On some Computational Method for the Explicit Forms of the Decompositions of Master Equations

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Introduction

Master Equations

The dynamics of quantum systems are described by a certain type of equations.
Let $\mathcal{H}_S$ be a complex Hilbert space and $\mathfrak{S}(\mathcal{H}_S)$ be the set of states:

$$\mathfrak{S}(\mathcal{H}_S) = \{\rho \in \mathcal{B}(\mathcal{H}_S) \mid \rho^* = \rho,\ \rho \geq 0,\ \mathrm{tr}\rho = 1\}.$$

In a (closed) system, the dynamics on $\mathfrak{S}(\mathcal{H}_S)$ is written as

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H_S(t), \rho_t],$$

where $H_S(t)$ is the system Hamiltonian and $\mathcal{I} = \mathbb{R}$ or $\mathcal{I} = [0, \infty)$.

Master Equations

However, our system suffers from the noise from the environment in general.
$\Rightarrow$ Different formulae including the noise effect need be considered.

Master Equations

In order to describe more general systems (open systems), consider the dynamics on $\mathfrak{S}\left(\mathcal{H}_S\right)$:

$$\frac{d}{dt}\rho_t = L\left(t\right)\rho_t,$$

where $L : \mathcal{I} \times \mathfrak{S}\left(\mathcal{H}_S\right) \to \mathfrak{S}\left(\mathcal{H}_S\right)$.

GKSL Equation

A well-known type of equations for open systems is the GKSL equation:

$$\frac{d}{dt}\rho_t = -i\left[H_S, \rho_t\right] + \frac{1}{2}\sum_k c_k\left(\left[F_k\rho_t, F_k^*\right] + \left[F_k, \rho_t F_k^*\right]\right),$$

$$c_k : \text{ scalar},$$
$$F_k \in \mathcal{B}\left(\mathcal{H}_S\right) \text{ s.t. } \left\{F_k\right\}_k \cup \left\{I\right\} \text{ forms a basis of } \mathcal{B}\left(\mathcal{H}_S\right).$$

Master Equations

If $\dim \mathcal{H}_S < \infty$ and we introduce the 'vectorisation' method, we have the 'matrix representation' of $L(t)$:

$$\mathrm{vec}\dot{\rho}_t = \tilde{L}(t)\,\mathrm{vec}\rho_t,$$
$$\tilde{L}(t) \in \mathcal{M}_{n^2}(\mathbb{C}).$$

Master Equations

Especially, when $\dim \mathcal{H}_S = n = 1$, the master equation has the form:

$$\frac{d}{dt}\rho_t = \tilde{\ell}(t)\,\rho_t,$$

which is a solvable system because it can be solved by the "separation of variables".

Master Equations

However, it can be difficult to 'exactly solve' the master equation if $n \geq 2$.
The analysis becomes far difficult if $n \geq 5$
(because the computation of the eigenvalues is difficult due to Abel-Ruffini theorem).
$\Rightarrow$What can we do?

Decomposition of Master Equation

Assume that the matrix form of $\tilde{L}(t)$ has a 'block-diagonal' form:

$$\tilde{L}(t) = \begin{bmatrix} \tilde{L_1}(t) & \\ & \tilde{L_2}(t) \end{bmatrix}.$$

Then, the total master equation can be considered as the union of independent systems:

$$\dot{\rho}_t = \tilde{L}(t)\,\rho_t \Longrightarrow \begin{cases} \dot{\rho}_{1,t} = \tilde{L_1}(t)\,\rho_{1,t} \\ \dot{\rho}_{2,t} = \tilde{L_2}(t)\,\rho_{2,t} \end{cases}$$

Decomposition of Master Equation

The sizes of $\tilde{L_1}, \tilde{L_2}$ are strictly less than n^2.
$\Rightarrow$ Reduction of the dimension
$\Rightarrow$ Simplification of analysis

Decomposition of Master Equation

Next, assume that $\tilde{L}(t)$ can be 'block-diagonalised' by a constant P:

$$P^{-1}\tilde{L}(t)P = \begin{bmatrix} \tilde{L_1}(t) & \\ & \tilde{L_2}(t) \end{bmatrix}.$$

Then, the transformed equation by $\rho_t = P\psi_t$ is

$$\dot{\psi}_t = \begin{bmatrix} \tilde{L_1}(t) & \\ & \tilde{L_2}(t) \end{bmatrix} \psi_t,$$

so again the total master equation can be considered as the union of independent systems.

Decomposition of Master Equation

However, it is not always possible to reduce $\tilde{L}(t)$ into a block-diagonal form.

Question

How to check if $\tilde{L}(t)$ is block-diagonalisable?

Question

If block-diagonalisable, how to compute the 'explicit' form of the decomposition?

Associative Algebra

Associative Algebra

- An **(associative) algebra** $\mathcal{A}$ is defined as a linear space over $\mathbb{C}$ which is closed under the multiplication with the associative law:
$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \;\; (A, B, C \in \mathcal{A}) .$$
- $N \in \mathcal{A}$ is said to be **nilpotent** if $\exists j \in \mathbb{N}$ such that $N^j = O$.
- $P \in \mathcal{A}$ is said to be **properly nilpotent** if PA is nilpotent for all $A \in \mathcal{A}$.

Associative Algebra

- $\mathrm{rad}\mathcal{A}$ is the **radical** of $\mathcal{A}$, which is the set of all properly nilpotent elements in $\mathcal{A}$.
- $\mathcal{A}$ is **semi-simple** if $\mathrm{rad}\mathcal{A} = \{O\}$.
- $\mathcal{A}$ is **simple** if $\mathcal{A}$ has no non-trivial ideal in $\mathcal{A}$.

Wedderburn Decomposition

An important result was shown by Wedderburn:

Theorem

(Wedderburn's theorem). If $\mathcal{A}$ is a finite-dimensional semi-simple algebra over $\mathbb{C}$, there are simple algebras $\mathcal{A}_k$ $(k = 1, \ldots, s)$ such that

$$\mathcal{A} = \bigoplus_{k=1}^{s} \mathcal{A}_k, \tag{1}$$

which is unique up to the order.

J.H.M. Wedderburn. Proc. Lond. Math. Soc. S2-6 (1908), pp. 77–118.
Yu.A. Drozd, V.V. Kirichenko. 'Finite Dimensional Algebras', Springer, 2012.

Wedderburn Decomposition

Relation between the Wedderburn decomposition and the block-diagonal reducibility is the following:

$$\tilde{L}(t): \text{ Generator of the master equation}$$

$$\mathcal{A}: \text{ Algebra generated by } \tilde{L}(t)$$

$$\mathcal{A} = \bigoplus_{k=1}^{s} \mathcal{A}_k \Longleftrightarrow \tilde{L}(t) \simeq \begin{bmatrix} \tilde{L}_1(t) & & \\ & \ddots & \\ & & \tilde{L}_s(t) \end{bmatrix}$$

Burnside's Theorem on Algebra

For the block-diagonal reduction, $s = 1$ or $s \geq 2$ is important.

Theorem

(Burnside's theorem). The only irreducible subalgebra of $\mathcal{M}_n(\mathbb{C})$ with $n \geq 2$ is $\mathcal{M}_n(\mathbb{C})$ itself. In other words, for an algebra $\mathcal{A} \subsetneq \mathcal{M}_n(\mathbb{C})$ with $n \geq 2$, there exists a non-trivial $\mathcal{A}$-invariant subspace.

Thus, $\dim \mathcal{A} < n^2$ iff $s \geq 2$.

W. Burnside. Proc. Lond. Math. Soc. 2 (1905), pp. 369–387.
Yu.A. Drozd, V.V. Kirichenko. 'Finite Dimensional Algebras', Springer, 2012.

Block-Diagonal Reduction of Generator

In order to test the block-diagonal reducibility of the generator $L(t)$,

1. Construct the algebra $\mathcal{A}$ generated by $\tilde{L}(t)$.
2. Construct a basis $\mathcal{B}$ of $\mathcal{A}$.
   - If $\dim \mathcal{B} = n^2$, then $\tilde{L}(t)$ cannot be block-diagonalised.
   - If $\dim \mathcal{B} < n^2$, then $\tilde{L}(t)$ **may be** block-diagonalised.
3. Check if $\mathcal{A}$ is semi-simple or not.
   - If $\mathcal{A}$ is semi-simple, then $\tilde{L}(t)$ can be block-diagonalised.
   - If $\mathcal{A}$ is not semi-simple, then $\tilde{L}(t)$ **cannot** be block-diagonalised.

Discriminant of Algebra

Question

How can we check if $\mathcal{A}$ is semi-simple?

$\Rightarrow$ The discriminant of an algebra can do.
Let $\mathcal{A}$ be the algebra generated by $\tilde{L}(t)$ and $\mathcal{B} = \{B_1, \dots, B_m\}$ be a basis of $\mathcal{A}$.

Theorem

The algebra $\mathcal{A}$ is semi-simple if and only if

$$\mathrm{disc}_{\mathcal{B}}\mathcal{A} = \det \begin{bmatrix} \mathrm{tr} B_1 B_1 & \cdots & \mathrm{tr} B_1 B_m \\ \vdots & \ddots & \vdots \\ \mathrm{tr} B_m B_1 & \cdots & \mathrm{tr} B_m B_m \end{bmatrix} \neq 0.$$

T. Kamizawa. Open Syst. Infor. Dyn. 24 (2017), 1750002.
Y.A. Mitropolsky, A.K. Lopatin. 'Nonlinear Mechanics, Groups and Symmetry'. Springer, 2013.

Explicit Decomposition

Explicit Decomposition of Semi-Simple Algebras

Question

Once an algebra $\mathcal{A}$ is known to be semi-simple and reducible, how can we compute each simple algebra of the Wedderburn decomposition?

$\Rightarrow$ It is enough to compute the identity elements $E_1, \ldots, E_s$ of simple algebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$, respectively, because

$$\mathcal{A}_k = E_k \mathcal{A}.$$

W. Eberly. Comput. Complex. 1 (1991), pp. 183–210.

Centre of Algebra

Moreover, it is enough to compute the identity elements of the 'centre' of simple algebras.

Let $\mathcal{A}$ be an algebra. The centre of $\mathcal{A}$ is defined by

$$Z(\mathcal{A}) = \{A \in \mathcal{A} \mid AB = BA \ (\forall B \in \mathcal{A})\}.$$

The centre is a commutative algebra and $E_k \in Z(\mathcal{A}_k)$.

Centre of Algebra

- Our algebra $\mathcal{A}$ is semi-simple:

$$\mathcal{A} \simeq \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_s,$$

  and in this case, we have

$$Z(\mathcal{A}) \simeq Z(\mathcal{A}_1) \oplus \cdots \oplus Z(\mathcal{A}_s).$$

- An identity element E_k of $\mathcal{A}_k$ is an identity element of $Z(\mathcal{A}_k)$.

Splitting Element

Question

How can we obtain the identity elements?

For the explicit decomposition, an element so-called the 'splitting element' plays an important role.

Splitting Element

Let $\mathcal{A}$ be an m-dimensional algebra over a number field $\mathbb{F}$ (finite extension of $\mathbb{Q}$).
An element $Q \in \mathcal{A}$ is a **splitting element** if its minimal polynomial ψ_Q over $\mathbb{F}$ is square-free (i.e. no multiple root) and $\deg \psi_Q = m$.

W. Eberly. Comput. Complex. 1 (1991), pp. 183–210.

Splitting Element

Splitting elements are very useful to compute algebraic structures. Indeed, it is relatively easy to find, if exists.

Theorem

(Lemma 2.1 in [Eberly91]). Let

$$\mathcal{A} \subset \mathcal{M}_n(\mathbb{F}) : \text{ m-dimensional algebra over a number field } \mathbb{F}$$

$$\mathcal{B} = \{B_k\}_{k=1}^{m} : \text{ Basis of } \mathcal{A}$$

$$c > 0$$

$$H \subset \mathbb{F} : \text{ Finite subset s.t. } |H| = \lceil cn(n-1) \rceil$$

Then, for random elements $\lambda_1, \ldots, \lambda_m \in H$, the element $Q = \sum \lambda_k B_k$ is a splitting element with the probability at least $1 - \frac{1}{c}$, or $\mathcal{A}$ does not contain a splitting element.

W. Eberly. Comput. Complex. 1 (1991), pp. 183–210.

Splitting Element

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Theorem

(Lemma 3.1 in [Eberly91]). Let

$$\mathcal{A} \subset \mathcal{M}_n(\mathbb{F}) : \text{ } m\text{-dimensional algebra over a number field } \mathbb{F}$$

$$\mathcal{B} = \{B_k\}_{k=1}^{m} : \text{ Basis of } \mathcal{A}$$

In addition, if $\mathbb{F}$ is a perfect field (every irreducible polynomial is separable over $\mathbb{F}$), $|\mathcal{A}| \geq m$ and $\mathcal{A}$ is a commutative semi-simple algebra over $\mathbb{F}$, then $\mathcal{A}$ has a splitting element.

W. Eberly. Comput. Complex. 1 (1991), pp. 183–210.

Splitting Element

Once a splitting element is found in an algebra $\mathcal{A}$ over a number field $\mathbb{F}$, one can compute the identity element in each simple algebra of the Wedderburn decomposition.

1. Compute the minimal polynomial ψ_Q of the splitting element Q.
2. Compute the factorisation $\psi_Q = \psi_1 \cdots \psi_s$ into distinct monic irreducible polynomials in $\mathbb{F}[x]$.
3. Compute the polynomials $g_k \in \mathbb{F}[x]$ s.t. $g_k \equiv 1 \pmod{\psi_k}$ and $g_k \equiv 0 \pmod{\psi_j}$ $(j \neq k)$.
4. $E_k = g_k(Q)$ is an identity element of $\mathcal{A}_k$ and

$$\mathcal{A} \simeq \bigoplus_k \mathcal{A}_k \simeq \bigoplus_k (E_k \mathcal{A}) .$$

Explicit Decomposition

To sum up, in order to obtain the explicit decomposition of $\tilde{L}(t)$:

1. Construct an algebra $\mathcal{A}$ over $\mathbb{C}$ generated by $\tilde{L}(t)$.
2. Compute the centre $Z(\mathcal{A})$ and some basis $\mathcal{B}_Z = \{C_k\}_{k=1}^{m_z}$ of $Z(\mathcal{A})$.
   - If there is a perfect number field $\mathbb{F}$ s.t. $\mathcal{B}_Z \subset \mathcal{M}_n(\mathbb{F})$,
3. Find a splitting element ($Q = \sum \lambda_k C_k$ is highly likely to be so).
4. Compute the minimal polynomial ψ_Q of Q and its factorisation $\psi_Q = \psi_1 \cdots \psi_q$.
5. Compute the polynomial $g_k \in \mathbb{F}[x]$ s.t. $g_k \equiv 1 \pmod{\psi_k}$ and $g_k \equiv 0 \pmod{\psi_j}$ $(j \neq k)$.
6. $E_k = g_k(Q)$ is an identity element of $\mathcal{A}_k$ and

$$\mathcal{A} \simeq \bigoplus_k \mathcal{A}_k \simeq \bigoplus_k (E_k \mathcal{A}).$$

Example

Let us consider a master equation:

$$\dot{\rho}_t = -i\left[H(t), \rho_t\right] + \sum_{j=1}^{2} \alpha_j(t) \left\{ \left[F_j(t)\rho_t, F_j^*(t)\right] + \left[F_j(t), \rho_t F_j^*(t)\right] \right\}, \tag{2}$$

where

$$\begin{aligned} J(t), \Gamma_k(t) &: \text{ Time-dependent scalar functions} \\ \sigma_k^j &: \text{ Pauli Matrix } \sigma^j \text{ on the } k\text{th particle} \\ \sigma_j^{\pm} = \sigma_j^x \pm i\sigma_j^y, H(t) &= J(t)\sigma_1^x\sigma_2^y \\ F_1(t) = \sqrt{\Gamma_1(t)}\sigma_1^-, F_2(t) &= \sqrt{\Gamma_2(t)}\sigma_1^+. \end{aligned}$$

T. Prosen. J. Stat. Mech: Theory Exp. 2010, P07020.

Example

1. Set

$$A_1 = \sigma_1^x \sigma_2^y,\ A_2 = \sigma_1^-,\ A_3 = \sigma_1^+,$$

and construct the algebra $\mathcal{A} = \mathcal{A}\left(A_1, A_2, A_3\right)$.

Example

2. Obtain a basis. One can see that $\mathcal{B} = \{B_k\}_{k=1}^{8}$ with

$$B_1 = A_1,\ B_2 = A_1^2,\ B_3 = A_2,\ B_4 = A_3,$$
$$B_5 = A_1A_2,\ B_6 = A_2A_1,\ B_7 = A_2A_3,$$
$$B_8 = A_1A_2A_3$$

forms a basis of $\mathcal{A}$ and the algebra is reducible.

Example

3. Compute the discriminant of $\mathcal{A}$.

$$\mathrm{disc}_{\mathcal{B}}\mathcal{A} = 16777216 \neq 0,$$

so the algebra is completely reducible.

Example

4. Calculate the centre of $\mathcal{A}$.
Simple linear equations reveals that the centre $Z(\mathcal{A})$ is

$$Z(\mathcal{A}) = \{c_1 I + c_2 V \mid c_1, c_2 \in \mathbb{C}\}$$

$$V = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}.$$

Example

5. Obtain a basis of $Z(\mathcal{A})$.

One can check that $\mathcal{B}_z = \{I, V\} \subset \mathcal{M}_4(\mathbb{Q})$ is a basis of $Z(\mathcal{A})$.

Example

6. Find a splitting element.
Put $Q = I + iV$. Then, the minimal polynomial of Q over $\mathbb{F} = \mathbb{Q}[i]$ is

$$\psi_Q(x) = x^2 - 2x = x(x-2) = \psi_1\psi_2,$$

so Q is a splitting element.

Example

7. Find the polynomials g_1, g_2.

One finds that $g_1 = \frac{1}{2}x$, $g_2 = -\frac{1}{2}x + 1$ satisfy

$$g_1 \equiv 0 \pmod{\psi_1},\ g_1 \equiv 1 \pmod{\psi_2}$$
$$g_2 \equiv 1 \pmod{\psi_1},\ g_2 \equiv 0 \pmod{\psi_2}$$

Example

8. Compute the identity elements.

Set $E_1 = g_1(Q)$ and $E_2 = g_2(Q)$, then we obtain the simple algebras $\mathcal{A}_1 = E_1\mathcal{A}$ and $\mathcal{A}_2 = E_2\mathcal{A}$.

Example

In fact, the kernels of E_1, E_2 are:

$$\ker E_1 = \operatorname{span}\left\{[0,0,-i,1]^T, [-i,1,0,0]^T\right\}$$
$$\ker E_2 = \operatorname{span}\left\{[0,0,i,1]^T, [i,1,0,0]^T\right\},$$

so by setting

$$P = \begin{bmatrix} 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

we can check that $H(t), F_1(t), F_2(t)$ can be simultaneously block-diagonalised.

References

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- W. Eberly. Comput. Complex. 1 (1991), pp. 211–234.
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