jan Haase** and **Susana Huelga**

*University of Ulm, Germany*

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Unia Europejska Europejski Fundusz Rozwoju Regionalnego

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**Quantum metrology** ⟷ **Open-system dynamics**

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**Quantum Fisher Information (QFI)**

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[*Haase et al*, *"Precision Limits in Quantum Metrology with Open Quantum Systems"*, **arXiv:1807.11882**
**Quantum Meas. Quantum Metrol. 5, 13 (2018)**]

# Quantum Metrology

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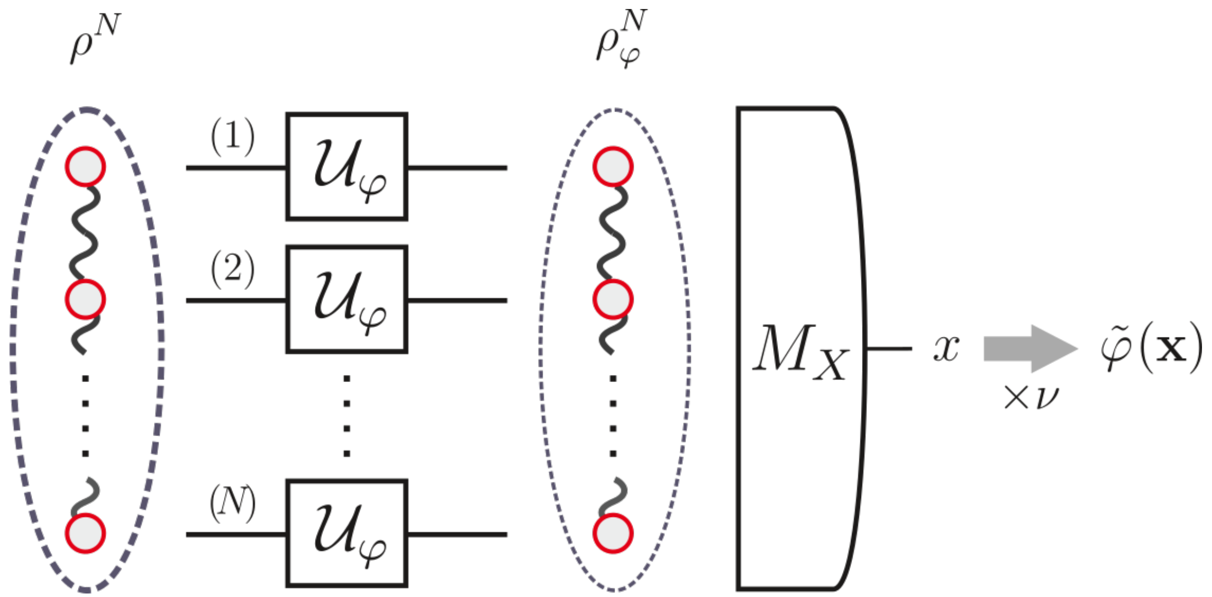

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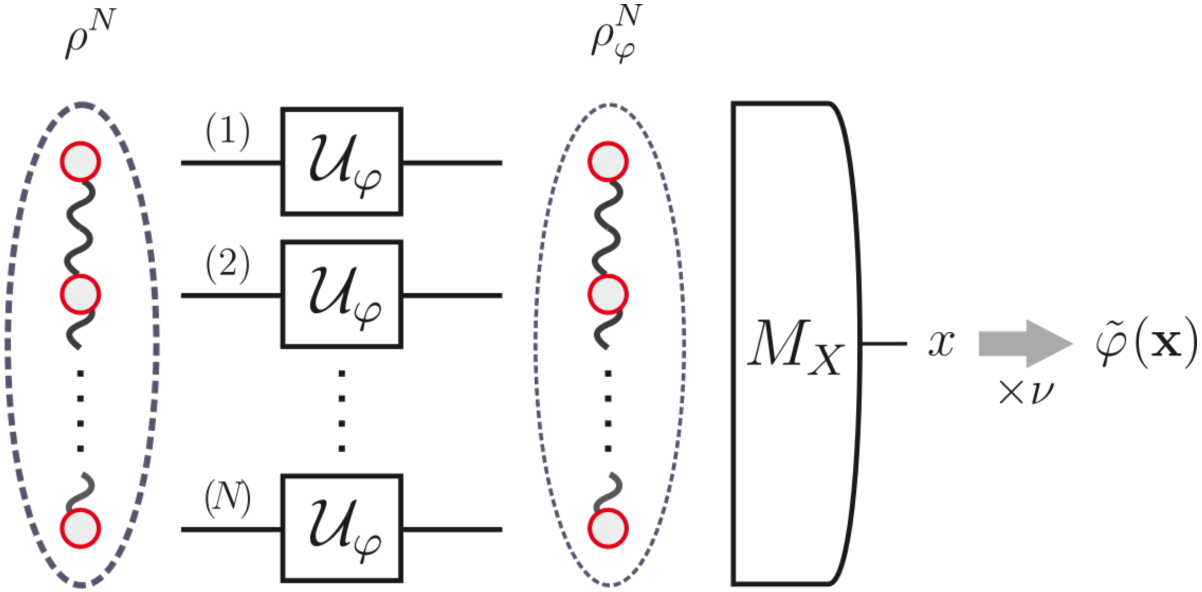


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$$\Delta^2\tilde{\varphi} = \langle(\tilde{\varphi} - \varphi)^2\rangle$$

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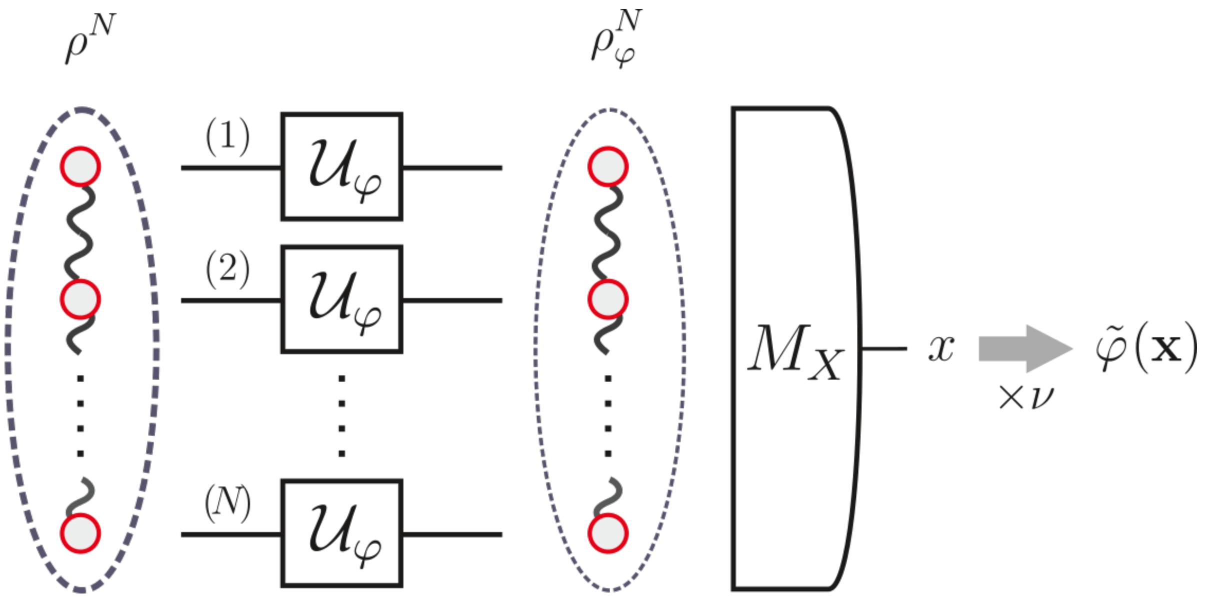


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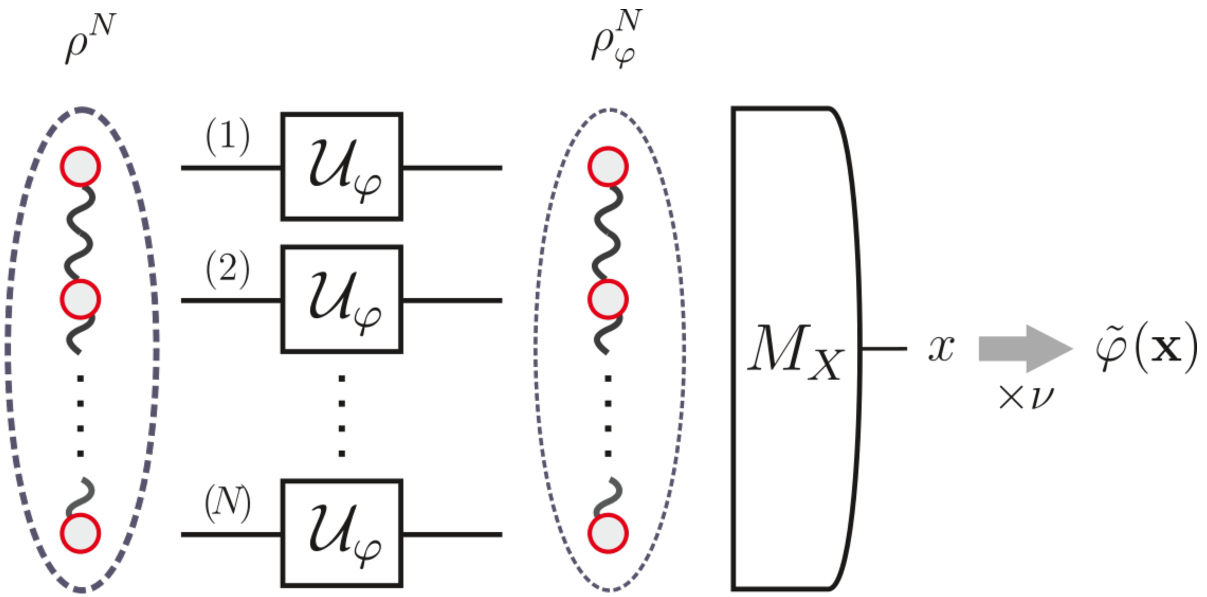


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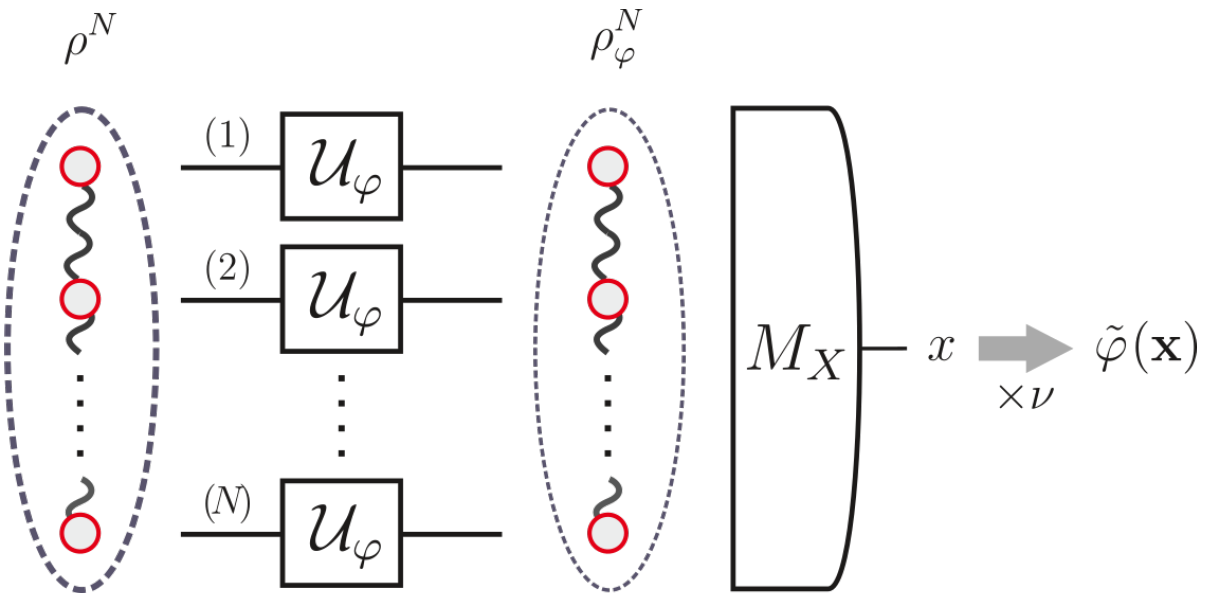


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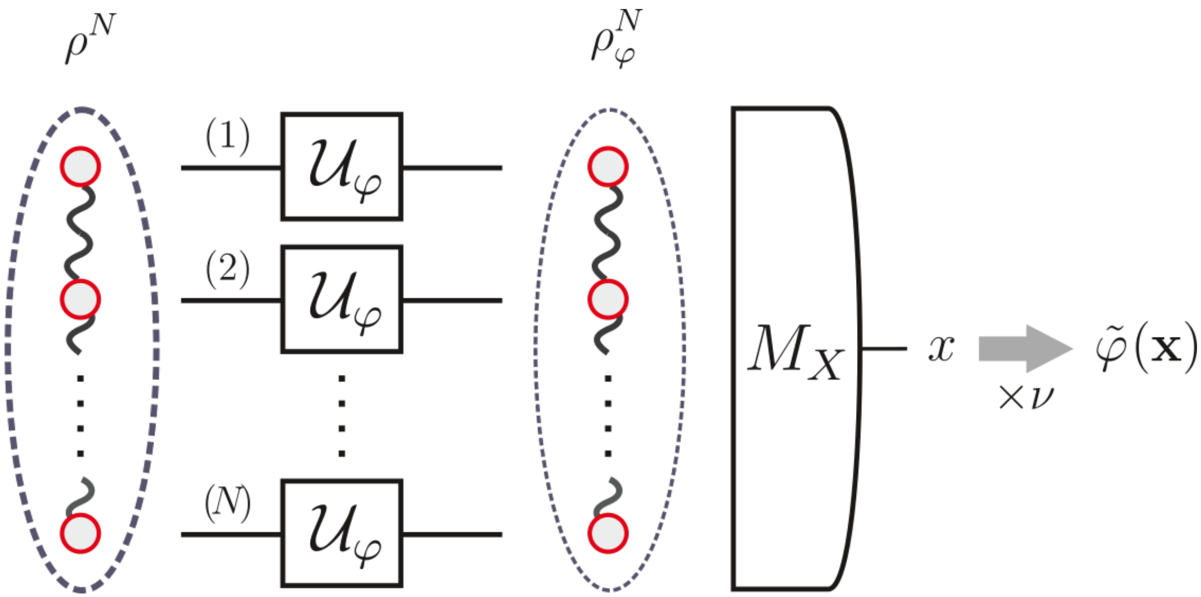


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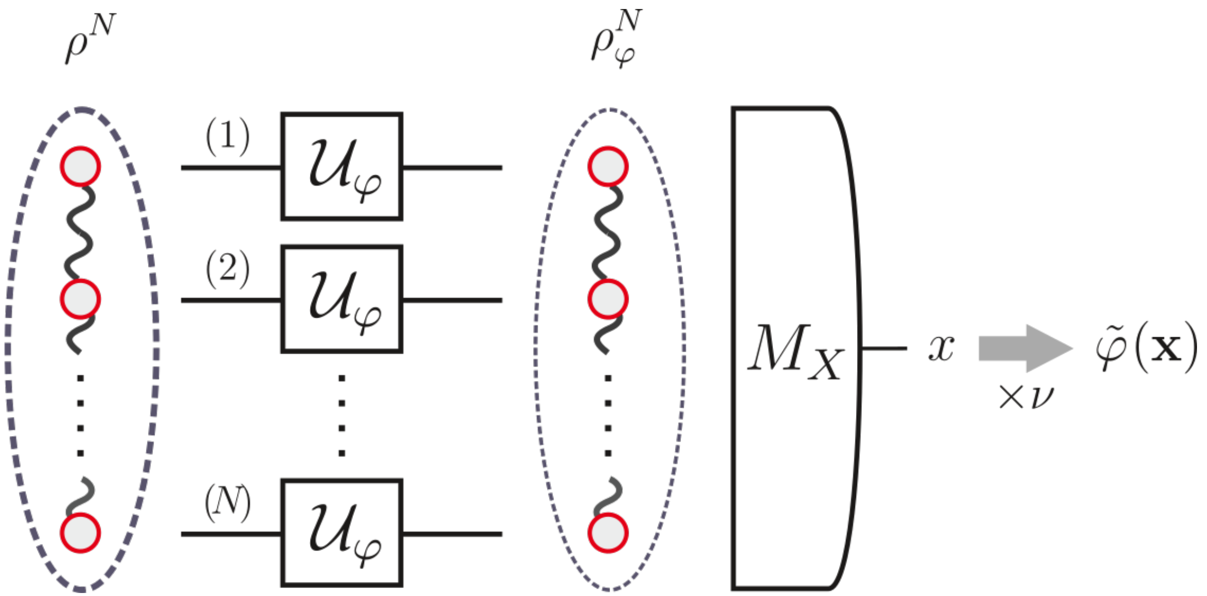


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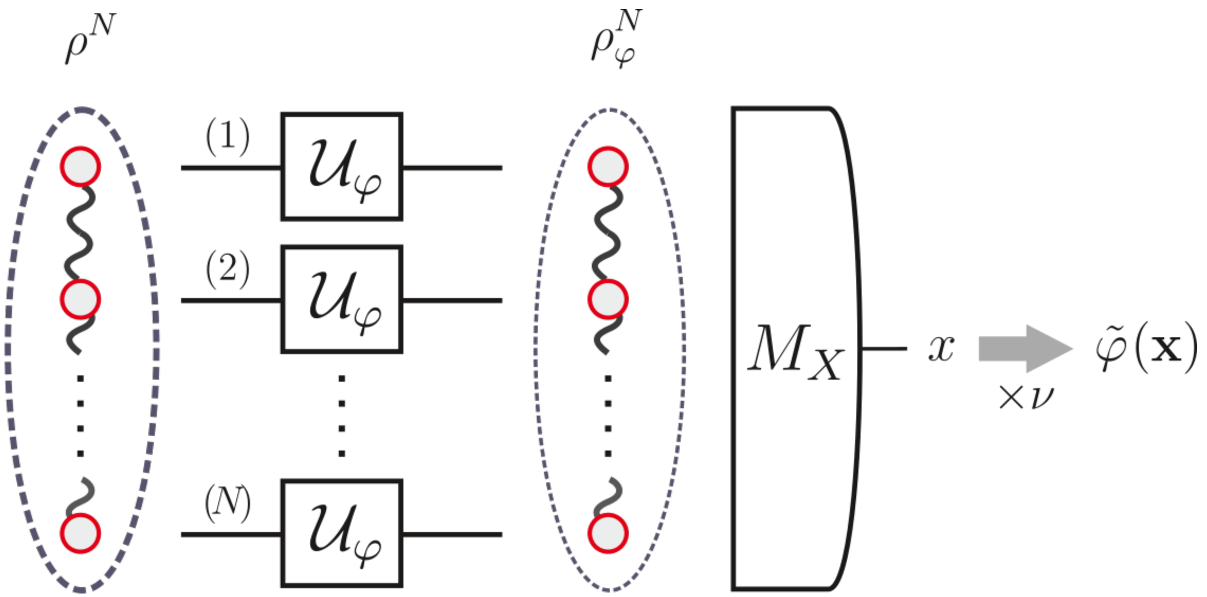


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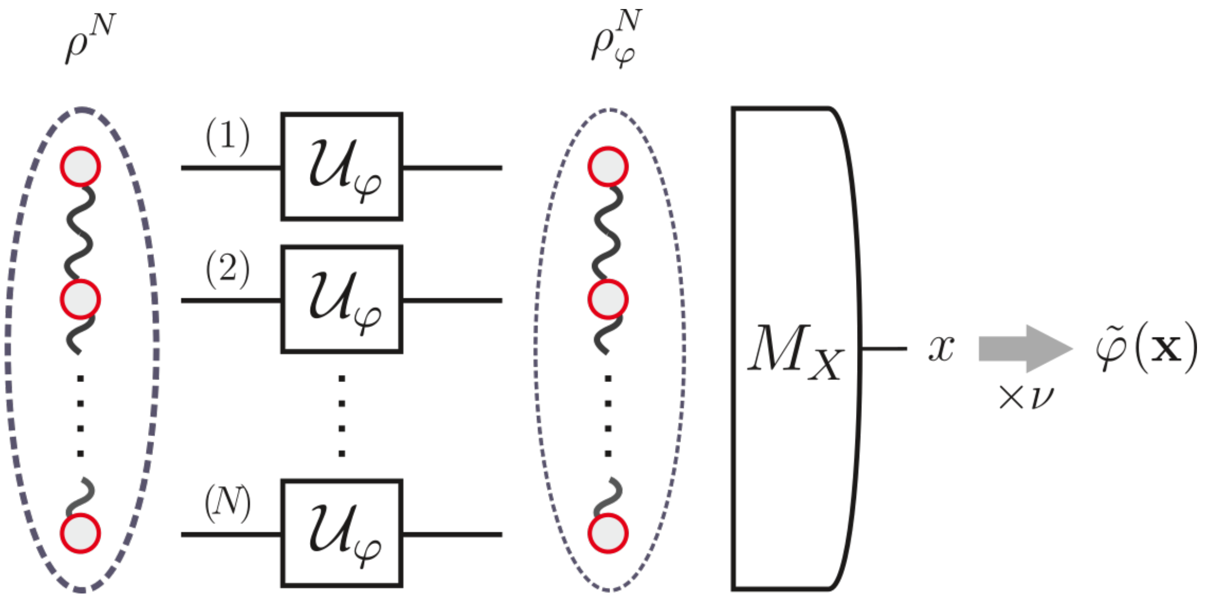

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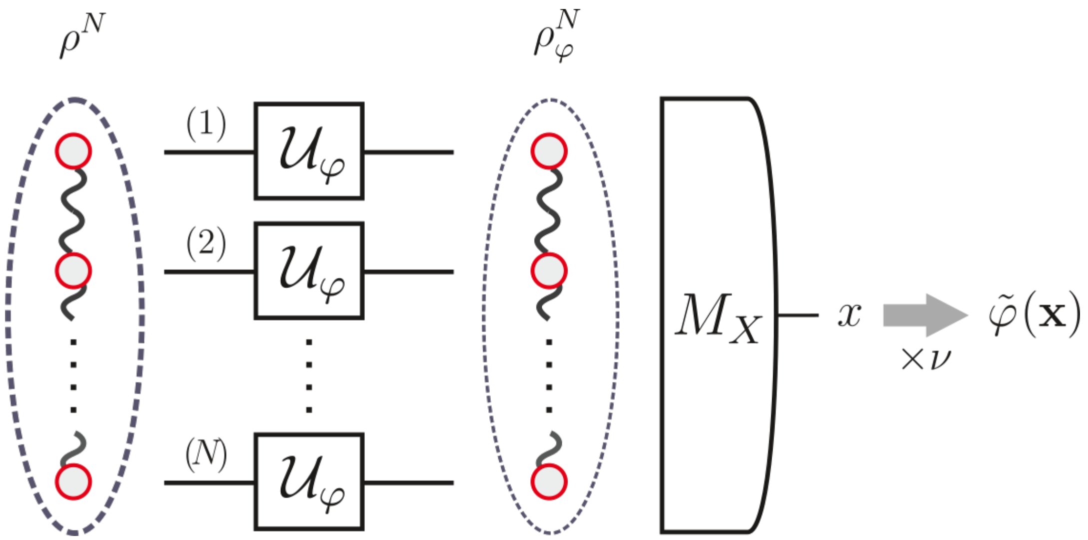
- ***Parameter-value independence of QFI*** due to unitary encoding. ***Not true in general***.

# Quantum Metrology

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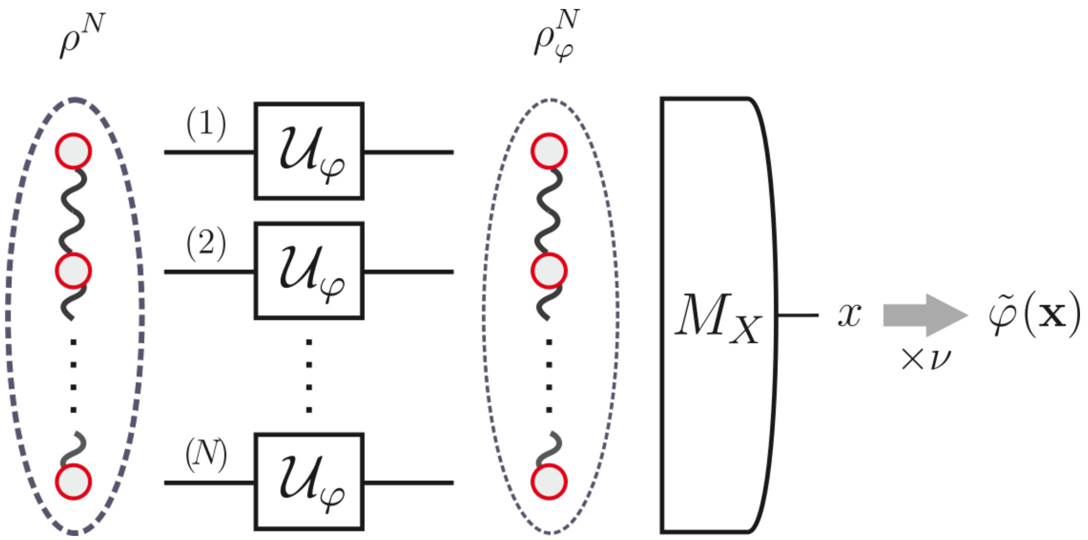


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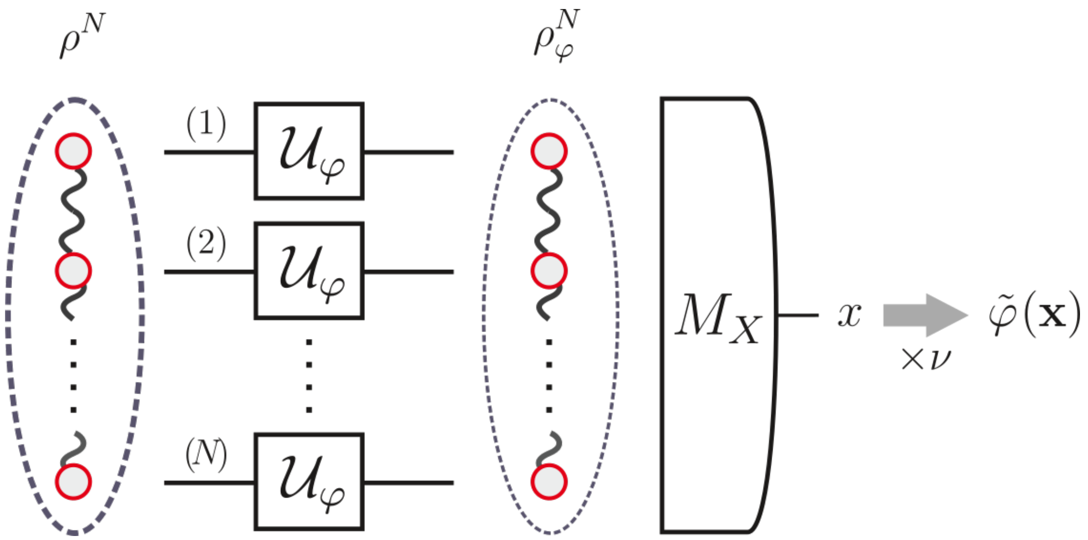


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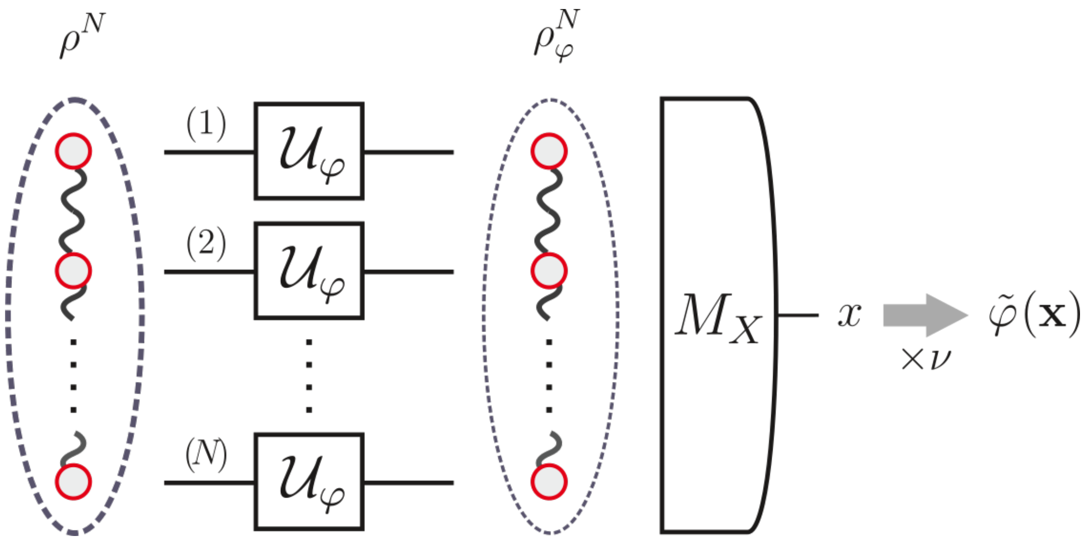


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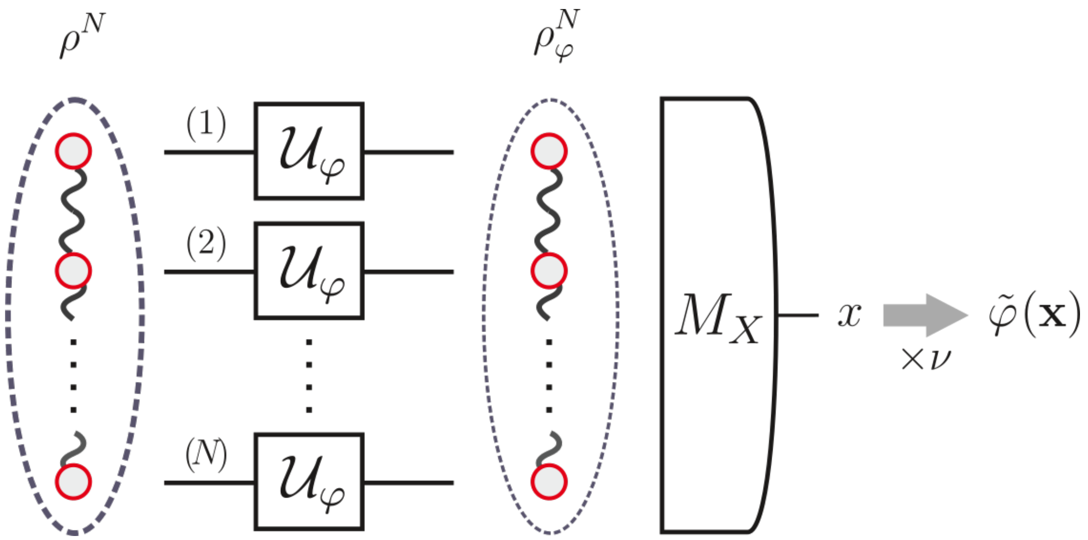


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For ***separable states*** precision is bounded by the **Standard Quantum Limit**:

$$F_Q[\rho^N_{\text{sep}}] \leq N \quad \Longrightarrow \quad \boxed{\Delta^2 \tilde\varphi \geq \frac{1}{N}}$$

**SQL**

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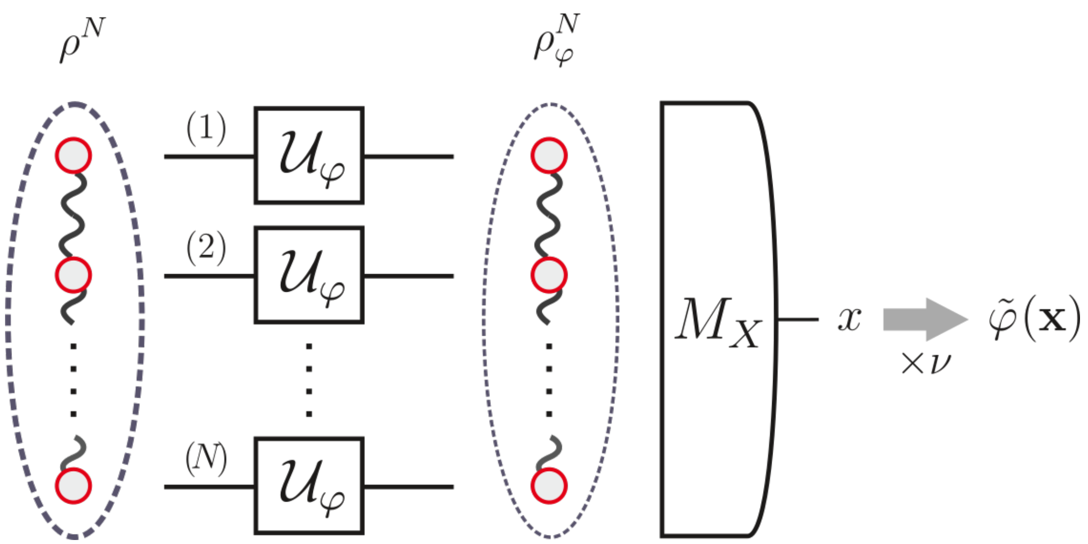


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$$F_Q[\psi^N_{\text{GHZ}}] = N^2 \quad \Longrightarrow \quad \boxed{\Delta^2\tilde{\varphi} \geq \frac{1}{N^2}}$$ **HL**

**GHZ (or N00N) state:** $|\psi^N_{\text{GHZ}}\rangle := \frac{1}{\sqrt{2}}(|0^N\rangle + |1^N\rangle)$

# Noisy Quantum Metrology

[*Akio Fujiwara and Hiroshi Imai*, **J. Phys. A: Math. Theor. 41 (2008)**][*Keiji Matsumoto*, **arXiv:1006.0300 [quant-ph] (2010)**]
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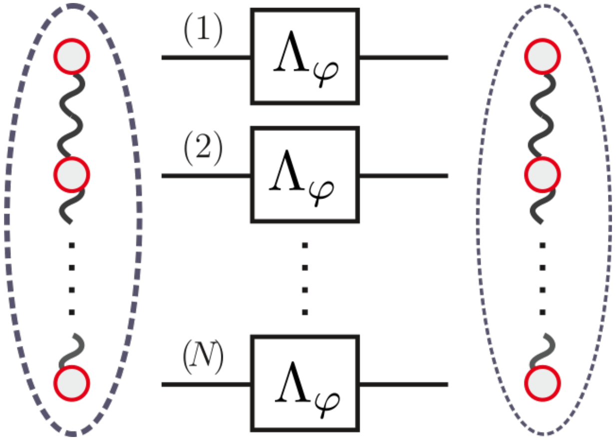


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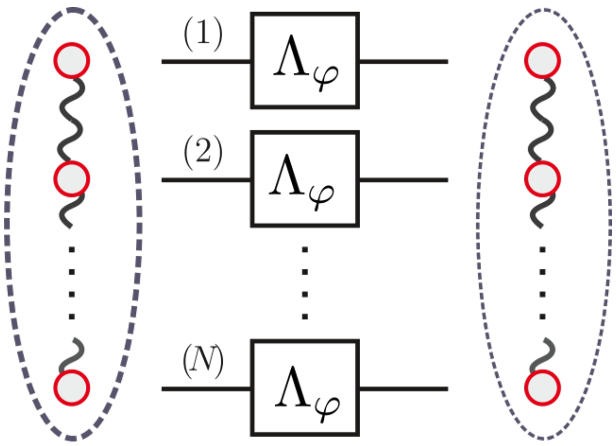


**The "curse of SQL":** $F_Q \;\leq\; \eta_{\text{noise}}\, N$

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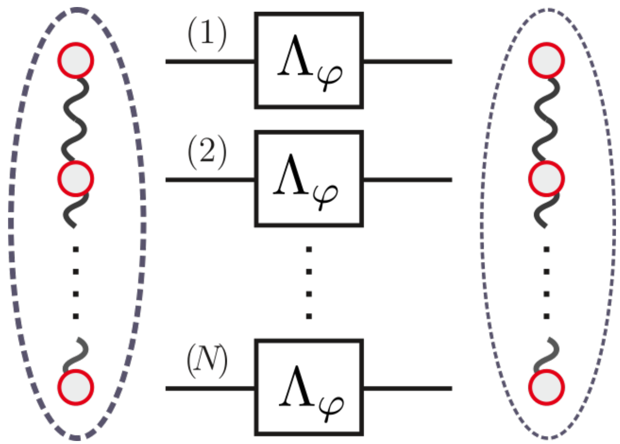


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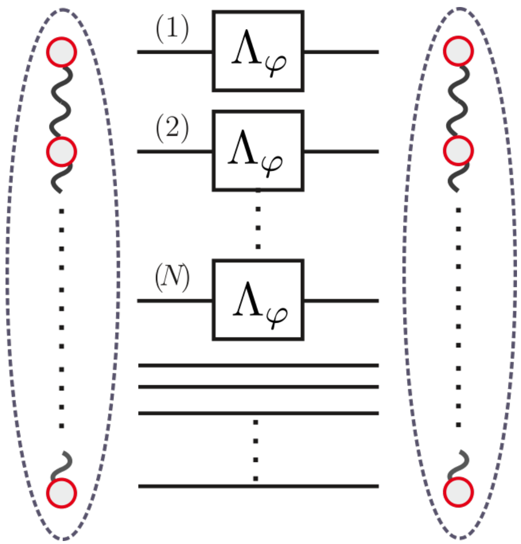


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$$\max_{|\psi_0^N\rangle} F_Q\left[\Lambda_\varphi^{\otimes N}\left[|\psi_0^N\rangle\right]\right] \leq \max_{|\psi_{\text{ext}}^N\rangle} F_Q\left[(\Lambda_\varphi \otimes \mathbb{I})^{\otimes N}\left[|\psi_{\text{ext}}^N\rangle\right]\right] \leq \min_K \left\{N\,\|\alpha_K\| + N\,(N-1)\,\|\beta_K\|^2\right\}$$

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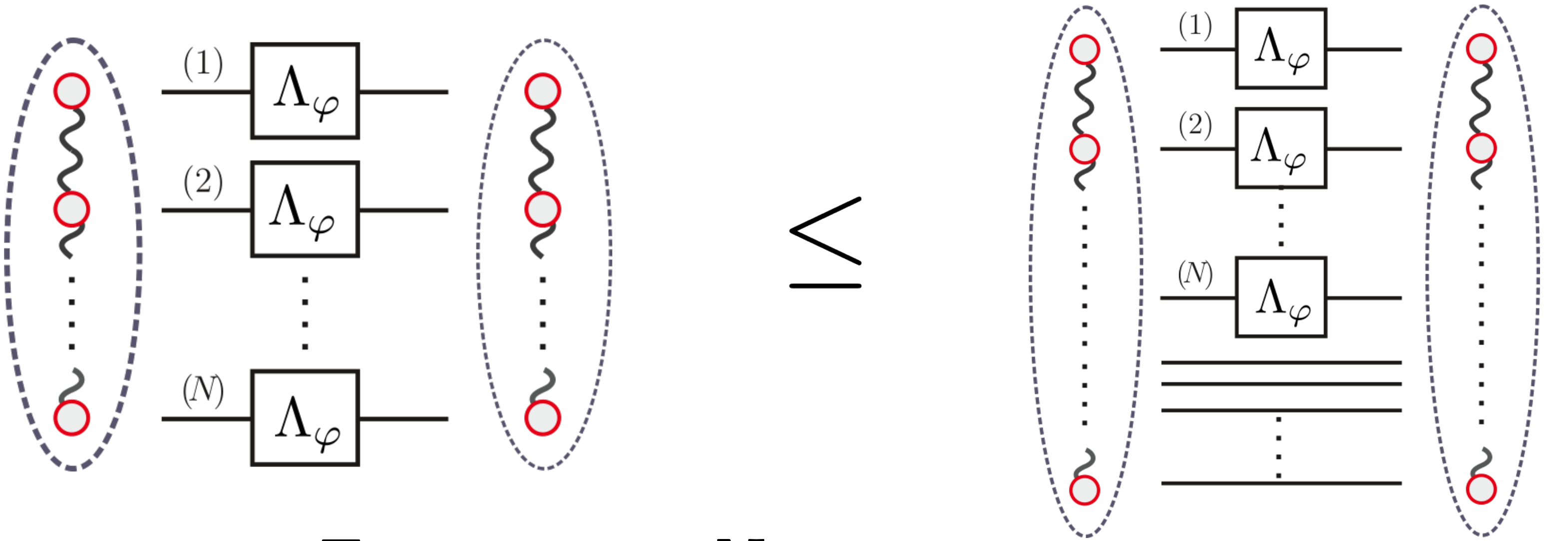


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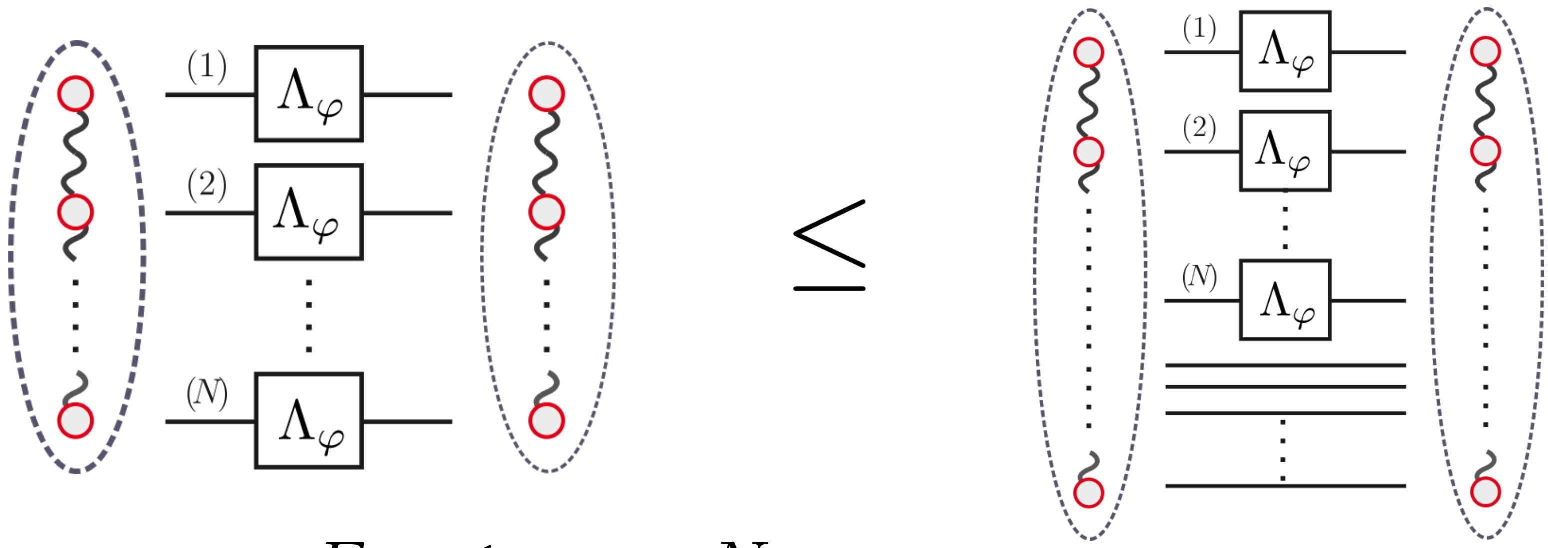


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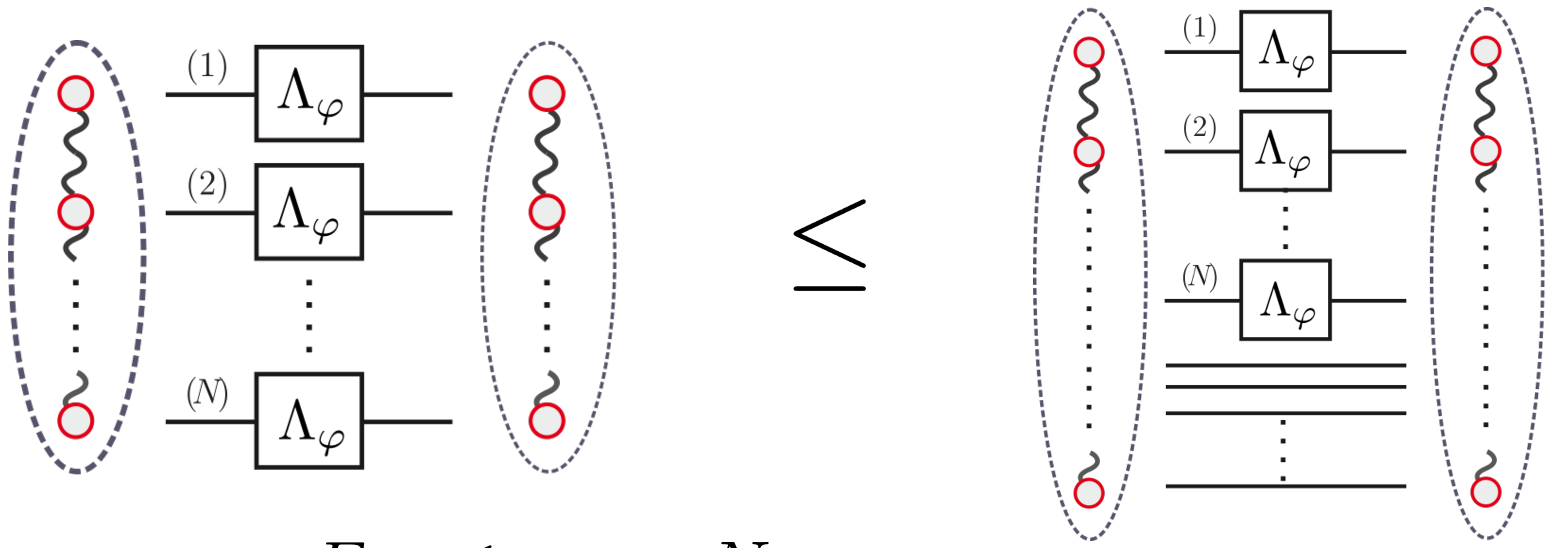
# Noisy Quantum Metrology




**The "curse of SQL":** $F_Q \leq \eta_{\text{noise}} N$

$$\max_{|\psi_0^N\rangle} F_Q\left[\Lambda_\varphi^{\otimes N}\left[|\psi_0^N\rangle\right]\right] \leq \max_{|\psi_{\text{ext}}^N\rangle} F_Q\left[(\Lambda_\varphi \otimes \mathbb{I})^{\otimes N}\left[|\psi_{\text{ext}}^N\rangle\right]\right] \leq \min_K \left\{N\,\|\alpha_K\| + N\,(N-1)\,\|\beta_K\|^2\right\}$$

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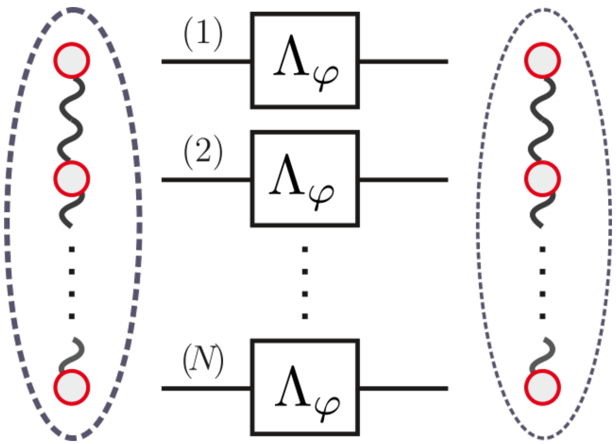


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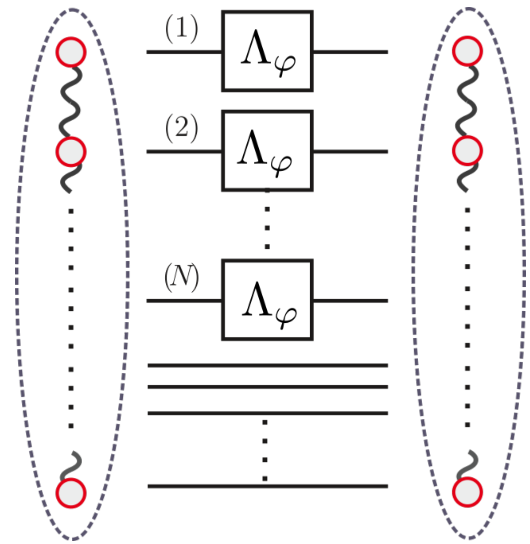


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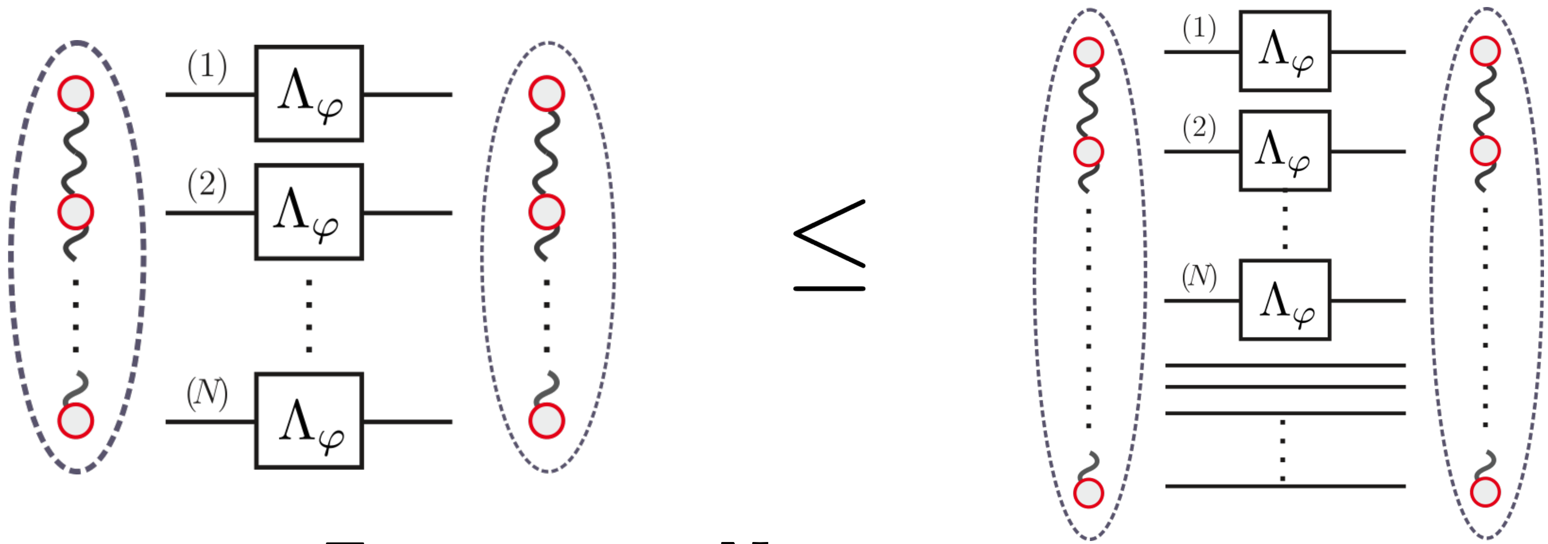


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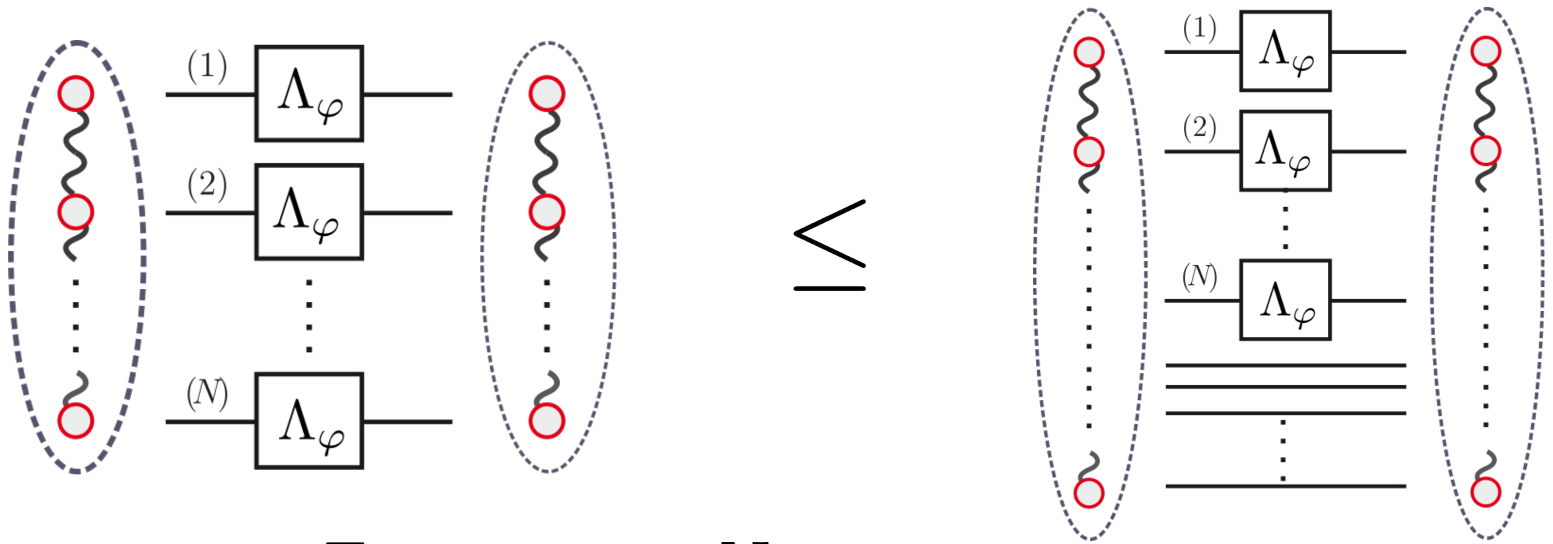


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*Ok if commute, but in general need to* ***add time d.o.f.***

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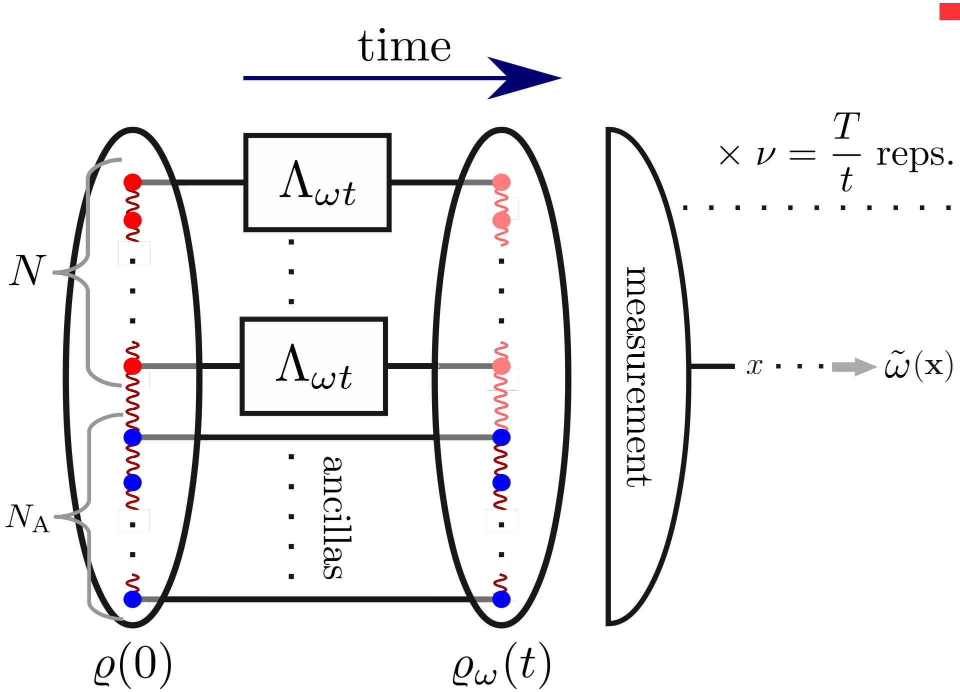

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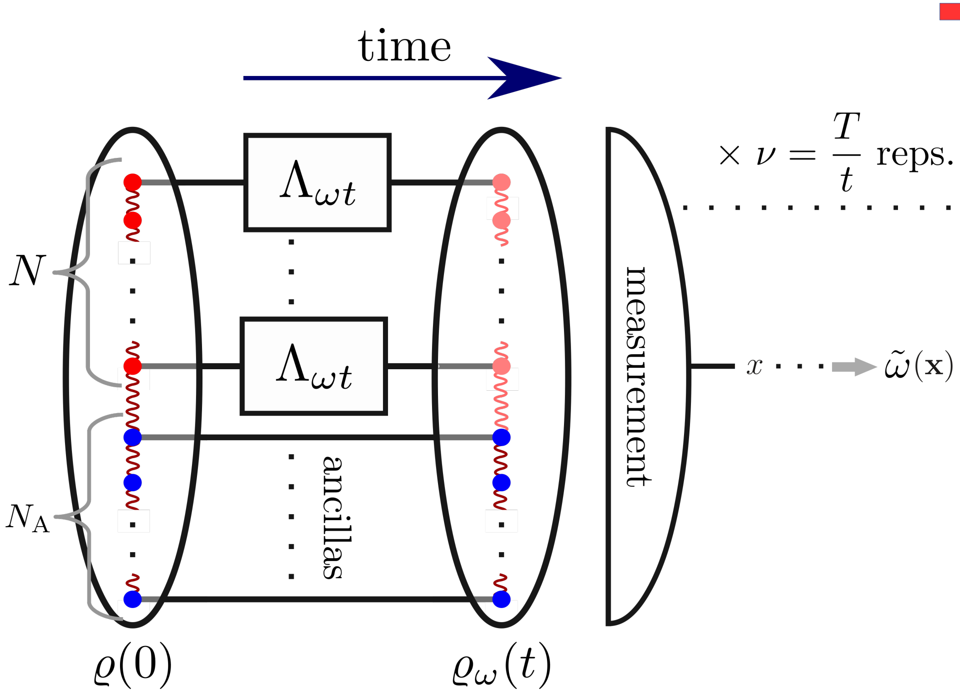


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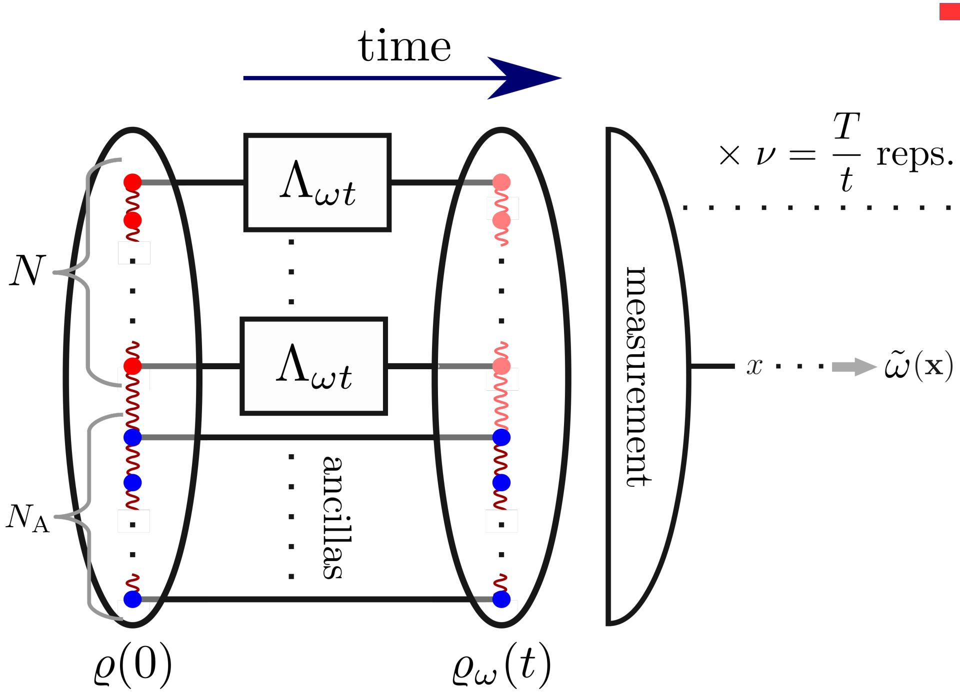


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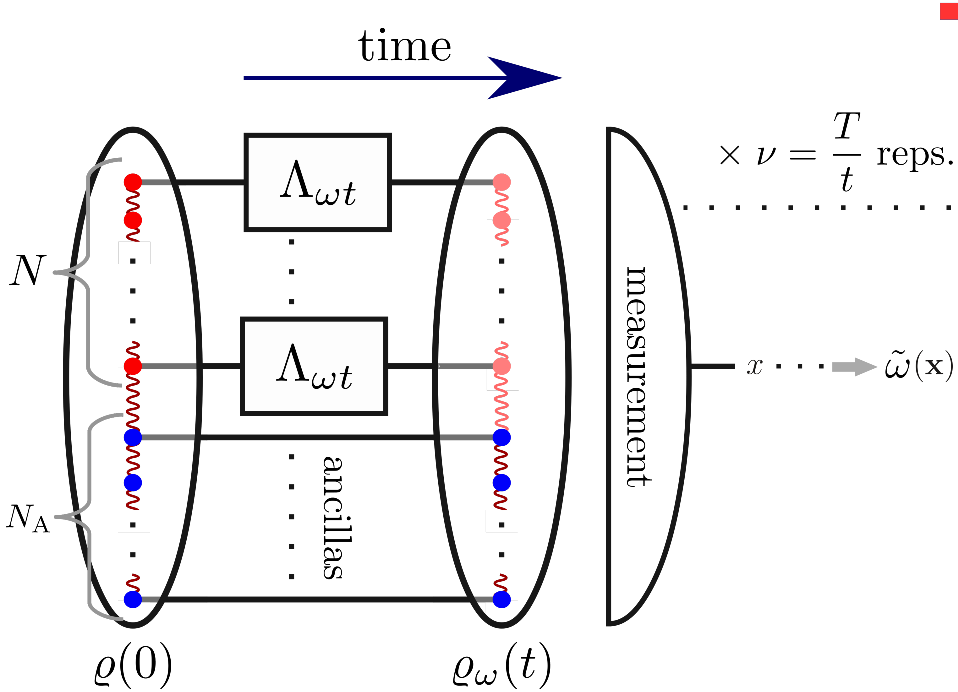


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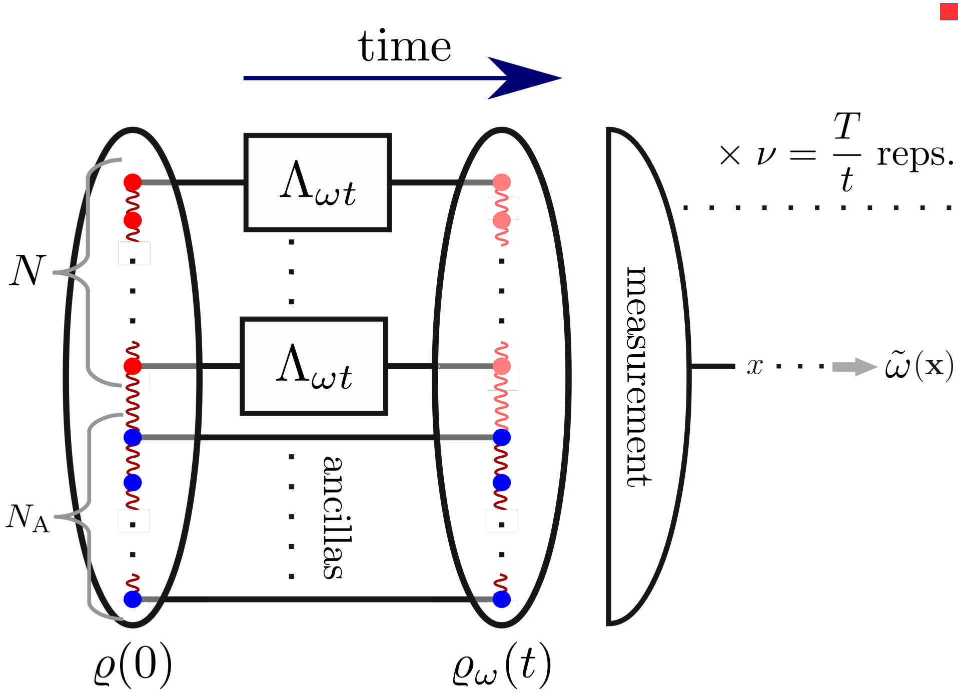


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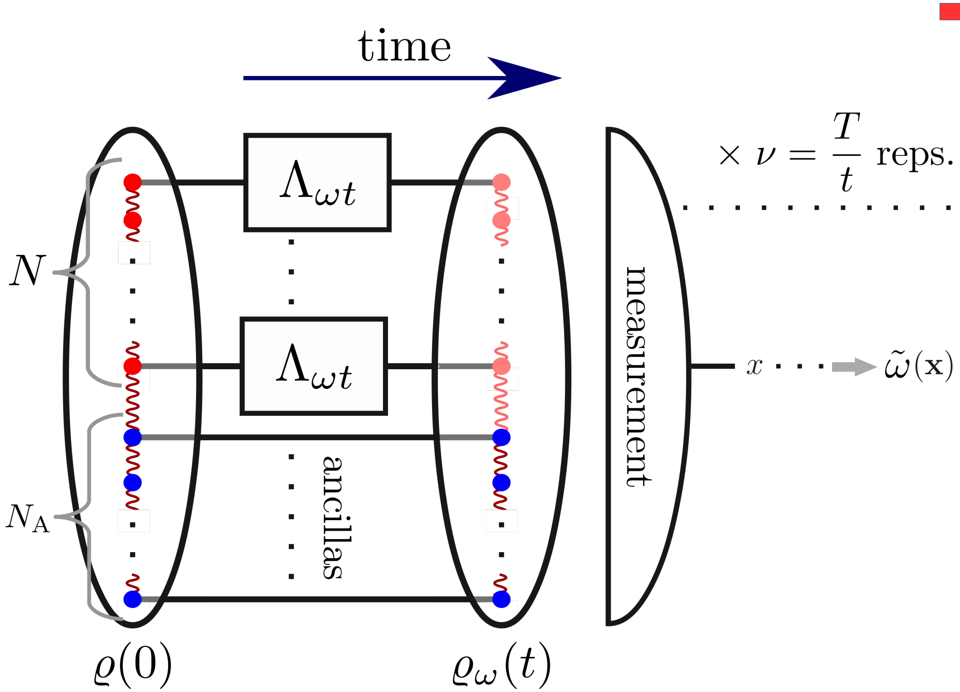


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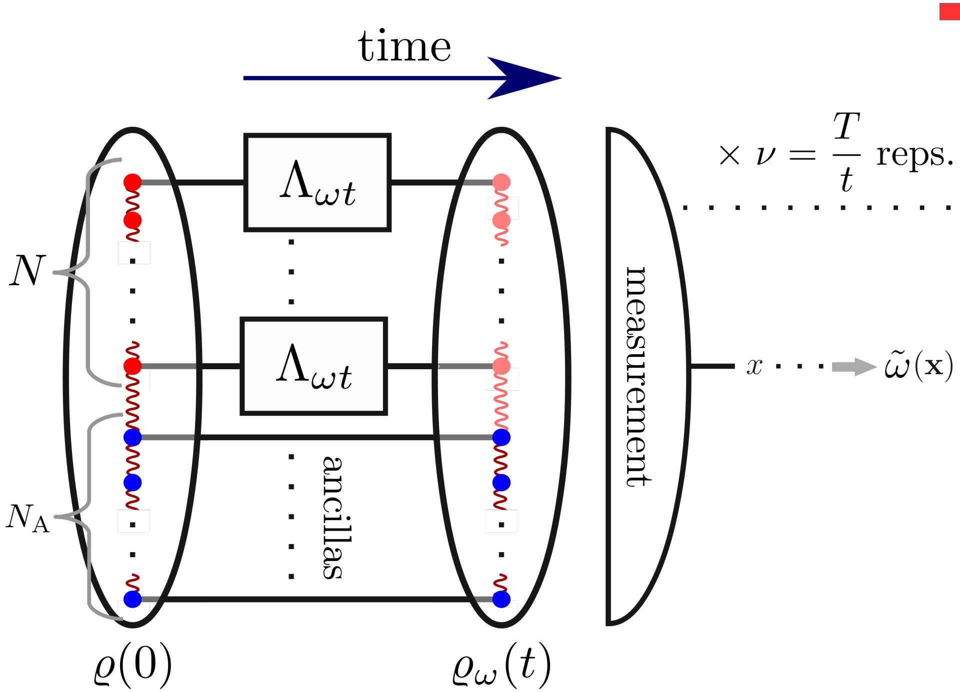


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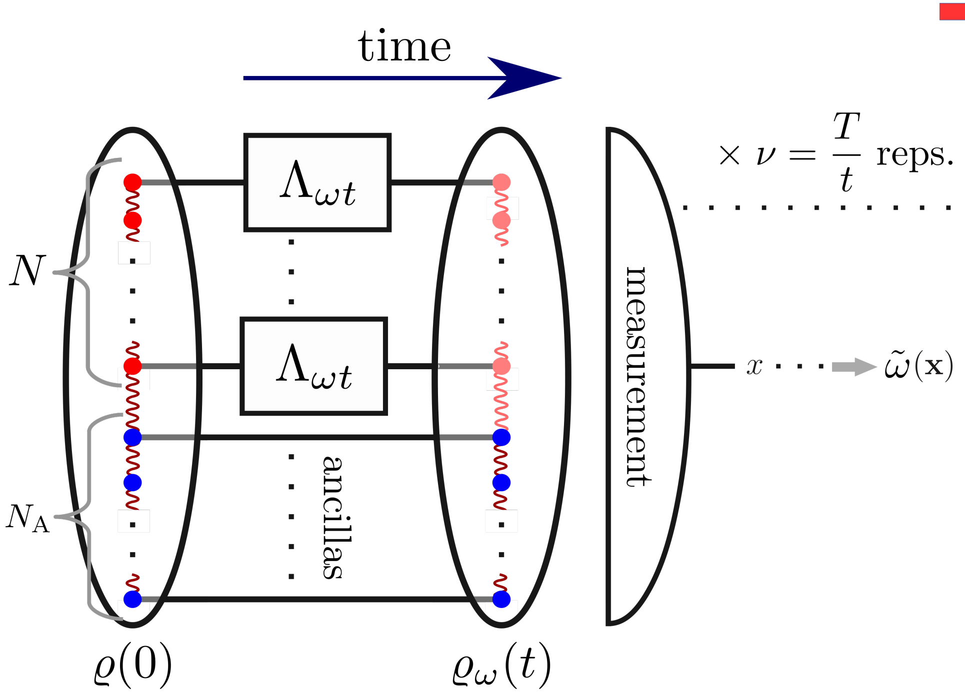


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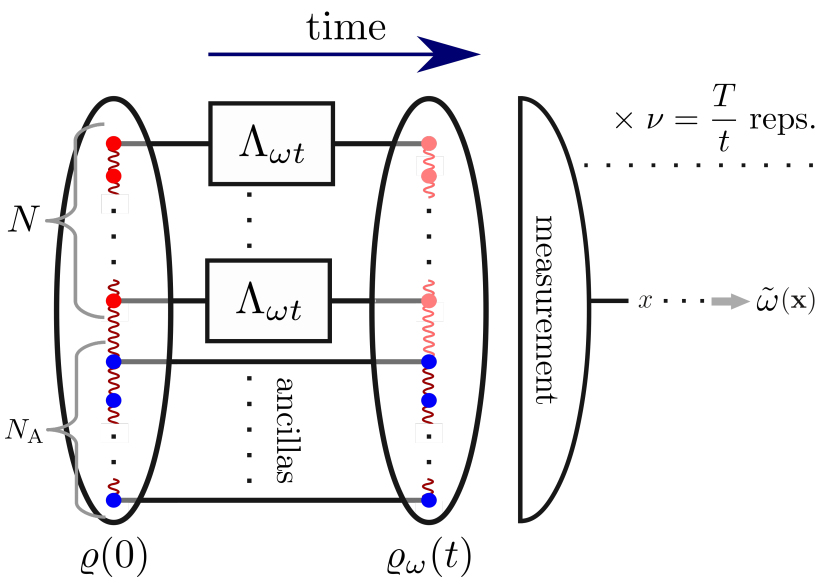


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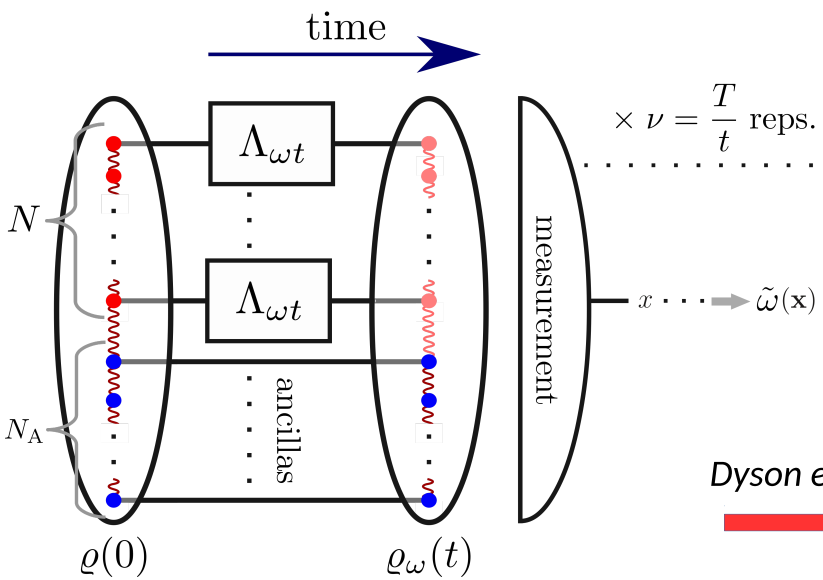


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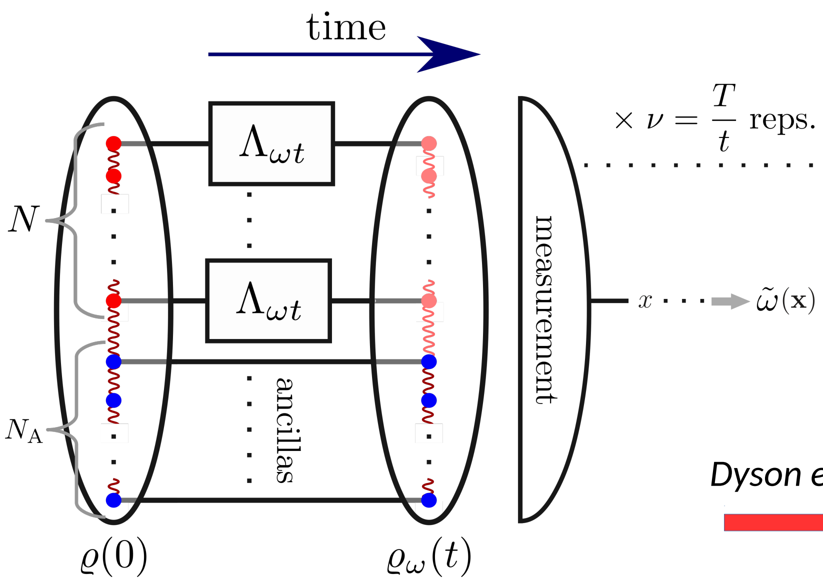


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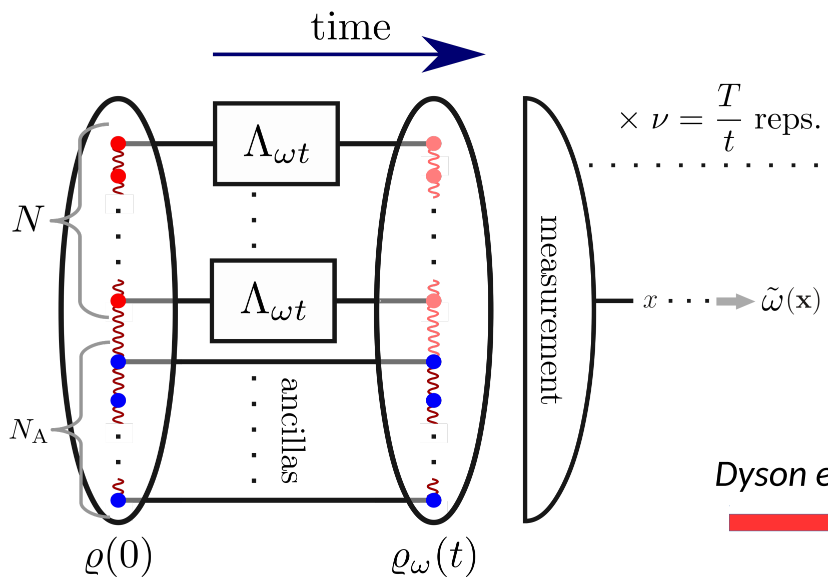


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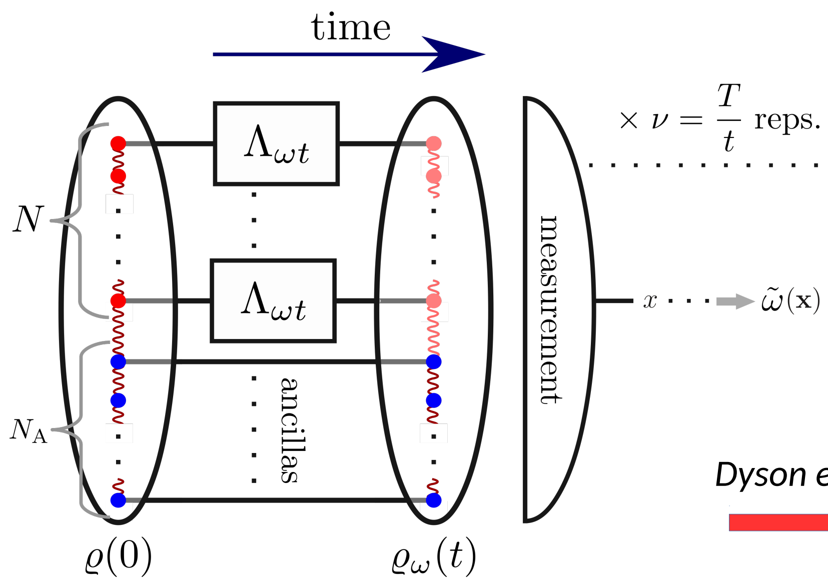


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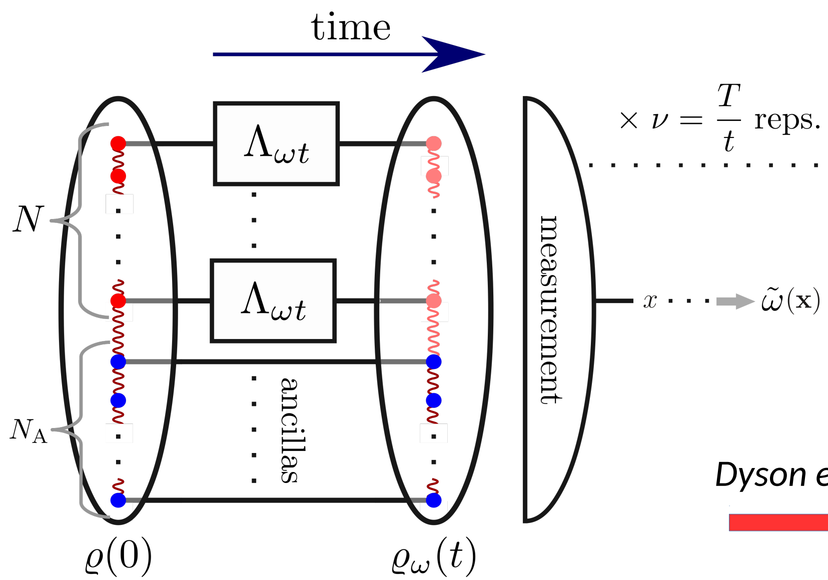


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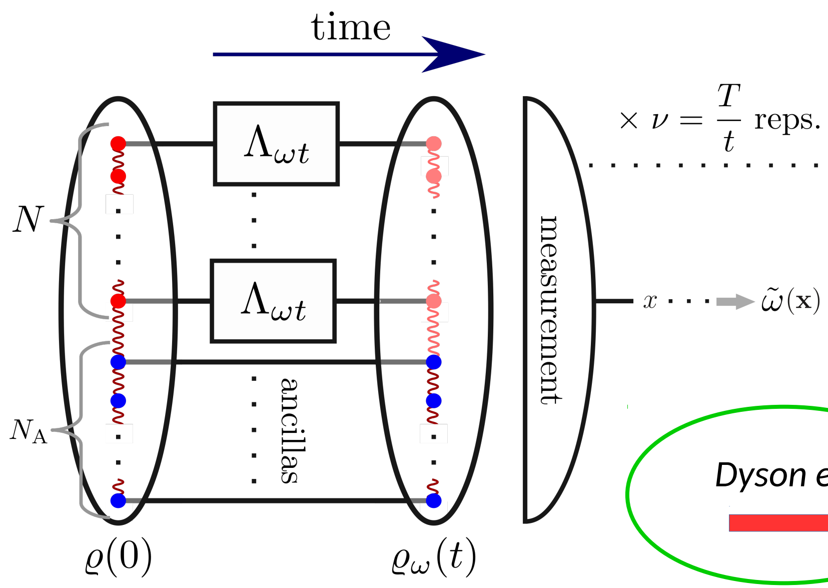


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**Catch...**

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We provide a general ("step by step") recipe how to construct a meaningful bound for a given dynamics.

**Summary of the procedure:**

$$\mathfrak{f}_{N,t} \leq \mathfrak{f}_{N,t}^{\uparrow} := 4 \min_{\mathbf{h}(t)} \left\{ N \left\| \frac{\alpha_{\mathbf{h}}(t)}{t} \right\| + N(N-1) \left\| \frac{\beta_{\mathbf{h}}(t)}{\sqrt{t}} \right\|^2 \right\}$$

4) Minimise the resulting bound by the $\mathbf{h}$-induced parameter : $\chi$

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