Schur-Weyl duality in the one-dimensional Hubbard model

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51 Symposium on Mathematical Physics

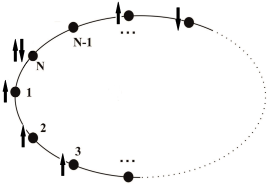
Toruń 2019

Contents

$$\hat{H} = \underbrace{t \sum_{i \in \tilde{2}} \sum_{j \in \tilde{N}} (\hat{c}^{\dagger}_{ji}\hat{c}_{j+1i} + \hat{c}^{\dagger}_{j+1i}\hat{c}_{ji})}_{\textit{waves}} + \underbrace{U \sum_{j \in \tilde{N}} \hat{n}_{j+}\hat{n}_{j-}}_{\textit{particles}}$$

$$\dim h_j = n = 4 \qquad h_j = lc_{\mathbb{C}}\{\uparrow\downarrow, \emptyset, \uparrow, \downarrow\}$$

$$\mathcal{H} = \prod_{j=1}^{N} \otimes\, h_j \qquad \mathcal{H} = \bigoplus_{N_e=0}^{2N} \mathcal{H}^{N_e}$$



The electron configurations

$$|f>=|f(1)f(2)\dots f(N)>=|i_1 i_2 \dots i_N>,\ i_j \in \tilde{4},\ j \in \tilde{N}$$

$$f : \tilde{N} \longrightarrow \tilde{4},\ \ \tilde{4} = \{\uparrow\downarrow, \emptyset, \uparrow, \downarrow\}$$

$$\tilde{4}^{\tilde{N}} = \{f : \tilde{N} \longrightarrow \tilde{4}\}$$

$$|\emptyset\rangle, \quad \hat{c}^{\dagger}_{\uparrow}\ |\emptyset\rangle = |\uparrow\rangle, \quad \hat{c}^{\dagger}_{\downarrow}|\emptyset\rangle = |\downarrow\rangle, \quad \hat{c}^{\dagger}_{\uparrow}\ \hat{c}^{\dagger}_{\downarrow}|\emptyset\rangle = |\uparrow\downarrow\rangle$$

$$N = 6$$

$$|\uparrow, \emptyset, \uparrow\downarrow, \downarrow, \uparrow, \downarrow\rangle \equiv c^{\dagger}_{1,\uparrow}\ c^{\dagger}_{3,\uparrow}\ c^{\dagger}_{3,\downarrow} c^{\dagger}_{4,\downarrow} c^{\dagger}_{5,\uparrow}\ c^{\dagger}_{6,\downarrow}|\emptyset\rangle$$

The translational symmetry

$$\hat{c}_{N+1,\sigma} = \hat{c}_{1,\sigma}$$

$|f_j; f^i\rangle$ - the j-th electron configuration of the orbit $\mathcal{O}_{f^i}$ of the cyclic group C_N

$$c_N |f_j; f^i\rangle = |f_{(j+1)_{modN}}; f^i\rangle$$

$$(c_N)^j |f_1; f^i\rangle = |f_j; f^i\rangle, \quad c_N \in C_N$$

$$\mathcal{O}_{f^i} = \{(c_N)^j |f_1; f^i\rangle | j \in \{0, 1, 2, ..., N-1\}\}$$

$$c_N (c^\dagger_{a,\sigma_a} c^\dagger_{b,\sigma_b} \cdots c^\dagger_{c,\sigma_c} |0\rangle) = c^\dagger_{(a+1)_{modN},\sigma_a} c^\dagger_{(b+1)_{modN},\sigma_b} \cdots c^\dagger_{(c+1)_{modN},\sigma_c} |0\rangle$$

$$a, b, c \in \tilde{N}, \quad \sigma_a, \sigma_b, \sigma_c \in \tilde{2} = \{\uparrow, \downarrow\}$$

The translational symmetry

The Hamiltonian for one electron is completely diagonalised by a Fourier transformation in the form

$$a_k^\dagger = \frac{1}{\sqrt{N}} \sum_{j \in \tilde{N}} \exp(i2\pi kj/N) a_j^\dagger, \; k \in B,$$

$$B = \{k = 0, \pm 1, \pm 2, \ldots, \begin{cases} \pm(N/2-1), & \text{for N even} \\ \pm(N-1)/2, & \text{for N odd} \end{cases}\}$$

$$H = \bigoplus_{k \in B} H(k)$$

Spin and pseudospin symmetry $SU(2) \times SU(2)$

$$S_z = \frac{1}{2}\sum_{j\in\tilde{N}}(c^\dagger_{j+}c_{j+} - c^\dagger_{j-}c_{j-})$$

$$S_+ = S_-{}^\dagger = \sum_{j\in\tilde{N}} c^\dagger_{j+}c_{j-}$$

$$c^\dagger_{j\sigma} \to (-1)^j c_{j\sigma}$$

$$J_z = \frac{1}{2}\sum_{j\in\tilde{N}}(c^\dagger_{j+}c_{j+} + c^\dagger_{j-}c_{j-} - 1)$$

$$J_+ = J_-{}^\dagger = \sum_{j\in\tilde{N}} (-1)^j c^\dagger_{j+}c^\dagger_{j-}$$

Spin and pseudospin symmetry $SU(2) \times SU(2)$

$$(\mathrm{S})^2 = \frac{1}{2}[(S_+)^2 + (S_-)^2] + (S_z)^2$$

$$(\mathrm{J})^2 = \frac{1}{2}[(J_+)^2 + (J_-)^2] + (J_z)^2$$

$$J^2|+\rangle_j = J^2|-\rangle_j = 0,$$

$$J^2\,\epsilon_j|\emptyset\rangle_j = \tfrac{3}{4}\epsilon_j|\emptyset\rangle_j,\;\; J^2|\pm\rangle_j = \tfrac{3}{4}|\pm\rangle_j$$

$$S^2\,\epsilon_j|\emptyset\rangle_j = S^2|\pm\rangle_j = 0,$$

$$S^2|+\rangle_j = \tfrac{3}{4}|+\rangle_j,\;\; S^2|-\rangle_j = \tfrac{3}{4}|-\rangle_j,$$

$$\epsilon_j = \begin{cases} 1 & \text{if } j \in A \\ -1 & \text{if } j \in B \end{cases}$$

The example of $N = 4, N_+ = 3, N_- = 1, u >> t$

μ	f^i	$\mathcal{O}_{f^i}$
$(3,1,0,0)$	$\vert -+++\rangle$	$c^\dagger_{1-}c^\dagger_{2+}c^\dagger_{3+}c^\dagger_{4+}\vert\emptyset\equiv\rangle\vert -+++\rangle$ $c^\dagger_{1+}c^\dagger_{2-}c^\dagger_{3+}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert +-++\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{3-}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert ++-+\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{3+}c^\dagger_{4-}\vert\emptyset\rangle\equiv\vert +++-\rangle$
$(2,0,1,1)$	$\vert \pm++\emptyset\rangle$	$c^\dagger_{1+}c^\dagger_{1-}c^\dagger_{2+}c^\dagger_{3+}\vert\emptyset\rangle\equiv\vert \pm++\emptyset\rangle$ $c^\dagger_{2+}c^\dagger_{2-}c^\dagger_{3+}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert\emptyset\pm++\rangle$ $c^\dagger_{1+}c^\dagger_{3+}c^\dagger_{3-}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert +\emptyset\pm+\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{4+}c^\dagger_{4-}\vert\emptyset\rangle\equiv\vert ++\emptyset\pm\rangle$
	$\vert\emptyset++\pm\rangle$	$c^\dagger_{2+}c^\dagger_{3+}c^\dagger_{4+}c^\dagger_{4-}\vert\emptyset\rangle\equiv\vert\emptyset++\pm\rangle$ $c^\dagger_{1+}c^\dagger_{1-}c^\dagger_{3+}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert \pm\emptyset++\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{2-}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert +\pm\emptyset+\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{3+}c^\dagger_{3-}\vert\emptyset\rangle\equiv\vert ++\pm\emptyset\rangle$
	$\vert \pm+\emptyset+\rangle$	$c^\dagger_{1+}c^\dagger_{1-}c^\dagger_{2+}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert \pm+\emptyset+\rangle$ $c^\dagger_{1+}c^\dagger_{2+}c^\dagger_{2-}c^\dagger_{3+}\vert\emptyset\rangle\equiv\vert +\pm+\emptyset\rangle$ $c^\dagger_{2+}c^\dagger_{3-}c^\dagger_{3+}c^\dagger_{4+}\vert\emptyset\rangle\equiv\vert\emptyset+\pm+\rangle$ $c^\dagger_{1+}c^\dagger_{3+}c^\dagger_{4+}c^\dagger_{4-}\vert\emptyset\rangle\equiv\vert +\emptyset+\pm\rangle$

The example of $N = 4, N_+ = 3, N_- = 1, k = 0$

The Hubbard Hamiltonian acting on the Hilbert space $\mathcal{H}^{k=0}$

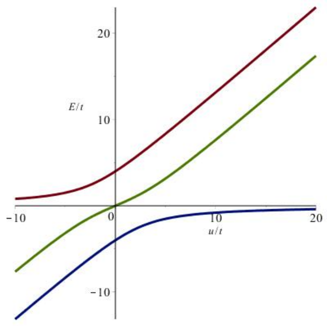
$$\begin{bmatrix} 0 & 2t & 2t & 0 \\ 2t & u & 0 & -2t \\ 2t & 0 & u & -2t \\ 0 & -2t & -2t & u \end{bmatrix}$$

The Hubbard Hamiltonian prepared in the basis
$|k = 0, S, J, S_z = 1, J_z = 0\rangle$

$$\left[\begin{array}{ccc|c} 0 & 2\sqrt{2}t & 0 & 0 \\ 2\sqrt{2}t & u & -2\sqrt{2}t & 0 \\ 0 & -2\sqrt{2}t & u & 0 \\ \hline 0 & 0 & 0 & u \end{array}\right].$$

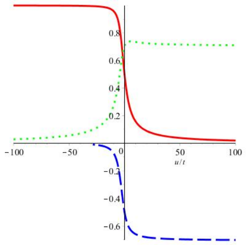
The example of $N = 4, N_+ = 3, N_- = 1, k = 0$

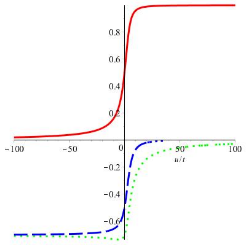
Spectrum of the submatrix of the dimension $\mathbf{3} \times \mathbf{3}$ of the Hubbard Hamiltonian

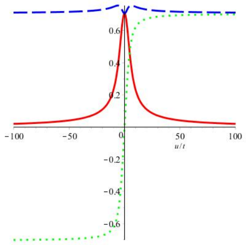


The example of $N = 4, N_+ = 3, N_- = 1, k = 0$

The three eigenvectors of the submatrix of the dimension $\mathbf{3} \times \mathbf{3}$ of the Hubbard Hamiltonian







The example of $N = 4, N_+ = 3, N_- = 1, k = 0$

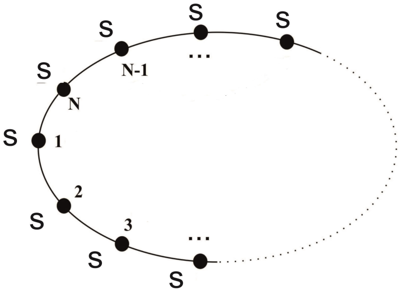
This result coincide with the eigenvalue $E = -2$ obtained by solving the antiferromagnetic Heisenberg model for the case of four nodes $N = 4$ of the magnetic chain, one spin deviation $r = 1$, quasimomentum $k = 2$, and with exchange coupling $J = 4t^2/u$:

$$J \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

D. Jakubczyk, The one-dimensional Hubbard model in the limit of $U >> t$, Reports on Mathematical Physics, 83, 139-162 (2019)

The Heisenberg XXX model

$$\hat{H} = \frac{J}{2}\sum_{n=1}^{N}(P_{n\,n+1} - I_{n\,n+1})$$



$$P_{n\,n+1} = \frac{1}{2}(\vec{\sigma}_n\vec{\sigma}_{n+1} + I_{n\,n+1})$$

Magnetic configurations

$$|f>=|f(1)f(2)\dots f(N)>=|i_1i_2\dots i_N>, \ i_j\in\tilde{2}, \ j\in\tilde{N}$$

$$f:\tilde{N}\longrightarrow\tilde{2}, \ \ \tilde{2}=\{\uparrow,\downarrow\}$$

$$\tilde{2}^{\tilde{N}}=\{f:\tilde{N}\longrightarrow\tilde{2}\}$$

$$\mathcal{H}=(\mathbb{C}^2)^{\otimes N}$$

$$\dim\mathcal{H}=2^N$$

Schur-Weyl duality

$$\mathcal{H} = h^{\otimes N} = \bigoplus_{\lambda \vdash N} (U^{\lambda} \otimes V^{\lambda})$$

$$A : \Sigma_N \times \mathcal{H} \to \mathcal{H}$$

$$B : U(n) \times \mathcal{H} \to \mathcal{H}$$

$$A(\sigma)|f\rangle = |i_{\sigma^{-1}(1)}, ..., i_{\sigma^{-1}(N)}\rangle, f \in \tilde{n}^{\tilde{N}}, \sigma \in \Sigma_N$$

$$B(u)|f\rangle = |ui_1, ..., ui_N\rangle, u \in U(n)$$

$$[A(\sigma), B(u)] = 0$$

Schur-Weyl duality

$$A = \sum_{\lambda \in D_W(N,n)} m(A, \Delta^\lambda)\ \Delta^\lambda$$

$$B = \sum_{\lambda \in D_W(N,n)} m(B, D^\lambda)\ D^\lambda$$

$$m(A, \Delta^\lambda) = \dim D^\lambda$$

$$m(B, D^\lambda) = \dim \Delta^\lambda$$

The actions

$$A:\Sigma_N\times\tilde{4}^{\tilde{N}}\longrightarrow\tilde{4}^{\tilde{N}},\;\; B:U(4)\times\tilde{4}^{\tilde{N}}\longrightarrow\tilde{4}^{\tilde{N}}$$

are replaced by

$$A':\Sigma_{N'}\times\tilde{2'}^{\tilde{N}'}\longrightarrow\tilde{2'}^{\tilde{N}'},\;\; B':SU(2)\times\tilde{2'}^{\tilde{N}'}\longrightarrow\tilde{2'}^{\tilde{N}'}$$

$$\mathcal{H}_s=lc_{\mathbb{C}}\;\tilde{2'}^{\tilde{N}'}=h_s^{\otimes N'},h_s\cong\mathbb{C}_2,\;\tilde{2'}=\{+,-\}$$

in the spin space

$$A'':\Sigma_{N''}\times\tilde{2''}^{\tilde{N}''}\longrightarrow\tilde{2''}^{\tilde{N}''},\;B'':SU(2)\times\tilde{2''}^{\tilde{N}''}\longrightarrow\tilde{2''}^{\tilde{N}''}$$

$$\mathcal{H}_p=lc_{\mathbb{C}}\;\tilde{2''}^{\tilde{N}''}=h_p^{\otimes N''},h_p\cong\mathbb{C}_2,\;\tilde{2''}=\{\pm,\emptyset\}$$

in the pseudo-spin space

Schur-Weyl duality in the Hubbard model

$$\mathcal{H}_{int} = \bigoplus_{(\tilde{N}', \tilde{N}'')} (\mathcal{H}_s \otimes \mathcal{H}_p) \cong \bigoplus_{(\tilde{N}', \tilde{N}'')} \left[(\mathbb{C}_2)^{\otimes N'} \otimes (\mathbb{C}_2)^{\otimes N''} \right]$$

$$\tilde{N}' \cup \tilde{N}'' = \tilde{N}, \tilde{N}' \cap \tilde{N}'' = \emptyset$$

$$\dim \mathcal{H}_{int} = \tilde{4}^{\tilde{N}}$$

$$\mathcal{H}_{int} = \bigoplus_{N_e=0}^{2N} \mathcal{H}^{N_e}$$

$$\mathcal{H}^{N_e} = \bigoplus_{(N_+, N_-)} \mathcal{H}^{N_e}_{(N_+, N_-)}, \; N_+ + N_- = N_e$$

$$R^{\Sigma_{N'}:(\Sigma_{\mu_1}\times\Sigma_{\mu_2})}$$

$$R^{\Sigma_{N'}:(\Sigma_{N'-\mu_2}\times\Sigma_{\mu_2})} \cong \sum_{r=0}^{\mu_2} \Delta^{\{N'-r,r\}}$$

$$S = \frac{N'}{2} - r, \quad 0 \le r \le \mu_2$$

$$S_z = \frac{N'}{2} - \mu_2$$

Schur-Weyl duality in the Hubbard model

$$R^{\Sigma_{N''}:(\Sigma_{\mu_3}\times\Sigma_{\mu_4})}$$

$$R^{\Sigma_{N''}:\left(\Sigma_{\frac{N''}{2}}\times\Sigma_{\frac{N''}{2}}\right)}\cong\sum_{r=0}^{\frac{N''}{2}}\Delta^{\{N''-r,r\}}$$

$$J=\frac{N''}{2}-r,\;\;0\leq r\leq\frac{N''}{2}$$

$$J_z=\frac{N''}{2}-\mu_4=0$$

$$N_+ \times N_- = \{(8,0),(7,1),(6,2),(5,3),(4,4),(3,5),$$

$$(2,6),(1,7),(0,8)\}$$

$$\dim \mathcal{H}_{int} = \sum_{N_e=0}^{16} \dim \mathcal{H}^{N_e} =$$

$$= 1 + 16 + 120 + 560 + 1820 + 4368 + 8008 + 11440+$$

$$+12870 + 11440 + 8008 + 4368 + 1820 + 560 + 120+$$

$$+16 + 1 = 65\,536 = 4^8$$

The example of $N = 8$

$$\mathcal{H} = \mathcal{H}^{N_e=8} = \bigoplus_{(N_+,N_-)} \mathcal{H}^{N_e=8}_{(N_+,N_-)}, \;\; N_+ + N_- = 8$$

$$\dim \mathcal{H} = \dim \mathcal{H}^8_{(8,0)} + \dim \mathcal{H}^8_{(7,1)} + \dim \mathcal{H}^8_{(6,2)} + \dim \mathcal{H}^8_{(5,3)} +$$

$$\dim \mathcal{H}^8_{(4,4)} + \dim \mathcal{H}^8_{(3,5)} + \dim \mathcal{H}^8_{(2,6)} + \dim \mathcal{H}^8_{(1,7)} +$$

$$\dim \mathcal{H}^8_{(0,8)},$$

$$\dim \mathcal{H} = 1 + 64 + 784 + 3136 + 4900 +$$

$$+3136 + 784 + 64 + 1 = 12\,870$$

The example of $N = 8$, $N_+ = 5$, $N_- = 3$, $S_z = 1$, $J_z = 0$

μ'	μ''	$\lambda' \vdash N'$	$\lambda'' \vdash N''$	S	J	$\dim\Delta^{\lambda'}$	$\dim\Delta^{\lambda''}$	τ	x_μ
$(5,3)$	$(0,0)$	$\{8\}$	$-$	4	$-$	1	$-$	1	56
		$\{7\,1\}$	$-$	3	$-$	7	$-$		
		$\{6\,2\}$	$-$	2	$-$	20	$-$		
		$\{5\,3\}$	$-$	1	$-$	28	$-$		
$(4,2)$	$(1,1)$	$\{6\}$	$\{2\}$	3	1	1	1	28	840
		$\{5\,1\}$	$\{1^2\}$	2	0	5	1		
		$\{4\,2\}$		1		9			
$(3,1)$	$(2,2)$	$\{4\}$	$\{4\}$	2	2	1	1	70	1 680
		$\{3\,1\}$	$\{3\,1\}$	1	1	3	3		
			$\{2^2\}$		0		2		
$(2,0)$	$(3,3)$	$\{2\}$	$\{6\}$	1	3	1	1	28	560
			$\{5\,1\}$		2		5		
			$\{4\,2\}$		1		9		
			$\{3^2\}$		0		5		

$= 3\,136$

Conclusions and remarks

We presented the application of the Schur-Weyl duality in the one-dimensional Hubbard model in the case of half-filling for any number of atoms.

We showed the way of using the Schur-Weyl duality in spin and pseudo-spin space in order to obtain the total spin $\boldsymbol{S}$ and the total pseudo-spin $\boldsymbol{J}$.

We created both the spin and pseudo-spin spaces as the appropriate tensor product of the one-node spin and pseudo-spin spaces and provide the space of all quantum states of the considered system taken over all possible locations of these spaces on the $\boldsymbol{N}$- atoms chain.

We used the concept of the initial Hilbert space which provides the proper Hilbert space of the considered system as its subspace, since there is the confinement of half-filling.

We gave the expressions for the total spin $\boldsymbol{S}$ and the total pseudo-spin $\boldsymbol{J}$ in context of the representation theory of the symmetric group.

The calculations are significant since there is a lack of analytical calculations in the literature of using of the symmetry $\boldsymbol{SU(2) \times SU(2)}$, which is crucial in understanding the Hubbard model.

Thank you for your attention!