On Entropy for Quantum Compound Systems

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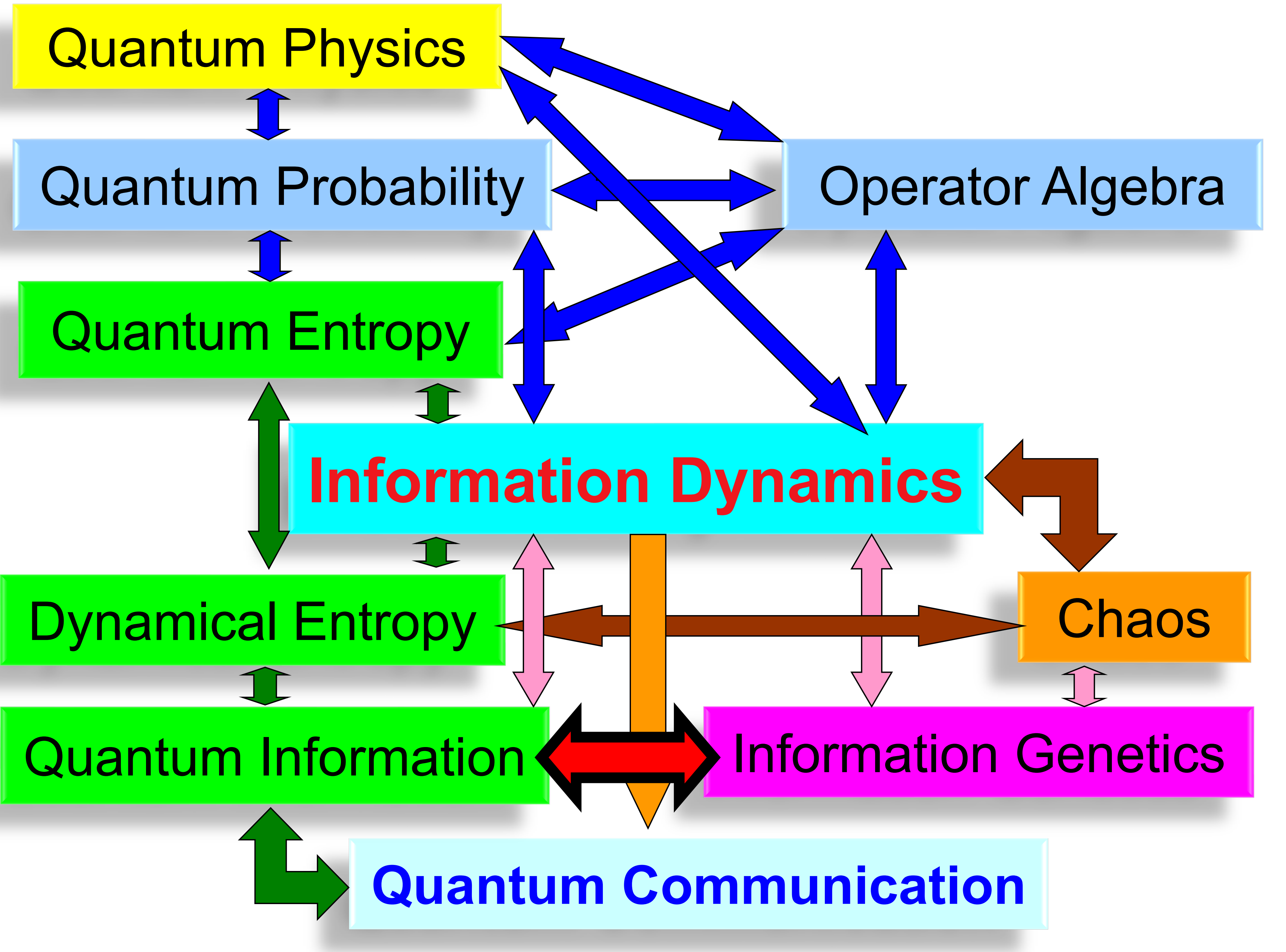
Introduction

1. Information Dynamics (M. Ohya)
2. Complexity (C)
3. Channel
4. Transmitted Complexity (T)
5. Other mutual entropy type complexities (T)
   How to construct compound states


Quantum Physics
Quantum Probability
Operator Algebra
Quantum Entropy
Information Dynamics
Dynamical Entropy
Chaos
Quantum Information
Information Genetics
Quantum Communication

1. Information Dynamics [M. Ohya]

Information dynamics is an attempt to provide a new view for the study of complex systems and their chaotic behavior

Information Dynamics = Synthesis of dynamics of state change and complexity of state

Two Complxities

$C^{\mathcal{S}}(\varphi)$

$T^{S}(\varphi;\Lambda^{*})$

(1)Complexity of a state describing the system

Complexity of state

(2) complexity of a dynamics describing the state change

Transmitted complexity

Examples of Complexity of state

entropy, fractal dimension, ergodicity, etc.

Examples of Transmitted Complexity

chaos degree, Lyapunov exponent, dynamical entropy, computational complexity, etc.

1. Information Dynamics [M. Ohya]

Information Dynamics

$$\Leftrightarrow \boxed{\left(\mathcal{A},\mathfrak{S},\mathcal{S},\alpha;\bar{\mathcal{A}},\bar{\mathfrak{S}},\bar{\mathcal{S}},\bar{\alpha};\Lambda^*;C^{\mathcal{S}}(\varphi),T^{\mathcal{S}}(\varphi;\Lambda^*);R\right)}$$

where R is a certain relation among above quantities.

Therefore, in Information Dynamics we have to

(i) **mathematically determine** $\mathcal{A},\mathfrak{S},\mathcal{S},\alpha;\bar{\mathcal{A}},\bar{\mathfrak{S}},\bar{\mathcal{S}},\bar{\alpha}$,

(ii) **choose** Λ^* **and** R,

(iii) **define** $C^{\mathcal{S}}(\varphi)$, $T^{\mathcal{S}}(\varphi;\Lambda^*)$.

⇒ **One can apply several fields including**

(1) Recognition of Chaos,
(2) Quantum SAT Algorithm,
(3) From DNA (Amino Acid Sequences) to Life Science
(4) Quantum Information Communication

1. Information Dynamics [M. Ohya]

Property of Two Complexities in ID [M. Ohya]

C **and** T **should satisfy the following properties :**

(1) $C^{\mathcal{S}}(\varphi) \geq 0$ and $T^{\mathcal{S}}(\varphi; \Lambda^*) \geq 0$ for any $\varphi \in \mathcal{S} \subset \mathfrak{S}$.

(2) For any **bijection** $j : ex\mathfrak{S} \to ex\mathfrak{S}$ (the set of all **extremal elements** of $\mathfrak{S}$),

- $C^{j(\mathcal{S})}(j(\varphi)) = C^{\mathcal{S}}(\varphi)$, $T^{j(\mathcal{S})}(j(\varphi); \Lambda^*) = T^{\mathcal{S}}(\varphi; \Lambda^*)$

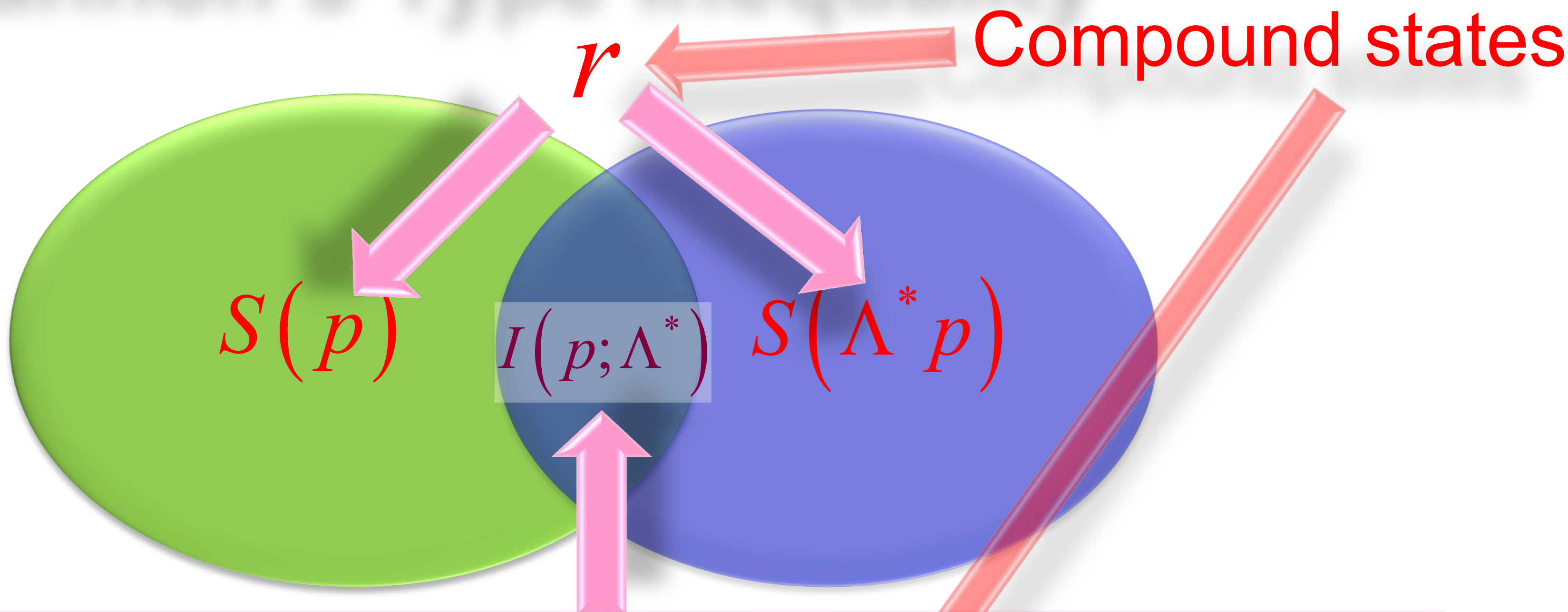
(3) For any $\tilde{\varphi} \in \tilde{\mathcal{S}} \subset \tilde{\mathfrak{S}}$, put $\varphi \equiv \tilde{\varphi} \uparrow \mathcal{A} \in \mathcal{S} = \tilde{\mathcal{S}} \uparrow \mathcal{A}$, $\overline{\varphi} \equiv \tilde{\varphi} \uparrow \overline{\mathcal{A}} \in \overline{\mathcal{S}} = \tilde{\mathcal{S}} \uparrow \overline{\mathcal{A}}$,

- $C^{\tilde{\mathcal{S}}}(\tilde{\varphi}) \leq C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\overline{\varphi})$, $C^{\mathcal{S} \otimes \overline{\mathcal{S}}}(\varphi \otimes \overline{\varphi}) = C^{\mathcal{S}}(\varphi) + C^{\overline{\mathcal{S}}}(\overline{\varphi})$

(4) $0 \leq T^{\mathcal{S}}(\varphi; \Lambda^*) \leq C^{\mathcal{S}}(\varphi)$

(5) $T^{\mathcal{S}}(\varphi; id) = C^{\mathcal{S}}(\varphi)$, $id =$ **identity channel**

1. Shannon's Type Inequality



$I(p;\Lambda^*)$ Mutual Entropy (Information)

Shannon's Type inequalities

$$0 \le I(p;\Lambda^*) = S(r, p \otimes \Lambda^* p) \le S(p)$$

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2. Complexity (C) (Entropy Type)

1) Shannon Entropy [Shannon]

$$C_1^S(p) \quad \Leftrightarrow \quad S(p) \equiv -\sum_k p_k \log p_k$$

2) Entrop of finite decomposition (straight extension of Shannon entropy) for Gaussian measure

$$C_2^S(\mu) \quad \Leftrightarrow \quad S(\mu) = \sup\left\{ -\sum_{A_k \in \tilde{\mathcal{A}}} \mu(A_k) \log \mu(A_k); \tilde{\mathcal{A}} \in \mathcal{P}(\mathcal{B}_2) \right\},$$

where $\mathcal{P}(\mathcal{B}_2)$ is the set of all finite partitions of $\mathcal{B}_2$.

3) Differential entropy [Gibbs]

$$C_3^S(\mu) \quad \Leftrightarrow \quad S(\mu) = -\int_{\mathbf{R}^2} \frac{d\mu}{dm} \log \frac{d\mu}{dm} dm$$

2. A. Entropy Type Complexity

4) Von Neumann Entropy [von Neumann]

$$C_4^S(\rho) \quad \Leftrightarrow \quad S(\rho) \equiv -tr\,\rho \log \rho$$

5) $\mathcal{S}$-mixing entropy [M. Ohya 1985] **(CNT entropy 1987)**

$$C_5^S(\rho) \quad \Leftrightarrow \quad S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu);\ \mu \in D_\varphi(\mathcal{S})\} & \left(D_\varphi(\mathcal{S}) \neq \emptyset\right), \\ \infty & \left(D_\varphi(\mathcal{S}) = \emptyset\right). \end{cases}$$

5) $\mathcal{S}$ -mixing entropy [Ohya]

$(\mathcal{A}, \mathfrak{S}(\mathcal{A}))$; a C*-system,

$\mathcal{S} \subset \mathfrak{S}(\mathcal{A})$; a weak* compact convex subset of $\mathfrak{S}(\mathcal{A})$

$\varphi \in \mathcal{S}$; It is decomposed such as $\varphi = \int_{\mathcal{S}} \omega \, d\mu$

where μ is maximum measures pseudosupported on ***exS*** (the set of all extremal points) in ***S*** . The measure μ giving the above decomposition is not unique.

$M_\varphi(\mathcal{S})$; the set of all such measures for $\varphi \in \mathcal{S}$

$$D_\varphi(\mathcal{S}) \equiv \left\{ \mu \in M_\varphi(\mathcal{S}); \exists\{\mu_k\} \subset \mathbf{R}^+ \ and \ \{\varphi_k\} \subset \mathrm{ex}\mathcal{S} \ s.t. \sum_k \mu_k = 1, \ \mu = \sum_k \mu_k \delta(\varphi_k) \right\},$$

where $\delta(\varphi)$ is Dirac measure concentrated on $\{\varphi\}$.

For a measure $\mu \in D_\varphi(\mathcal{S})$, $H(\mu) = -\sum_k \mu_k \log \mu_k$

S-mixing entropy of a general state $\varphi \in \mathcal{S}$ w.r.t. $\mathcal{S}$ is defined by

$$C_5^S(\rho) \quad \Leftrightarrow \quad S^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); \mu \in D_\varphi(\mathcal{S})\} & \left(D_\varphi(\mathcal{S}) \neq \emptyset\right), \\ \infty & \left(D_\varphi(\mathcal{S}) = \emptyset\right). \end{cases}$$

5) $\mathcal{S}$ -mixing entropy [Ohya]

Theorem [Ohya] **Properties of $\mathcal{S}$-mixing entropy**

If $\mathcal{A} = \mathbf{B}(\mathcal{H})$ and $\alpha_t = Ad(U_t)$ (i.e., $\alpha_t(A) = U_t^* A U_t$ for any $A \in \mathcal{A}$) with a unitary operator U_t, for any state φ, given by $\varphi(\cdot) = tr\rho\,\cdot$ with a density operator ρ , the following facts hold:

1. $S(\varphi) = -tr\rho\log\rho = S(\rho)$; v.N. entropy
2. If φ is an α-invariant faithful state and every eigenvalue of ρ is **non-degenerate**, then $S^{I(\alpha)}(\varphi) = S(\varphi)$, where $I(\alpha)$ is the set of all α-invariant faithful states.
3. $\varphi \in K(\alpha)$, then $S^{K(\alpha)}(\varphi) = 0$, where $K(\alpha)$ is the set of all KMS states.

Theorem [Ohya] For any $\varphi \in K(\alpha)$, we have

1. $S^{K(\alpha)}(\varphi) \le S^{I(\alpha)}(\varphi)$
2. $S^{K(\alpha)}(\varphi) \le S(\varphi)$

(1) This $\mathcal{S}$ - mixing entropy gives a **measure of the uncertainty observed from the reference system.**
(2) This entropy can be applied to characterize **normal states** and **quantum Markov chains** in von Neumann algebras.

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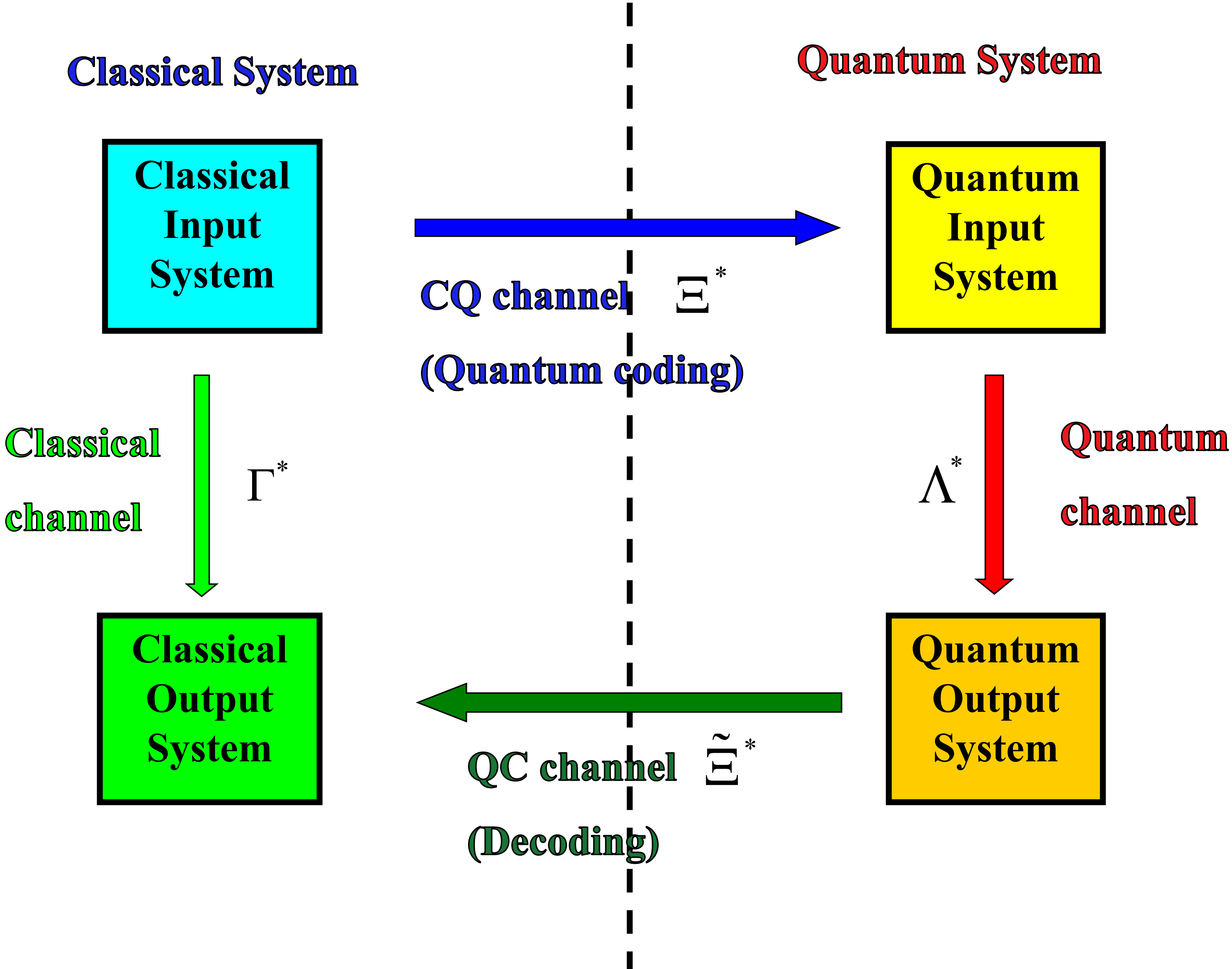
Communication Process



3. Channel

A. Classical Channel

1) Transition Probability Matrix

$$\Lambda^* : \Delta_n \to \Delta_m,\ \Delta_n = \left\{ p = \{p_1, \cdots, p_n\};\ p_i \geq 0\,(\forall i),\ \sum_{i=1}^{n} p_i = 1 \right\}$$

$$\Lambda^* = \begin{pmatrix} p(1|1) & p(1|2) & p(1|3) & \cdots & p(1|n) \\ p(2|1) & p(2|2) & p(2|3) & \cdots & p(2|n) \\ p(3|1) & p(3|2) & p(3|3) & \cdots & p(3|n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p(m|1) & p(m|2) & p(m|3) & \cdots & p(m|n) \end{pmatrix}, \quad \sum_{j=1}^{m} p(j|k) = 1, (k = 1, 2, \cdots, n)$$

2) Gaussian channels

Γ : $\mathbf{P}_G^{(1)} \to \mathbf{P}_G^{(2)}$; a map from $\mathbf{P}_G^{(1)}$ to $\mathbf{P}_G^{(2)}$ is defined by the Gaussian channel

$\lambda : \mathcal{H}_1 \times \mathcal{B}_2 \to [0,1]$ such as $\Gamma(\mu_1)(Q) \equiv \int_{\mathcal{H}_1} \lambda(x,Q) d\mu_1(x)$

$$\lambda(x,Q) \equiv \mu_0(Q^x),\ Q^x \equiv \{y \in \mathcal{H}_2;\ Ax + y \in Q\},\ x \in \mathcal{H}_1,\ Q \in \mathcal{B}_2,$$

where A is a linear transformation from $\mathcal{H}_1$ to $\mathcal{H}_2$, λ satisfies the following conditions:

(1) $\lambda(x,\bullet) \in \mathbf{P}_G^{(2)}$ for exch fixed $x \in \mathcal{H}_1$

(2) $\lambda(\bullet,Q)$ is a measurable function on $(\mathcal{H}_1, \mathcal{B}_1)$ for each fixed $Q \in \mathcal{B}_2$

B. Quantum Channel

$\mathcal{A}_k$; the set of v.N. alg. on $\mathcal{H}_k$ $(k=1,2)$

$\mathfrak{S}(\mathcal{A}_k)$; the set of all normal state on $\mathcal{A}_k$ $(k=1,2)$

• $\Lambda^*:\mathfrak{S}(\mathcal{A}_1)\to\mathfrak{S}(\mathcal{A}_2)$; **Quantum Channel**

(1) Λ^* satisfying the **affine property**

$$\sum_k \lambda_k = 1(\lambda_k \geq 0) \Rightarrow \Lambda^*\left(\sum_k \lambda_k \varphi_k\right) = \sum_k \lambda_k \Lambda^*(\varphi_k), \forall \varphi_k \in \mathfrak{S}(\mathcal{A}_1)$$

is called a **linear Channel**

(2) Predual map Λ of Λ^* satisfying the **completely positivity**

$$\sum_{j,k=1}^{n} B_j^* \Lambda\left(A_j^* A_k\right) B_k \geq 0, \forall n \in \mathbf{N}, \forall B_k \in \mathcal{A}_1, \forall A_k \in \mathcal{A}_2$$

is called a **completely positive (CP) channel**

(3) Λ^* is **Schwarz type** if $\Lambda(\overline{A}^*) = \Lambda(\overline{A})^*$ and $\Lambda^*(\overline{A})^*\Lambda^*(\overline{A}) \leq \Lambda^*(\overline{A}^*\overline{A})$.

(4) Λ^* is **stationary** if $\Lambda \circ \alpha_t = \overline{\alpha}_t \circ \Lambda$ for any $t \in \mathbf{R}$.
(Here α_t and $\overline{\alpha}_t$ are groups of automorphisms of the algebra $\mathcal{A}$ and $\overline{\mathcal{A}}$ respectively.)

(5) Λ^* is **ergodic** if Λ^* is stationary and $\Lambda^*(exI(\alpha)) \subset exI(\overline{\alpha})$.
(Here $exI(\alpha)$ is the set of extreme points of the set of all stationary states $I(\alpha)$.)

(6) Λ^* is **orthogonal** if any two orthogonal states $\varphi_1, \varphi_2 \in \mathfrak{S}(\mathcal{A})$ (denoted by $\varphi_1 \perp \varphi_2$) implies $\Lambda^*\varphi_1 \perp \Lambda^*\varphi_2$.

B. Quantum Channel

(7) Λ^* is **deterministic** if Λ^* is orthogonal and bijective.

(8) For a subset $\mathfrak{S}_0$ of $\mathfrak{S}(\mathcal{A})$, Λ^* is **chaotic** for $\mathfrak{S}_0$ if $\Lambda^*\varphi_1 = \Lambda^*\varphi_2$ for any $\varphi_1, \varphi_2 \in \mathfrak{S}_0$.

(9) Λ^* is **chaotic** if Λ^* is chaotic for $\mathfrak{S}(\mathcal{A})$.

(10) **Stinespring-Sudarshan-Kraus representation** a completely positive channel Λ^* can be represented as

$$\Lambda^*\rho = \sum_i A_i \rho A_i^*, \qquad \sum_i A_i^* A_i \leqslant 1.$$

Here A_i are bounded operators in H.

Most of channels appeared in physical processes are CP channels. Examples of such channels are the followings: Take a density operator $\rho \in \mathfrak{S}(\mathcal{H})$ as an input state.

(1) Unitary evolution: Let H be the Hamiltonian of a system.

$$\rho \to \Lambda^*\rho = U_t \rho U_t^*,$$

where $t \in \mathbf{R}$, $U_t = \exp(-itH)$.

(2) Semigroup evolution Let $V_t (t \in \mathbf{R}^+)$ be an one parameter semigroup on $\mathcal{H}$.

$$\rho \to \Lambda^*\rho = V_t \rho V_t^*, \quad t \in \mathbf{R}^+.$$

B. Quantum Channel

(3) **Quantum measurement:** If a measuring apparatus is prepared by an positive operator valued measure $\{Q_n\}$ then the state ρ changes to a state $\Lambda^*\rho$ after this measurement,

$$\rho \to \Lambda^*\rho = \sum_n Q_n \rho Q_n.$$

(4) **Reduction (Open System):** If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and ξ, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between two systems is given by

$$\theta_t \equiv U_t(\rho \otimes \xi)U_t^*,$$

where $U_t = \exp(-itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho \to \Lambda_t^*\rho \equiv \mathrm{tr}_{\mathcal{K}}\theta_t.$$

GKSL Master Equation

$$\frac{d}{dt}\rho = -[H,\rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^* - \frac{1}{2}\{L_\alpha^* L_\alpha, \rho\} \right)$$

$$L_\alpha \in B(\mathcal{H})$$

B. Quantum Channel

Open System Dynamics

Based on [4], let H_1 be the Hamiltonian of a system S_1 described by a Hilbert space $\mathcal{H}_1$. If a system S_1 interacts with an external system (heat bath) S_2 with the Hamiltonian H_2 described by another Hilbert space $\mathcal{H}_2$ and the initial states of S_1 and S_2 are ρ and ξ, respectively, then the compound state σ_t of S_1 and S_2 at time t after the interaction between two systems is given by

$$\sigma_t \equiv U_t(\rho \otimes \xi)U_t^*,$$

where $U_t = \exp(-itH)$ with the total Hamiltonian H of S_1 and S_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{H}_2$ such as

$$\rho \to \Lambda^* \rho \equiv \mathrm{tr}_{\mathcal{H}_2} \sigma_t.$$

The total Hamiltonian H is described by

$$\begin{aligned} H &= H_1 \otimes I + I \otimes H_2 + H_{in}, \\ H_1 &= a^* a, \\ H_2 &= \sum_{j=1} b_j^* b_j \end{aligned}$$

B. Quantum Channel

Open System Dynamics

For simplify, we assume that the system Σ_2 is given by a single mode

$$\begin{aligned} H_2 &= b^* b, \\ H_{in} &= \varepsilon \left(b a^* + b^* a \right) \end{aligned}$$

For a given state $\xi \in \mathfrak{S}(\mathcal{H}_2)$, the quantum channel Λ_t^* at time t for the open system is denoted by Stinespring - Sudarshan - Kraus representation such as

$$\Lambda_t^* (\rho) = \sum_{i=0}^{\infty} O_i(t) (\rho \otimes \xi) O_i^*(t), \quad (\forall \rho \in \mathfrak{S}(\mathcal{H}_1))$$

where O_i is a partial isometric operator given by

$$O_i(t) = \sum^{\infty} |k-i\rangle \left\langle \Phi_{k-i}^{(k)}(t) \right|,$$

$$\left| \Phi_{k-i}^{(k)}(t) \right\rangle = \sum_{j=0}^{k} \tau_{k-i,j}^{(k)}(t) |j\rangle \otimes |k-j\rangle \in [\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)} \subset \mathcal{H}_1 \otimes \mathcal{H}_2,$$

$$\tau_{k-i,j}^{(k)}(t) = \sum_{\ell=0}^{k} \exp\left(-it\varepsilon\lambda_\ell^{(k)}\right) C_{k-i}^{k,\ell} \overline{C_j^{k,\ell}},$$

B. Quantum Channel

Open System Dynamics

where $\lambda_\ell^{(k)}$ is given in 2 and

$$
\begin{aligned}
C_{k-i}^{k,\ell} &= \sum_{r=L}^{k-i} (-1)^{2k-i-r} \frac{\sqrt{k!\ell!\,(k-i)!(\ell+i)!}}{r!i!(k-i-r)!(\ell+i-k+r)!} \\
&\quad \times \alpha^{\ell-(k-i)+2r} \left(-\bar{\beta}\right)^{2k-i-2r}, \left(|\alpha|^2 + |\beta|^2 = 1\right),
\end{aligned}
$$

For any k, $[\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)}$ is a subspace spanned by a subset $\{|j\rangle \otimes |k-j\rangle\,; j = 0, 1, 2, \cdots, k\}$. Then

$$H_{in}\,[\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)} \subseteq [\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)}$$

is hold. Let $H_{in}^{(k)}$ be a restriction of H_{in} into $[\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)}$. $H_{in}^{(k)}$ is a finite dimensional selfajoint operator on $\mathbb{C}^{k+1}$ satisfying

$$
\begin{aligned}
H_{in}^{(k)} \Psi_\ell^{(k)} &= \lambda_\ell^{(k)} \Psi_\ell^{(k)}, \quad (\ell = 0, 1, 2, \cdots, k), \\
\Psi_\ell^{(k)} &= \sum_{m=0}^{k} C_m^{k,\ell} \, |m\rangle \otimes |k-m\rangle \in [\mathcal{H}_1 \otimes \mathcal{H}_2]^{(k)}.
\end{aligned}
$$

B. Quantum Channel

(5) **Optical communication processes:** Quantum communication process is described by the following scheme.

$$\begin{array}{ccc} & \text{Noise } \nu \in \mathfrak{S}(\mathcal{K}_1) & \\ & \downarrow & \\ \mathfrak{S}(\mathcal{H}_1) \ni \rho & \longrightarrow & \bar{\rho} = \Lambda^* \rho \in \mathfrak{S}(\mathcal{H}_2) \\ & \downarrow & \\ & \text{Loss } \mathfrak{S}(\mathcal{K}_2) & \end{array}$$

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\ \gamma^* \downarrow & & \uparrow a^* \\ \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow[\pi^*]{} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2) \end{array}$$

The above maps γ^*, a^* are given as

$$\begin{aligned} \gamma^*(\rho) &= \rho \otimes \nu, \quad \rho \in \mathfrak{S}(\mathcal{H}_1), \\ a^*(\theta) &= \mathrm{tr}_{\mathcal{K}_2} \theta, \quad \theta \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2), \end{aligned}$$

where ν is a noise coming from the outside of the system. The map π^* is a certain channel determined by physical properties of the device transmitting information. Hence the channel for the above process is given as

$$\Lambda^* \rho \equiv \mathrm{tr}_{\mathcal{K}_2} \pi^* (\rho \otimes \nu) = (a^* \circ \pi^* \circ \gamma^*)(\rho).$$

B. Quantum Channel

Attenuation channel and beam Splitter [Ohya, 1983]

(6) **Attenuation process:** Based on the construction of the optical communication processes of (5), the attenuation channel is defined as follows: Take $\nu_0 = |0\rangle\langle 0| =$ vacuum state and $\pi_0^*(\cdot) \equiv V_0(\cdot)V_0^*$ given by

$$V_0(|n\rangle \otimes |0\rangle) \equiv \sum_{j=0}^{n} C_j^n |j\rangle \otimes |n-j\rangle,$$

$$C_j^n \equiv \sqrt{\frac{n!}{j!(n-j)!}}\alpha^j \beta^{n-j}, \quad \left(|\alpha|^2 + |\beta|^2 = 1\right)$$

Then the output state of the attenuation channel Λ_0^* is obtained by

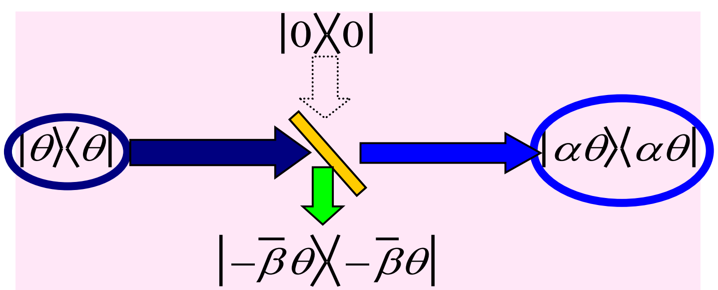
$$\Lambda_0^*\rho \equiv \mathrm{tr}_{\mathcal{K}_2}\pi_0^*(\rho \otimes \nu) = \mathrm{tr}_{\mathcal{K}_2}\pi_0^*(\rho \otimes |0\rangle\langle 0|),$$

$\eta = |\alpha|^2$ $(0 \le \eta \le 1)$ is called a transmission rate of the attenuation channel Λ_0^*. In particular, for a coherent input state $\rho = |\theta\rangle\langle\theta|$, one has

$$\pi_0^*(|\theta\rangle\langle\theta| \otimes |0\rangle\langle 0|) = |\alpha\theta\rangle\langle\alpha\theta| \otimes |-\beta\theta\rangle\langle-\beta\theta|,$$

which is called a beam splitting operator.

π_0^* beam splitter



B. Quantum Channel Noisy optical channel and generalized beam Splitter[Ohya NW,1984]

(7) **Noisy optical channel:** Based on (5), the noisy optical channel is defined as follows: Take a noise state $\nu = |m_1\rangle\langle m_1|$, m_1 photon number state of $\mathcal{K}_1$ and a linear mapping $V : \mathcal{H}_1 \otimes \mathcal{K}_1 \to \mathcal{H}_2 \otimes \mathcal{K}_2$ as

$$V\left(|n_1\rangle \otimes |m_1\rangle\right) \equiv \sum_{j=0}^{n_1+m_1} C_j^{n_1,m_1} |j\rangle \otimes |n_1+m_1-j\rangle$$

with

$$C_j^{n_1,m_1} \equiv \sum_{r=l}^{K} (-1)^{n_1+j-r} \frac{\sqrt{n_1!m_1!j!\left(n_1+m_1-j\right)!}}{r!\left(n_1-j\right)!\left(j-r\right)!\left(m_1-j+r\right)!} \alpha^{m_1-j+2r}\beta^{n_1+j-2r}$$

where

$$K = \min\{n_1, j\}, \quad L \equiv \max\{m_1-j, 0\}.$$

Then the output state of the noisy optical channel Λ^* is defined by

$$\Lambda^*\rho \equiv \mathrm{tr}_{\mathcal{K}_2}\pi^*\left(\rho\otimes\nu\right) = \mathrm{tr}_{\mathcal{K}} V\left(\rho\otimes|m_1\rangle\langle m_1|\right)V^*$$

for the input state $\rho \in \mathfrak{S}(\mathcal{H}_1)$. In particular, for a coherent input state $\rho = |\theta\rangle\langle\theta|$ and a coherent noise state $\nu = |\gamma\rangle\langle\gamma|$, we obtain

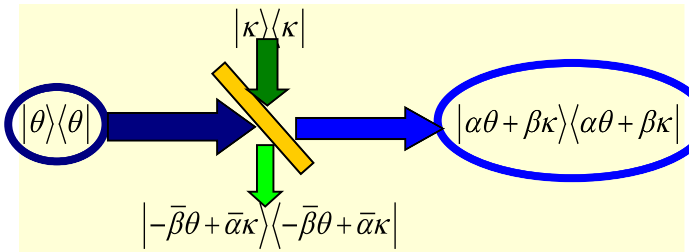
$$\pi^*\left(|\theta\rangle\langle\theta|\otimes|\gamma\rangle\langle\gamma|\right) = |\alpha\theta+\beta\gamma\rangle\langle\alpha\theta+\beta\gamma|\otimes|-\beta\theta+\alpha\gamma\rangle\langle-\beta\theta+\alpha\gamma|,$$

B. Quantum Channel Noisy optical channel and generalized beam Splitter[Ohya NW,1984]

$$\pi^* \left(|\theta\rangle \langle\theta| \otimes |\gamma\rangle \langle\gamma| \right) = |\alpha\theta + \beta\gamma\rangle \langle\alpha\theta + \beta\gamma| \otimes |-\beta\theta + \alpha\gamma\rangle \langle -\beta\theta + \alpha\gamma| ,$$

which is called a generalized beam splitting operator because it means that two coherent states are splitted to two coherent states by passing through

π^* generalized beam splitter



the CP channel π. The mathematical formulations of beam splitting are studied in [?] based on liftings in the sense of Accardi and Ohya [?] is denoted by

$$\mathcal{E}_0^* \left(|\xi\rangle \langle\xi| \right) = |\alpha\xi\rangle \langle\alpha\xi| \otimes |\beta\xi\rangle \langle\beta\xi|$$

where $\mathcal{E}_0^*$ is a mapping from $\mathfrak{S}(\mathcal{H})$ to $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, and in [?] on generalized Fock spaces.

B. Quantum Channel

(8) **Connected channel:** Based on (5), the noisy optical channel is defined as follows: For $n \in \mathbf{N}$, a n-connected channel $\Lambda^*_{(n)}$ with a fixed noise state ζ was defined by

$$\Lambda^*_{(n)}(\rho) \equiv tr_{\mathcal{K}_2} \pi^{*n}(\rho \otimes \zeta) = tr_{\mathcal{K}_2} V^n (\rho \otimes \zeta) V^{*n},$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$, where π^{*n} is n -folds composition of the CP channels π^*, V^n and V^{*n} are also n -folds composition of V and V^*, respectively.

Theorem

The n-connected channel $\Lambda^*_{(n)}$ with noise state $|m\rangle\langle m|$ is described by

$$\Lambda^*_{(n)}(\rho) = \sum_{i=0}^{\infty} O_i V^n Q^{(m)} \rho Q^{(m)*} V^{*n} O_i^*,$$

where $Q^{(m)} \equiv \sum_{l=0}^{\infty} (|y_l\rangle \otimes |m\rangle) \langle y_l|$, $O_i \equiv \sum_{k=0}^{\infty} |z_k\rangle (\langle z_k| \otimes \langle i|)$, $\{|y_l\rangle\}$ and $\{|z_k\rangle\}$ are CONS in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. $\{|i\rangle\}$ is the set of number states in $\mathcal{K}_2$.

Theorem

For the n-connected channel $\Lambda^*_{(n)}$ with $\alpha = \frac{1}{2}$, if n is given by $6\ell \quad (\ell \in \mathbf{N})$, then Π^{*n} is the identity channel id, that is

$$\Lambda^*_{(n)}(\rho) = \rho$$

is held for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$, and if n is given by $6\ell + \frac{3}{2} \quad (\ell \in \mathbf{N})$, then Π^{*n} is the exchanged channel, that is

$$\Lambda^*_{(n)}(\rho) = \zeta$$

is held for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$.

If n is given by $6\ell + \frac{3}{4} \quad (\ell \in \mathbf{N})$, one can obtain the output and loss states of n-folds composition of the CP channels Π^* such as the maximal entangled state

$$\Pi^{*n}(\rho \otimes \zeta) = \frac{1}{2}(|1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle)(\langle 1| \otimes \langle 0| + \langle 0| \otimes \langle 1|)$$

for the input state $\rho = |1\rangle\langle 1|$ and the noise state $\zeta = |0\rangle\langle 0|$. It means that $\Pi^{*n} \left(n = 6\ell + \frac{3}{4} \quad (\ell \in \mathbf{N})\right)$ generates the maximal entangled state.

Introduction

1. Information Dynamics (M. Ohya)
2. Complexity (C)
3. Channel
4. Transmitted Complexity (T)
5. Other mutual entropy type complexities (T)

   How to construct compound states

B. Transmitted Complexity (T)

1) Mutual Entropy [Shannon]

$$T_1^S\left(p;\Lambda^*\right) \Leftrightarrow I\left(p;\Lambda^*\right) \equiv \sum_{i,j} r_{ij} \log \frac{r_{ij}}{p_i q_j} = S\left(r, p \otimes q\right)$$

$$0 \le T_1^S\left(p;\Lambda^*\right) \le \min\left\{C_1^S\left(p\right), C_1^S\left(\Lambda^* p\right)\right\}$$

2) Mutual entropy (information) [GKY] with respect to μ_1 and Γ^* is defined by the **Kullback - Leibler information** such as

$$T_2^S\left(\mu_1;\lambda\right) \quad \Leftrightarrow \quad \mathrm{I}\left(\mu_1;\lambda\right) = S\left(\mu_{12} \mid \mu_1 \otimes \mu_2\right)$$

$$= \begin{cases} \int_{\mathcal{H}_1 \times \mathcal{H}_2} \frac{d\mu_{12}}{d\mu_1 \otimes \mu_2} \log \frac{d\mu_{12}}{d\mu_1 \otimes \mu_2} d\mu_1 \otimes \mu_2 & \left(\mu_{12} \ll \mu_1 \otimes \mu_2\right) \\ \infty & \left(\mu_{12} \not\ll \mu_1 \otimes \mu_2\right) \end{cases}$$

where $\frac{d\mu_{12}}{d\mu_1 \otimes \mu_2}$ is the Radon-Nikodym derivative of μ_{12} w.r.t. $\mu_1 \otimes \mu_2$

Theorem [OW]

(1) $C_3^S\left(\mu_1\right) < T_2^S\left(\mu_1;\lambda\right)$, (2) $C_2^S\left(\mu_1\right) = +\infty$

B. Transmitted Complexity (T)

3) Ohya Mutual Entropy for density operator [Ohya]

(**Classical System**)

Classical mutual entropy $\Leftarrow$ constructed by **Joint prob. measure** $r = \{r_{ij}\}$

In general, there does not exist the joint states in the quantum systems [Urbanik]

How to define the quantum mutual entropy?

(**Quantum System**)

$\rho = \sum_n \lambda_n E_n$; **Schatten decomposition of** ρ

(one dimensional orthogonal decomposition of ρ)

Remark Schatten decomposition is not uniqe in general

Ohya Compound state $\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$

$$id \otimes \Lambda^* \left(\sum_n \lambda_n E_n \otimes E_n \right) \to \sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$$ (by Jamiolkowski isometric channel)

B. Transmitted Complexity (T)

3) Ohya Mutual Entropy for density operator [Ohya]

Two compound states

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n \text{ (Ohya compound state)}$$

$$\sigma_0 = \rho \otimes \Lambda^* \rho \text{ (Trivial compound state)}$$

Relative Entropy [Umegaki]

$$T_3^S\left(\rho;\Lambda^*\right) \Leftrightarrow I\left(\rho;\Lambda^*\right) \equiv \sup_E \left\{ S\left(\sigma_E, \rho \otimes \Lambda^* \rho\right)\right\}$$

Theorem [Ohya]

(1) $T_3^S\left(\rho; id\right) = C_4^S\left(\rho\right)$ (von Neumann entropy)

(2) If the system is classical, then the quantum mutual entropy equals to the classical mutual entropy.

(3) $0 \le T_3^S\left(\rho;\Lambda^*\right) \le \min\left\{ C_4^S\left(\rho\right), C_4^S\left(\Lambda^*\rho\right)\right\}$

Lifting in the sense of Accardi and Ohya [AO]

$$\mathcal{E}^*_{E,\Lambda^*} : \mathfrak{S}\left(\mathcal{H}_1\right) \to \mathfrak{S}\left(\mathcal{H}_1 \otimes \mathcal{H}_2\right)$$

$$\mathcal{E}^*_{E,\Lambda^*}\left(\rho\right) \rightarrow \sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n \text{ (nonlinear lifting)}$$

B. Transmitted Complexity (T)

3) Ohya Mutual Entropy for density operator [Ohya]

Two compound states

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n \text{ (Ohya compound state)}$$

$$\sigma_0 = \rho \otimes \Lambda^* \rho \text{ (Trivial compound state)}$$

Relative Entropy [Umegaki]

$$T_3^S\left(\rho;\Lambda^*\right) \quad \Leftrightarrow \quad I\left(\rho;\Lambda^*\right) \equiv \sup_E \left\{ S\left(\sigma_E, \rho \otimes \Lambda^* \rho\right) \right\}$$

$$0 \leq T_3^S\left(\rho;\Lambda^*\right) \leq \min\left\{ C_4^S\left(\rho\right), C_4^S\left(\Lambda^* \rho\right) \right\}$$

Theorem [Ohya]

(1) If Λ^* is **deterministic** , then $I\left(\rho;\Lambda^*\right) = S\left(\rho\right)$.

(2) If Λ^* is **chaiotic** , then $I\left(\rho;\Lambda^*\right) = 0$.

(3) If ρ is faithful stationary state and all eigenvales of ρ are not degenerate, and Λ^* is **ergodic** , then $I\left(\rho;\Lambda^*\right) = S\left(\Lambda^* \rho\right)$.

B. Transmitted Complexity (T)

Relative Entropy for density operator [Umegaki]

$$S(\rho,\sigma) \equiv \begin{cases} tr\rho(\log\rho - \log\sigma) & s(\rho) \le s(\sigma) \\ \infty & (\text{otherwise}) \end{cases}$$

← support projection of σ

Theorem

1) *Positivity* : $S(\phi,\psi) \ge 0$ **and** $S(\phi,\psi) = 0$ **iff** $\phi = \psi$.

2) *Joint Convexity* : $S(\lambda\psi_1 + (1-\lambda)\psi_2, \lambda\phi_1 + (1-\lambda)\phi_2) \le \lambda S(\psi_1,\phi_1) + (1-\lambda)S(\psi_2,\phi_2)$

for any $\lambda \in [0,1]$.

3) *Additivity* : $S(\psi_1 \otimes \psi_2, \phi_1 \otimes \phi_2) = S(\psi_1,\phi_1) + S(\psi_2,\phi_2)$.

4) *Lower Semicontinuity* : *If* $\lim_{n\to\infty} \|\psi_n - \psi\| = 0$ *and* $\lim_{n\to\infty} \|\phi_n \to \phi\| = 0$, *then* $S(\psi,\phi) \le \lim_{n\to\infty} \inf S(\psi_n,\phi_n)$.

Moreover, if there exists a positive number λ *satisfying* $\psi_n \le \lambda\phi_n$, *then* $\lim_{n\to\infty} S(\psi_n,\phi_n) = S(\psi,\phi)$.

5) *Monotonicity* : *For a channel* Λ^* *from* $\mathfrak{S}$ *to* $\overline{\mathfrak{S}}$,

$S(\Lambda^*\psi, \Lambda^*\phi) \le S(\psi,\phi)$.

6) *Lower Bound* : $\|\psi - \phi\|^2 / 4 \le S(\psi,\phi)$.

It was extended by Araki (for von Neumann alg.) and by Uhlmann (for *-algebra).

B. Transmitted Complexity (T)

4) Ohya Mutual Entropy for general C*- systems [Ohya]

For $\varphi \in \mathcal{S} \subset \mathfrak{S}(\mathcal{A})$ and $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\overline{\mathcal{A}})$,

Two compound states for general C*-systems

$$\Phi_\mu^{\mathcal{S}} = \int_S \omega \otimes \Lambda^* \omega d\mu \text{ (Ohya compound state)}$$

$$\Phi_0 = \varphi \otimes \Lambda^* \varphi \text{ (Trivial compound state)}$$

$$T_4^S\left(\rho;\Lambda^*\right) \Leftrightarrow I^S(\varphi;\Lambda^*) = \lim_{\varepsilon\to 0}\sup\left\{I_\mu^S(\varphi;\Lambda^*); \mu \in F_\varphi^\varepsilon(S)\right\}$$

$I_\mu^{\mathcal{S}}(\varphi;\Lambda^*) = S(\Phi_\mu^{\mathcal{S}},\Phi_0)$ ⟵ **Relative Entropy [Araki & Uhlmann]**

where

$$F_\varphi^\varepsilon(\mathcal{S}) = \begin{cases} \left\{\mu \in D_\varphi(\mathcal{S}); S^{\mathcal{S}}(\varphi) \le H(\mu) \le S^{\mathcal{S}}(\varphi) + \varepsilon < +\infty\right\} \\ M_\varphi(\mathcal{S}) \quad if \quad S^{\mathcal{S}}(\varphi) = +\infty \end{cases}$$

$$0 \le T_4^S\left(\varphi;\Lambda^*\right) \le \min\left\{C_5^S\left(\varphi\right), C_5^S\left(\Lambda^*\varphi\right)\right\}$$

$$T_4^S\left(\varphi; id\right) = C_5^S\left(\varphi\right) \text{ (S - mixing entropy)}$$

B. Transmitted Complexity (T)

4) Ohya Mutual Entropy for general C*- systems [Ohya]

When the **input system is classical**, an input state ρ is given by a probability distribution or a probability measure. In either case, **the Schatten decomposition of ρ is unique**, namely, for the case of probability distribution ; $\rho = \{\mu_k\}$,

$$\rho = \sum_k \mu_k \delta_k,$$

where δ_k is the delta measure, that is,

$$\delta_k(j) = \delta_{k,j} = \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases}, \forall j.$$

Therefore for any channel Λ^*, the **mutual entropy** becomes

$$I(\rho; \Lambda^*) = \sum_k \mu_k S(\Lambda^* \delta_k, \Lambda^* \rho),$$

Ohya quntum mutual entropy

which equals to the following usual expression when one of the **two terms is finite for an infinite dimentional Hilbert space**:

$$I(\rho; \Lambda^*) = S(\Lambda^* \rho) - \sum_k \mu_k S(\Lambda^* \delta_k).$$

Levitin and Holevo's (LH for short) semi-quantum mutual entropy

Thus the Ohya's q. mutual entropy contaions the LH s-q. mutual entropy as a special case.

Entropy and mutual entropy type complexities of Classical and Quantum Systems

Entropy type complexities of Classical and Quantum Systems

↙ Semi-classical E ← v.N. E

↑ $C_4^S(\rho)$

Shannon's E

$C_1^S(p)$

↖ E (D, Finite P.) ← Ohya $\mathcal{S}$-mixing E for GQS

$C_2^S(\mu), C_3^S(\mu)$

$C_5^S(\varphi)$

S-mixing entropy of quantum channel Generalized by Prof. Mukhamedov and NW

S-mixing Renyi entropy Generalized by Prof. Mukhamedov, Ohmura and NW

Mutual Entropy type complexities of Classical and Quantum Systems

↙ Semi-quantum ME ← Ohya ME

↑ $T_3^S(\rho;\Lambda^*)$

Shannon's ME

$T_1^S(p;\Lambda^*)$

↖ ME (GKY) ← Ohya ME for GQS

$T_2^S(\mu;\Lambda^*)$

$T_4^S(\varphi;\Lambda^*)$

(3) Information Transmission of Gaussian Communication Process

Information Transmission of Gaussian Communication Process

- Input state $\mu \in P_G^{(1)}$
- Gaussian Channel λ
- Entropy of Finite partition

$$C_2^S(\mu) \Leftrightarrow S_{\tilde{\mathcal{A}}}(\mu) \equiv \sup\left\{ -\sum_{A_k \in \tilde{\mathcal{A}}} \mu(A_k) \log \mu(A_k); \quad \tilde{\mathcal{A}} \in \mathcal{P}(\Omega) \right\}$$

- Differential Entropy

$$C_3^S(\mu) \Leftrightarrow S(\mu) \equiv -\int_{\mathbf{R}} \frac{d\mu}{dm} \log \frac{d\mu}{dm} dm$$

- Mutual Entropy by GKY

$$T_2^S(\mu_1;\lambda) \Leftrightarrow I(\mu_1;\lambda) = S(\mu_{12} \mid \mu_1 \otimes \mu_2)$$

- Inequality

$$T_2^S(\mu_1;\lambda) \le C_2^S(\mu_1) = +\infty$$

$$T_2^S(\mu_1;\lambda) \ge C_3^S(\mu_1)$$

Information Transmission of General Quantum Communication Process

- Input state $\varphi \in \mathfrak{S}(\mathcal{A}_1)$
- Quantum channel Λ^*
- C*-mixing Entropy

$$C_4^S(\varphi) \Leftrightarrow S^S(\varphi) = \inf\left\{ H(\nu); \nu \in D_\varphi(S) \right\}$$

- State Decomposition $\varphi = \int_{\mathcal{S}} \omega \, d\nu$
- Quantum Mutual Entropy

$$T_4^S(\varphi;\Lambda^*) \Leftrightarrow I^{\mathcal{S}}(\varphi;\Lambda^*) = \limsup_{\varepsilon \to 0} \left\{ I_\mu^{\mathcal{S}}(\varphi;\Lambda^*); \mu \in F_\varphi^\varepsilon(\mathcal{S}) \right\}$$

- Inequality

$$0 \le T_4^S(\varphi;\Lambda^*) \le C_5^S(\varphi)$$

New Treatment of Information Transmission of Gaussian Communication Process

New Treatment of Information Transmission of Gaussian Communication Process

- Input state $\mu \in P_G^{(1)}$
- Gaussian Channel λ
- Entropy Func. Str. Eq.

$$C_{SE}^S(\mu_1) \Leftrightarrow \tilde{S}_{SE}(\mu_1) = -tr\frac{\Xi_1^*(\mu_1)}{tr\left[\Xi_1^*(\mu_1)\right]}\log\frac{\Xi_1^*(\mu_1)}{tr\left[\Xi_1^*(\mu_1)\right]}.$$

- Mutual Ent. Func. Str. Eq.

$$T_{SE}^S(\mu_1;\lambda) \Leftrightarrow \tilde{I}_{SE}(\mu_1;\lambda) = \sup_E S\left(\sigma_E, \frac{\Xi_1^*(\mu_1)\otimes\Pi^*\circ\Xi_1^*(\mu_1)}{tr\left[\Xi_1^*(\mu_1)\otimes\Pi^*\circ\Xi_1^*(\mu_1)\right]}\right).$$

- Inequality

$$0 \le T_{SE}^S(\mu_1;\lambda) \le C_{SE}^S(\mu_1)$$

Information Transmission of Quantum Communication Process

- Input state $\rho \in \mathfrak{S}(\mathcal{H}_1)$
- Quantum channel Λ^*
- V.N. Entropy

$$C_4^S(\rho) \Leftrightarrow S(\rho) = -tr\rho\log\rho$$

- Schatten Decomposition $\rho = \sum_n \lambda_n E_n = \sum_n \lambda_n |x_n\rangle\langle x_n|$
- Quantum Mutual Entropy

$$T_3^S(\rho;\Lambda^*) \Leftrightarrow I(\rho;\Lambda^*) = \sup_E S(\sigma_E, \rho\otimes\Lambda^*\rho)$$

- Inequality

$$0 \le T_3^S(\rho;\Lambda^*) \le C_4^S(\rho)$$

$$
\begin{array}{ccc}
\mu_1 \in P_G^{(1)} & \xrightarrow{\Gamma^*} & \Gamma^*(\mu_1) = \mu_2 \in P_G^{(2)} \\
\updownarrow \Xi_1^* & & \updownarrow \Xi_2^* \\
\Xi_1^*(\mu_1) \in \mathbf{T}(\mathcal{H}_1)_+ & & \Xi_2^*(\Gamma^*(\mu_1)) \in \mathbf{T}(\mathcal{H}_2)_+ \\
\updownarrow \Pi_1^* & & \updownarrow \Pi_2^* \\
\varphi_1 = \Pi_1^* \circ \Xi_1^*(\mu_1) \in \mathcal{A}'_{1*,+} & \xrightarrow{\Theta^*} & \Theta^*(\varphi_1) = \Pi_2^* \circ \Xi_2^* \circ \Gamma^*(\mu_1) \in \mathcal{A}'_{2*,+}
\end{array}
$$

which is denoted by

$$
\boxed{\Theta^*(\varphi_1) = \Pi_2^*\left(A(\Pi_1^*)^{-1}(\varphi_1)A^* + \Xi_2^*(\mu_0)\right),} \quad \left(\forall \varphi_1 \in \mathcal{A}'_{1*,+}\right)
$$

In this paper, we suppose two conditions.

- **A treatment I**
  1. Linearity condition (linear approximation)
  2. Trace preserving condition : $\Theta^*(\varphi)(I) = \varphi(I)$ is hold for any $\varphi \in \mathcal{A}'_{1*,+}$
- **A treatment II**
  1. Linearity condition (linear approximation)
  2. Weak trace preserving condition : $\Theta^*(\varphi)(I) = \Theta^*(\varphi')(I)$ is hold for any $\varphi, \varphi' \in \mathcal{A}'_{1*,+}$ satisfying $\varphi(I) = \varphi'(I)$

- **Treatment [Ohya and Watanabe]**

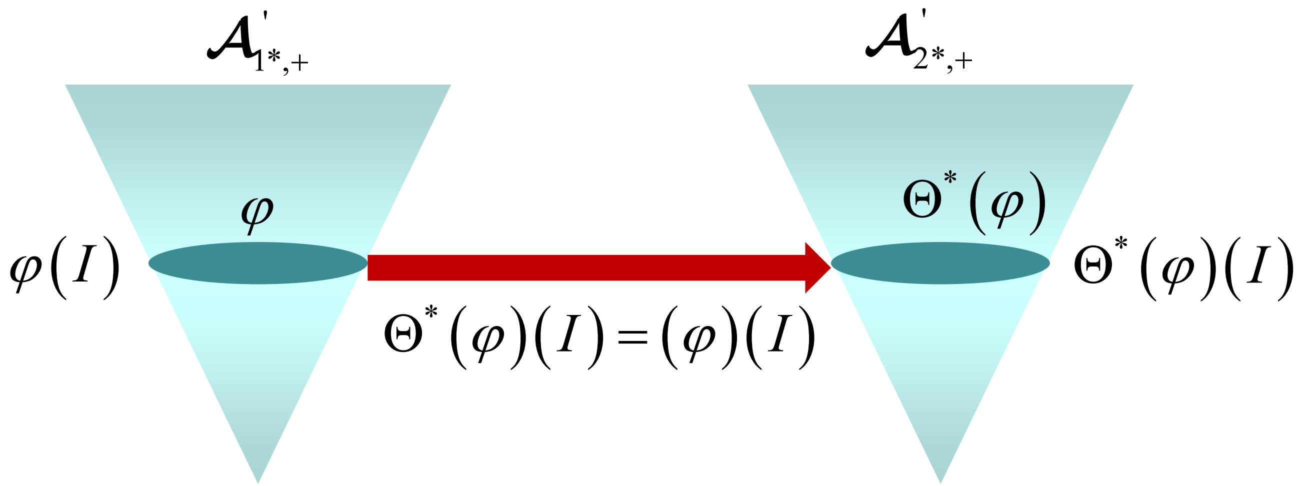
(1) Linearity condition (Linear Approximation)

(2) Trace preserving condition :

$\Theta^*(\varphi)(I) = (\varphi)(I)$ is hold for any $\varphi \in \mathcal{A}'_{1*,+}$

$A^* A = (1 - trR_0) I$ ⇒ $\varphi(I) = 1$ **(Ohya & NW)**

$A^* A = (\alpha - trR_0) \frac{I}{\alpha}$ ⇒ $\varphi(I) = \alpha$ **(Makiwara & NW)**



• A new treatment [Watanabe]

(1) Linearity condition (Linear Approximation)

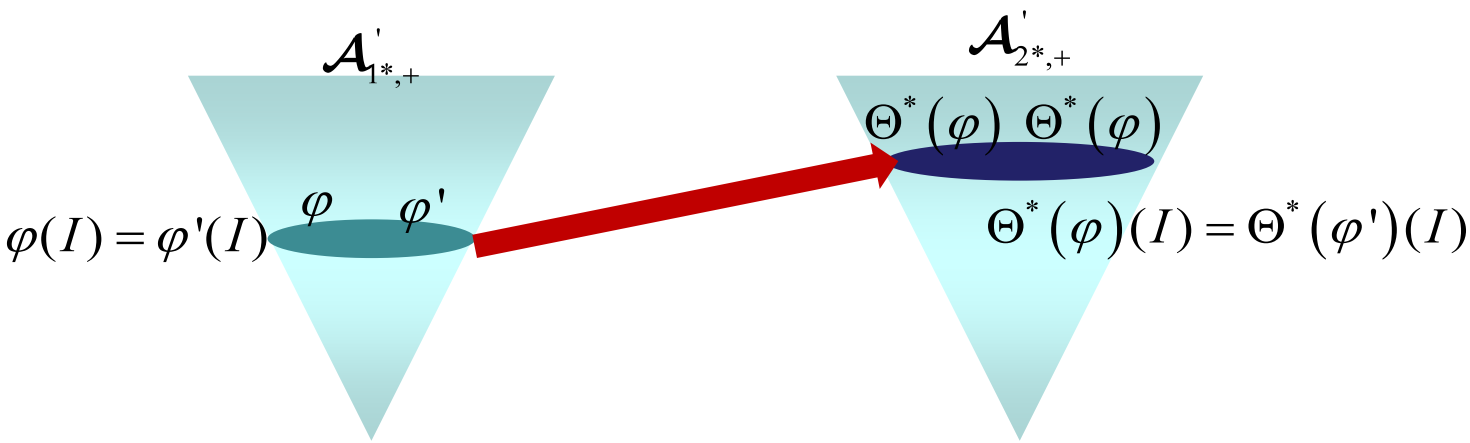
(2) Weak trace preserving condition :

$\Theta^*(\varphi)(I) = \Theta^*(\varphi')(I)$ is hold for any $\varphi, \varphi' \in \mathcal{A}'_{1*,+}$

satisfying $\varphi(I) = \varphi'(I)$

$$A^* A = (1 - trR_0) I \Rightarrow \varphi(I) = 1 \quad \textbf{(NW)}$$

$$A^* A = (\alpha - trR_0) \frac{I}{\alpha} \Rightarrow \varphi(I) = \alpha \quad \textbf{(NW)}$$



Now we introduce a concept of the structure equivalent class in the Gaussian communication processes.

Definition

Structure equivalent of $\mathcal{A}'_{1*,+}$ and $\mathbf{P}_G^{(1)}$

1. φ_1 and φ_2 are structure equivalent (i.e., $\varphi_1 \overset{s}{\sim} \varphi_2$) if there exists a positive number $\lambda > 0$ such that $\varphi_1(I) = \lambda \varphi_2(I)$ holds,
2. $\mu_1 = \left[0, (\Pi_1^*)^{-1}(\varphi_1)\right]$ and $\mu_2 = \left[0, (\Pi_2^*)^{-1}(\varphi_2)\right]$ are structure equivalent (i.e., $\mu_1 \overset{s}{\sim} \mu_2$) if $\varphi_1 \overset{s}{\sim} \varphi_2$ is satisfied.

Definition

Structure equivalent class of $\mathcal{A}'_{k*,+}$ and $\mathbf{P}_G^{(k)}$

1. $\widetilde{\varphi_k} \equiv \left\{\psi \in \mathcal{A}'_{k*,+}; \quad \varphi_k \overset{s}{\sim} \psi\right\},$
2. $\widetilde{\mu_k} \equiv \left\{\nu \in \mathbf{P}_G^{(k)}; \quad \mu_k \overset{s}{\sim} \nu\right\} \quad (k = 1, 2).$

Introduction

1. Information Dynamics (M. Ohya)
2. Complexity (C)
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4. Transmitted Complexity (T)
5. Other mutual entropy type complexities (T)
   How to construct compound states

5. Other Mutual Entropy Type Complexities

1) **Ohya mutual entropy w.r.t.** φ,Λ^* **[Ohya,1983, 1989]**

$$T_4^{\mathcal{S}}\left(\varphi;\Lambda^*\right)\Leftrightarrow I\left(\varphi;\Lambda^*\right)\equiv\sup\left\{\int_{\mathcal{S}} S^{Araki}\left(\Lambda^*\omega,\Lambda^*\varphi\right)d\mu;\varphi=\int_{ex\mathcal{S}}\omega d\mu\right\}$$

2) **Lindblad-Nielsen's entropy w.r.t.** ρ,Λ^*

$$T_5^{S}\left(\varphi;\Lambda^*\right)\Leftrightarrow I_{L-N}\left(\rho;\Lambda^*\right)\equiv S\left(\Lambda^*\rho\right)+S\left(\rho\right)-S_e\left(\rho,\Lambda^*\right)$$

3) **Coherent information w.r.t.** ρ,Λ^*

$$T_6^{S}\left(\varphi;\Lambda^*\right)\Leftrightarrow I_C\left(\rho;\Lambda^*\right)\equiv S\left(\Lambda^*\rho\right)-S_e\left(\rho,\Lambda^*\right)$$

In the above two cases, the entropy exchange $S_e\left(\rho,\Lambda^*\right)$ w.r.t. ρ,Λ^* is defined as follows: When Λ^* is given by

$$S_e\left(\rho,\Lambda^*\right)\equiv S\left(W\right),$$

$$\Lambda^*\left(\rho\right)\equiv\sum_i A_i\rho A_i^*,\left(\sum_i A_i^* A_i=I\right)\qquad W\equiv\left(W_{ij}\right),W_{ij}\equiv trA_i\rho A_j^*$$

5. Other Mutual Entropy Type Complexities

A. Comparison among these quantum mutual entropy type complexities

Theorem [OW] ***For the attenuation channel*** $\Lambda_0^*(\rho) \equiv tr_{\mathcal{K}_2} V(\rho \otimes |0\rangle\langle 0|) V^* = \sum_{n=0} A_n \rho A_n^*$,

$$\left(A_n \equiv Q_n V W = \left\{ \sum_{j=0} |z_j\rangle (\langle z_j| \otimes \langle n|) \right\} V \left\{ \sum_{i=0} (|y_i\rangle \otimes |0\rangle) \langle y_i| \right\} \right)$$

one can obtain the following results for any input states $\rho = \sum_n \lambda_n E_n,\ E_n = |n\rangle\langle n|$,

(1) $0 \le T_3^S(\rho; \Lambda_0^*) \le \min\left\{ C_4^S(\rho), C_4^S(\Lambda_0^* \rho) \right\}$, **Ohya Mutual Entropy**

(2) $T_5^S(\rho; \Lambda_0^*) = S(\rho)$, **Lindblad Entropy**

(3) $T_6^S(\rho; \Lambda_0^*) = 0$. **Coherent Entropy**

Theorem [OW] *For the attenuation channel* Λ_0^* *and the input state*

$$\rho = \lambda |0\rangle\langle 0| + (1-\lambda) |\theta\rangle\langle \theta|,$$

we have

(1) $0 \le T_3^S(\rho; \Lambda_0^*) \le \min\left\{ C_4^S(\rho), C_4^S(\Lambda_0^* \rho) \right\}$ **Ohya Mutual Entropy**

(2) $0 \le T_5^S(\rho; \Lambda_0^*) \le 2C_4^S(\rho)$ **Lindblad Entropy**

(3) $-C_4^S(\rho) \le T_6^S(\rho; \Lambda_0^*) \le C_4^S(\rho)$ **Coherent Entropy**

Calculation of Mutual Entropy-type Complexity for Attenuation Channel

Lemma For the attenuation channel Λ_0^* and the input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

there exists a unitary operator U such that

$$UWU^* = \lambda|0\rangle\langle 0| + (1-\lambda)\left|-\overline{\beta}\theta\right\rangle\left\langle -\overline{\beta}\theta\right|,$$

Theorem $[OW]$ For the attenuation channel Λ_0^* and the input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

the entropy exchange is obtained by

$$S_e\left(\rho, \Lambda_0^*\right) = -trW\log W = -\sum_{j=0}^{1}\mu_j \log \mu_j,$$

where

$$\mu_j = \frac{1}{2}\left\{1 + (-1)^j\sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\beta|^2|\theta|^2\right)\right)}\right\} \quad (j = 0,1).$$

Calculation of Mutual Entropy-type Complexity for Attenuation Channel

Theorem $[OW]$ For the attenuation channel Λ_0^* and the input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

(1) if $|\alpha| > |\beta|$ then the cohent entropy $I_C(\rho;\Lambda_0^*) > 0$ is holds,

(2) ~~if $|\alpha| < |\beta|$ then the cohent entropy $I_C(\rho;\Lambda_0^*) < 0$ is holds,~~

(3) if $|\alpha| < |\beta|$ then the cohent entropy $I_C(\rho;\Lambda_0^*) = 0$ is holds.

Theorem $[OW]$ For the attenuation channel Λ_0^* and the input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

(1) ~~if $|\alpha| > |\beta|$ then the Lindblad-Nielsen's entropy $I_{LN}(\rho;\Lambda_0^*) > S(\rho)$ is holds,~~

(2) if $|\alpha| < |\beta|$ then the Lindblad-Nielsen's entropy $I_{LN}(\rho;\Lambda_0^*) < S(\rho)$ is holds,

(3) if $|\alpha| < |\beta|$ then the Lindblad-Nielsen's entropy $I_{LN}(\rho;\Lambda_0^*) = S(\rho)$ is holds.

5. Compound States

Compound State (signal is transmitted)

Let us consider construct of the compound state.

If the initial state ρ and the quantum channel Λ^* (signal is transmitted from the initial system to the final system) are given, compound states Φ should satisfy the following marginal conditions:

(1) $tr_2 \, \Phi = \rho$, (Marginal condition 1)

(2) $tr_1 \, \Phi = \Lambda^* \rho$ (Marginal condition 2)

5. Compound States

Compound State (signal is transmitted)

Φ

$tr_{\mathcal{H}_2}\Phi;$ CP channel

$tr_{\mathcal{H}_1}\Phi;$ CP channel

$\rho = \sum_n \lambda_n E_n$

Λ^* CP channel

$\Lambda^*\rho = \sum_n \lambda_n \Lambda^* E_n$

(1) **Trivial compound state** $\sigma_0 = \rho \otimes \Lambda^*\rho$

(2) **Ohya compound state** $\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$

5. Compound States

Compound State (signal is transmitted)

$\mathcal{E}^*_{E,\Lambda^*}$ Liftings

σ_E

$\mathcal{E}^*_{E,\Lambda^*}$ nonlinear liftings

$\rho = \sum_n \lambda_n E_n$

Λ^* CP channel

$\Lambda^* \rho = \sum_n \lambda_n \Lambda^* E_n$

(1) **Trivial compound state** $\sigma_0 = \rho \otimes \Lambda^* \rho$

(2) **Ohya compound state** $\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$

Lifting in the sense of Accardi and Ohya [AO]

$$\sigma_E = \mathcal{E}^*_{E,\Lambda^*}(\rho), \quad \mathcal{E}^*_{E,\Lambda^*} : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ^* *and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$$
$$= \|\rho\|\left(a|0\rangle + b|\theta\rangle\right)\left(\bar{a}\langle 0| + \bar{b}\langle\theta|\right)$$
$$+\left(1-\|\rho\|\right)\left(c|0\rangle + d|\theta\rangle\right)\left(\bar{c}\langle 0| + \bar{d}\langle\theta|\right)$$
$$\Lambda^*\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\alpha\theta\rangle\langle\alpha\theta|$$
$$= \left\|\Lambda^*\rho\right\|\left(a'|0\rangle + b'|\alpha\theta\rangle\right)\left(\bar{a}'\langle 0| + \bar{b}'\langle\alpha\theta|\right)$$
$$+\left(1-\left\|\Lambda^*\rho\right\|\right)\left(c'|0\rangle + d'|\alpha\theta\rangle\right)\left(\bar{c}'\langle 0| + \bar{d}'\langle\alpha\theta|\right)$$

if $\lambda = \frac{1}{2}$ and $\beta = \sqrt{\frac{2}{3}}$, then there exists a compound state Φ satisfying

$$I\left(\rho;\Lambda^*\right) = S\left(\Phi, \rho\otimes\Lambda^*\rho\right) = I_{LN}\left(\rho;\Lambda^*\right).$$

5. Compound States

$$|b|^2 = \frac{1}{\tau^2 + 2\exp\left(-\frac{1}{2}|\theta|^2\right)\tau \ +1}, \quad |c|^2 = \frac{1}{t^2 + 2\exp\left(-\frac{1}{2}|\theta|^2\right)t \ +1},$$

$$|a|^2 = \tau^2 |b|^2, \quad |c|^2 = t^2 |d|^2,$$

$$a\bar{b} = \bar{a}b = \tau |b|^2, \quad c\bar{d} = \bar{c}d = t|c|^2,$$

$$\tau = \frac{-(1-2\lambda) + \sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\theta|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\theta|^2\right)}$$

$$t = \frac{-(1-2\lambda) - \sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\theta|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\theta|^2\right)}$$

5. Compound States

$$|b'|^2 = \frac{1}{\tau'^2 + 2\exp\left(-\frac{1}{2}|\alpha\theta|^2\right)\tau' + 1}, \quad |c'|^2 = \frac{1}{t'^2 + 2\exp\left(-\frac{1}{2}|\alpha\theta|^2\right)t' + 1},$$

$$|a'|^2 = \tau'^2 |b'|^2, \quad |c'|^2 = t'^2 |d'|^2,$$

$$a'\overline{b'} = \overline{a'}b' = \tau' |b'|^2, \quad c'\overline{d'} = \overline{c'}d' = t' |c'|^2,$$

$$\tau' = \frac{-(1-2\lambda) + \sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\alpha\theta|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\alpha\theta|^2\right)}$$

$$t' = \frac{-(1-2\lambda) - \sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\alpha\theta|^2\right)\right)}}{2(1-\lambda)\exp\left(-\frac{1}{2}|\alpha\theta|^2\right)}$$

5. Compound States

$$
\begin{aligned}
\Phi = & \Big[\Big(\sqrt{\|\rho\|}\,(a\,|0\rangle + b\,|\theta\rangle)\otimes\sqrt{\|\Lambda^*\rho\|}\,(a'\,|0\rangle + b'\,|\alpha\theta\rangle)\Big) \\
& + \Big(\sqrt{(1-\|\rho\|)}\,(c\,|0\rangle + d\,|\theta\rangle)\otimes\sqrt{1-\|\Lambda^*\rho\|}\,(c'\,|0\rangle + d'\,|\alpha\theta\rangle)\Big)\Big] \\
& \Big[\Big(\sqrt{\|\rho\|}\,(\overline{a}\,\langle 0| + \overline{b}\,\langle\theta|)\otimes\sqrt{\|\Lambda^*\rho\|}\,(\overline{a'}\,\langle 0| + \overline{b'}\,\langle\alpha\theta|)\Big) \\
& + \Big(\sqrt{(1-\|\rho\|)}\,(\overline{c}\,\langle 0| + \overline{d}\,\langle\theta|)\otimes\sqrt{1-\|\Lambda^*\rho\|}\,(\overline{c'}\,\langle 0| + \overline{d'}\,\langle\alpha\theta|)\Big)\Big] \\
& + \Big[\Big(\sqrt{\|\rho\|}\,(a\,|0\rangle + b\,|\theta\rangle)\otimes\sqrt{1-\|\Lambda^*\rho\|}\,(c'\,|0\rangle + d'\,|\alpha\theta\rangle)\Big) \\
& + \Big(\sqrt{(1-\|\rho\|)}\,(c\,|0\rangle + d\,|\theta\rangle)\otimes\sqrt{\|\Lambda^*\rho\|}\,(a'\,|0\rangle + b'\,|\alpha\theta\rangle)\Big)\Big] \\
& \Big[\Big(\sqrt{\|\rho\|}\,(\overline{a}\,\langle 0| + \overline{b}\,\langle\theta|)\otimes\sqrt{1-\|\Lambda^*\rho\|}\,(\overline{c'}\,\langle 0| + \overline{d'}\,\langle\alpha\theta|)\Big) \\
& + \Big(\sqrt{(1-\|\rho\|)}\,(\overline{c}\,\langle 0| + \overline{d}\,\langle\theta|)\otimes\sqrt{\|\Lambda^*\rho\|}\,(\overline{a'}\,\langle 0| + \overline{b'}\,\langle\alpha\theta|)\Big)\Big]
\end{aligned}
$$

$tr_{\mathcal{H}_2}\Phi;$ CP channel

$tr_{\mathcal{H}_1}\Phi;$ CP channel

$\rho = \sum_n \lambda_n E_n$

Λ^* CP channel

$\Lambda^*\rho = \sum_n \lambda_n \Lambda^* E_n$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ^* *and the input state*

$$\rho = \lambda |0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if $\lambda = \frac{1}{2}$ and α=1, then there exists a compound state Φ satisfying

$$S\left(\Phi, \rho \otimes \Lambda^* \rho\right) = S(\rho).$$

Problem

How to construct the following compound states?

1) Separable compound state

2) Entangled compound state

5. Compound States

For a given initial state ρ,

the Schatten decomposition of ρ is given by

$$\rho = \sum_n \lambda_n E_n$$

Separable compound state

$$\omega_E = \sum_n \lambda_n E_n \otimes E_n$$

Entangled compound state

$$\Phi_E = \left[\sum_k \sqrt{\lambda_k} \left| x_k \right\rangle \otimes \left| x_k \right\rangle \right] \left[\sum_{k'} \sqrt{\lambda_{k'}} \left\langle x_{k'} \right| \otimes \left\langle x_{k'} \right| \right]$$

5. Compound States

Separable compound state

$$\omega_E = \sum_n \lambda_n E_n \otimes E_n$$

Jamiolkowski isomorphism

channel (CP channel)

$$\sum_n \left(I \otimes V_n^*\right)\left(I \otimes V_n\right) = I \otimes I \text{ and}$$

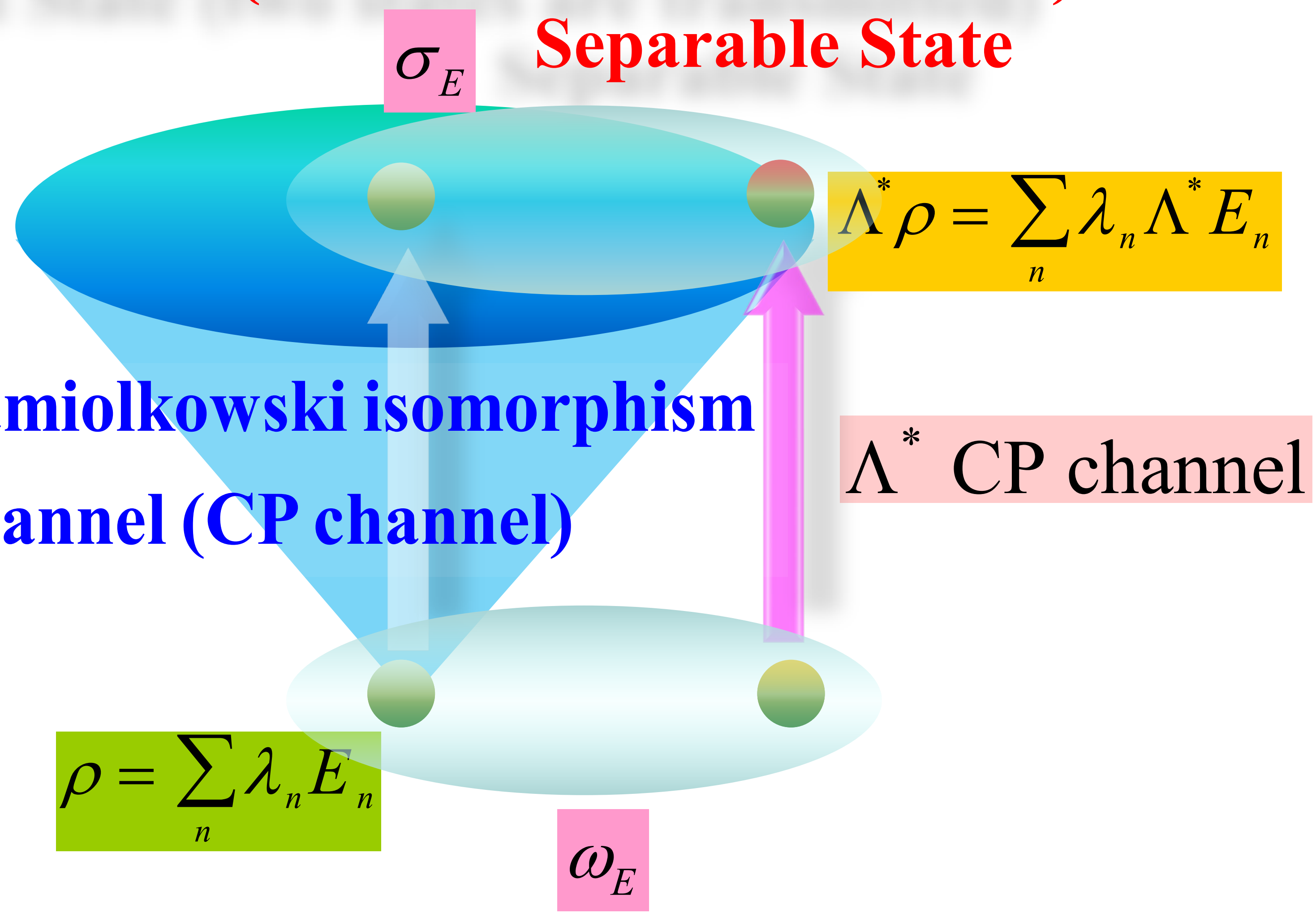
$$\Lambda^* \text{ is given by } \Lambda^*(\rho) = \sum_n V_n \rho V_n^*$$

Ohya compound state (Separable)

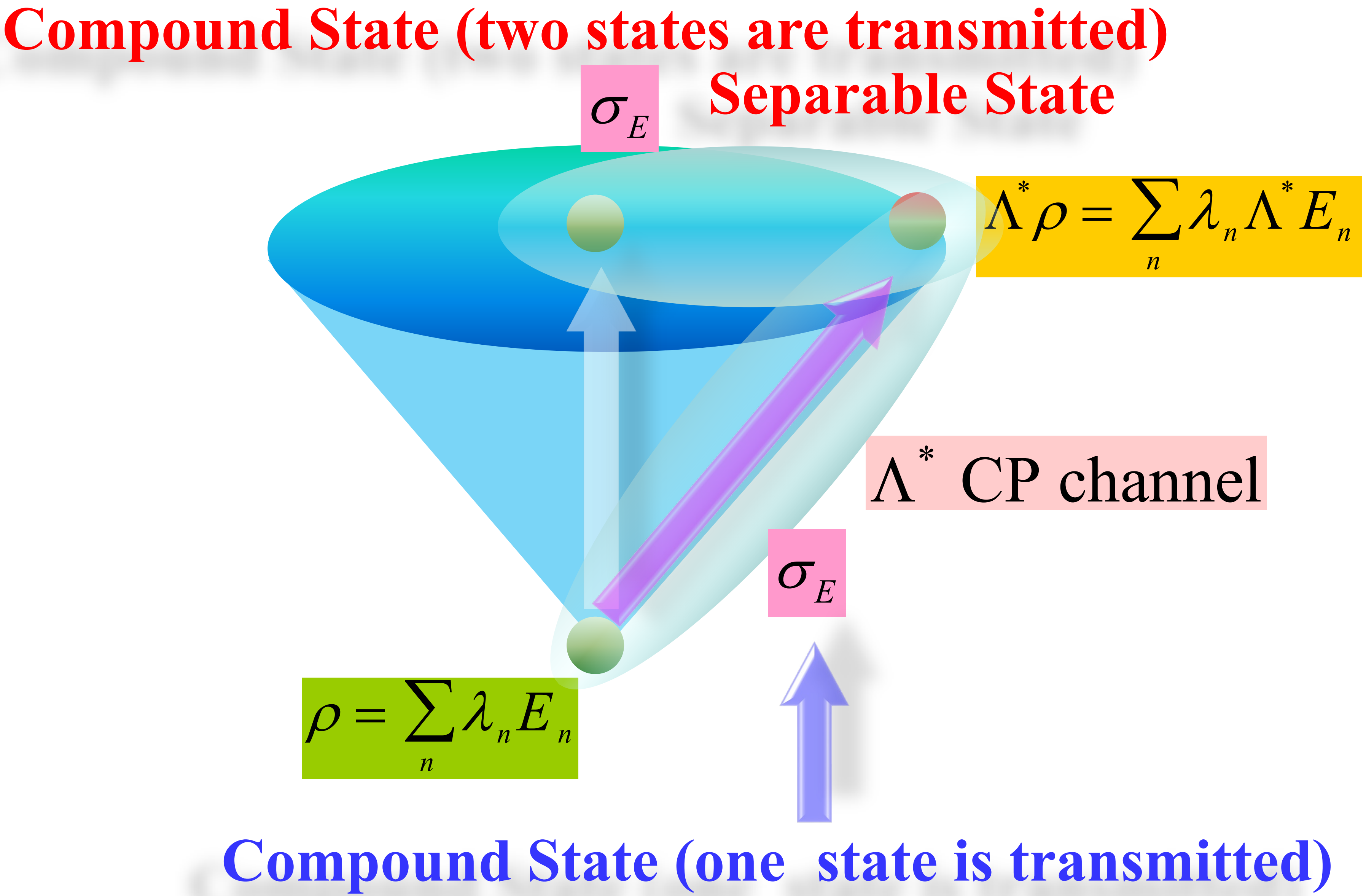
$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$$

5. Compound States

Compound State (two states are transmitted)



5. Compound States



5. Compound States

Entangled compound state

$$\Phi_E = \left[\sum_k \sqrt{\lambda_k} \left| x_k \right\rangle \otimes \left| x_k \right\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}} \left\langle x_{k'} \right| \otimes \left\langle x_{k'} \right|\right]$$

Jamiolkowski isomorphism channel

(CP channel)

$$\sum_n \left(I \otimes V_n^*\right)\left(I \otimes V_n\right) = I \otimes I \text{ and}$$

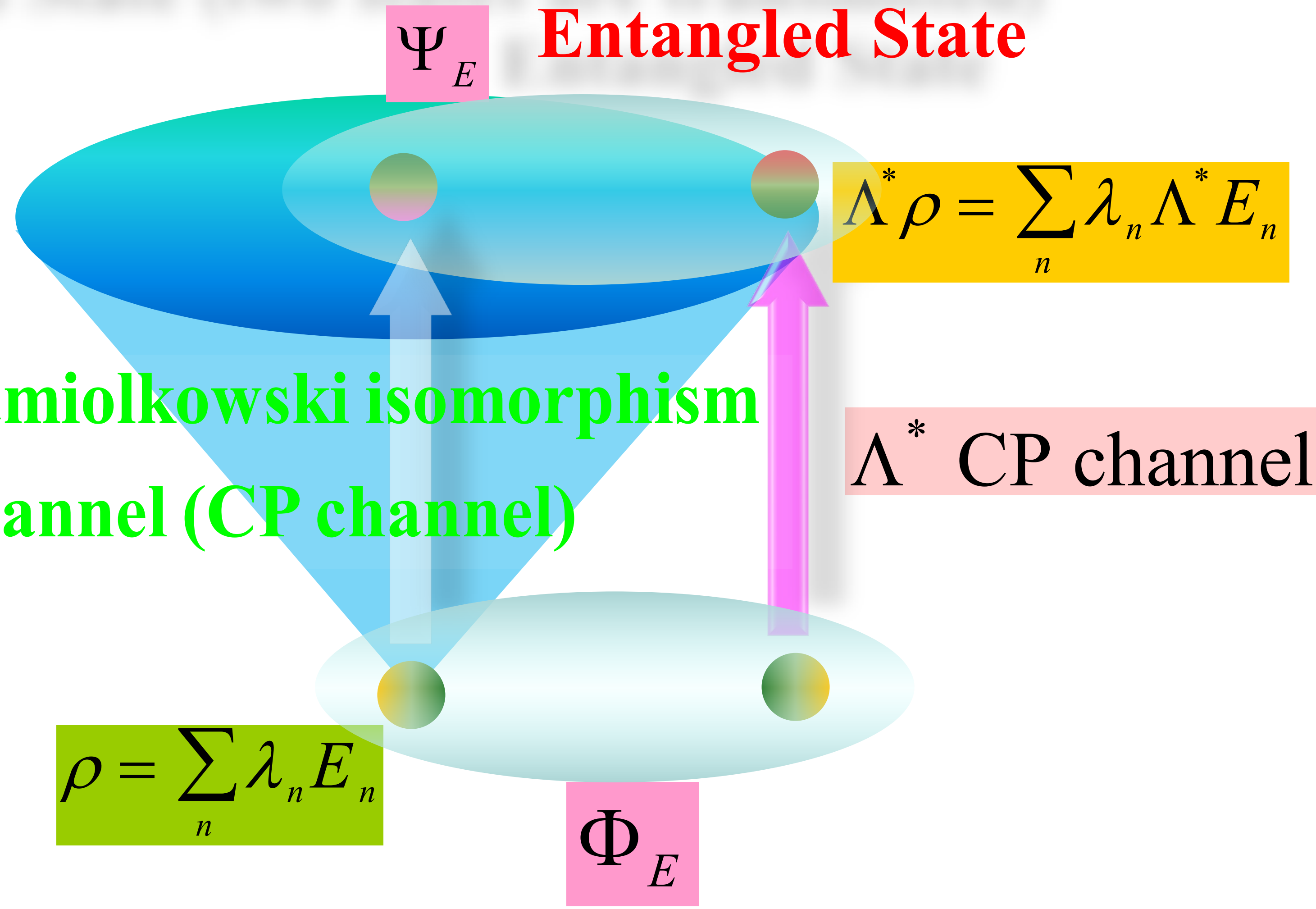
Λ^* is given by $\Lambda^*(\rho) = \sum_n V_n \rho V_n^*$

Entangled compound state

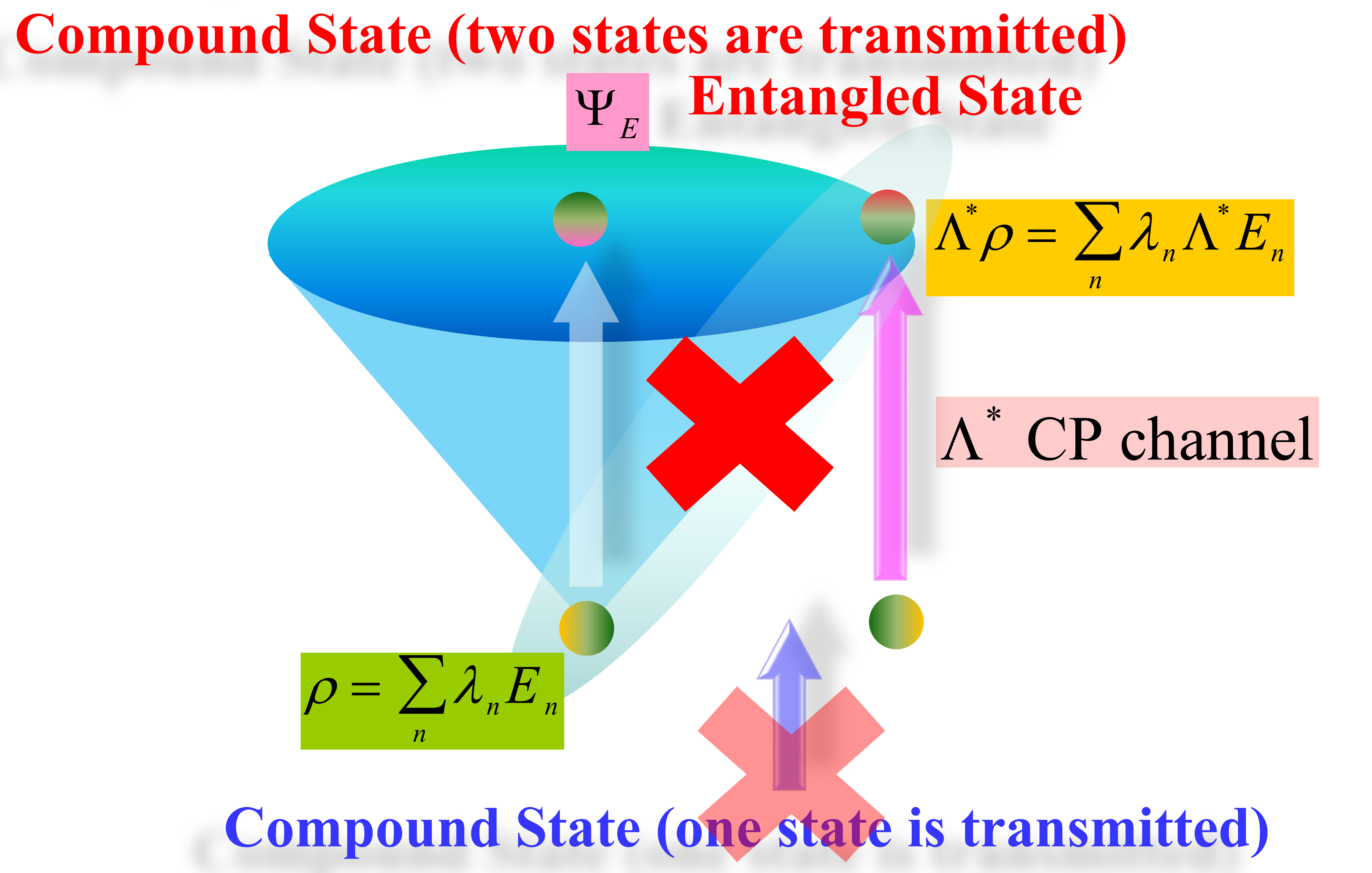
$$\Psi_E = \sum_n \left(I \otimes V_n\right)\left[\sum_k \sqrt{\lambda_k} \left| x_k \right\rangle \otimes \left| x_k \right\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}} \left\langle x_{k'} \right| \otimes \left\langle x_{k'} \right|\right]\left(I \otimes V_n^*\right)$$

5. Compound States

Compound State (two states are transmitted)



5. Compound States



5. Compound States

Theorem [NW] *Let* Ψ_E *be a compound state w.r.t. the initial state* ρ, *the quantum CP channel* Λ^* *and a Schatten decomposition of* $\rho = \sum_k \lambda_k E_k$ *defined by*

$$\Psi_E = \sum_n (I \otimes V_n)\left[\sum_k \sqrt{\lambda_k}\,|x_k\rangle \otimes |x_k\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}}\,\langle x_{k'}| \otimes \langle x_{k'}|\right](I \otimes V_n^*)$$

under the condition

$$\sum_n (I \otimes V_n^*)(I \otimes V_n) = I \otimes I \text{ and } \Lambda^* \text{ is given by } \Lambda^*(\rho) = \sum_n V_n \rho V_n^*.$$

One can obtain two marginal states as follows

$$tr_{\mathcal{H}_2}\Psi_E = S(\rho), \quad tr_{\mathcal{H}_1}\Psi_E = S(\Lambda^*\rho).$$

Upper bound: $\mathrm{S}(\Psi_E, \rho \otimes \Lambda^*\rho) \le 2S(\rho)$

5. Compound States

Corollary [NW] *Let* Ψ_E *be a compound state w.r.t. the initial state* ρ.

1) Pure entangled compound state w.r.t. *the quantum CP channel* Λ^* *and a Schatten decomposition of* $\rho = \sum_k \lambda_k E_k$ *defined by*

$$\Psi_E = (I \otimes V)\left[\sum_k \sqrt{\lambda_k}\, |x_k\rangle \otimes |x_k\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}}\, \langle x_{k'}| \otimes \langle x_{k'}|\right](I \otimes V^*)$$

under the condition

$$(I \otimes V^*)(I \otimes V) = I \otimes I \text{ and } \Lambda^* \text{ is given by } \Lambda^*(\rho) = V\rho V^*.$$

One can obtain two marginal states as follows

$$tr_{\mathcal{H}_2} \Psi_E = S(\rho), \quad tr_{\mathcal{H}_1} \Psi_E = S(\Lambda^* \rho).$$

Upper bound: $\mathrm{S}(\Psi_E, \rho \otimes \Lambda^* \rho) \leq 2S(\rho)$

5. Compound States

Corollary [NW] *Let Ψ_E be a compound state w.r.t. the initial state ρ.*

2) Mixed entangled compound state w.r.t. *the quantum CP channel Λ^* and a Schatten decomposition of $\rho = \sum_k \lambda_k E_k$ defined by*

$$\Psi_E = \sum_n (I \otimes V_n) \left[\sum_k \sqrt{\lambda_k} |x_k\rangle \otimes |x_k\rangle \right] \left[\sum_{k'} \sqrt{\lambda_{k'}} \langle x_{k'}| \otimes \langle x_{k'}| \right] (I \otimes V_n^*)$$

under the condition $V_n = \sum_m \mu_n \left| y_m^{(n)} \right\rangle \langle x_m |$

$$\sum_n (I \otimes V_n^*)(I \otimes V_n) = I \otimes I \text{ and } \Lambda^* \text{ is given by } \Lambda^*(\rho) = \sum_n V_n \rho V_n^*.$$

One can obtain two marginal states as follows

$$tr_{\mathcal{H}_2} \Psi_E = S(\rho), \quad tr_{\mathcal{H}_1} \Psi_E = S(\Lambda^* \rho).$$

$a)\ y_m^{(n)} \perp y_m^{(n')}\ (\forall m, \forall n \neq n') \Rightarrow$ Upper bound: $\mathrm{S}(\Psi_E, \rho \otimes \Lambda^* \rho) \leq 2S(\rho)$

5. Compound States

Lemma [NW] *Let* Ψ_E *be a compound state w.r.t. the initial state* ρ.

2) Mixed entangled compound state w.r.t. *the quantum CP channel* Λ^* *and a Schatten decomposition of* $\rho = \sum_k \lambda_k E_k$ *defined by*

$$\Psi_E = (I \otimes V)\left[\sum_k \sqrt{\lambda_k}\,|x_k\rangle \otimes |x_k\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}}\,\langle x_{k'}| \otimes \langle x_{k'}|\right](I \otimes V^*)$$

under the condition

$$(I \otimes V^*)(I \otimes V) = I \otimes I \text{ and } \Lambda^* \text{ is given by } \Lambda^*(\rho) = V\rho V^*.$$

One can obtain two marginal states as follows

$$tr_{\mathcal{H}_2}\Psi_E = S(\rho), \quad tr_{\mathcal{H}_1}\Psi_E = S(\Lambda^*\rho).$$

Upper bound: $\mathrm{S}(\Psi_E, \rho \otimes \Lambda^*\rho) \le 2S(\rho)$

5. Compound States

Theorem [NW] Ψ_E *is the compound state given above.*

For any $\mu \in [0,1]$, *let* $\Psi_{E,\mu}$ *be a compound state defined by*

$$\Psi_{E,\mu} = \mu\,\sigma_E + (1-\mu)\Psi_E.$$

One can obtain two marginal states as follows

$$tr_{\mathcal{H}_2}\Psi_{E,\mu} = S(\rho),\quad tr_{\mathcal{H}_1}\Psi_{E,\mu} = S(\Lambda^*\rho).$$

Upper bound : $\mathrm{S}(\Psi_{E,\mu}, \rho\otimes\Lambda^*\rho) \le (2-\mu)S(\rho).$

5. Compound States

Theorem [NW] One can define a linear CP channel Ξ^* depending on the Schatten decomposition of ρ

from $\Phi_E = \left[\sum_k \sqrt{\lambda_k} \left| x_k \right\rangle \otimes \left| x_k \right\rangle \right] \left[\sum_{k'} \sqrt{\lambda_{k'}} \left\langle x_{k'} \right| \otimes \left\langle x_{k'} \right| \right]$

to $\omega_E = \sum_n \lambda_n E_n \otimes E_n$ as follows:

$$\Xi^*(\bullet) = \sum_{n,n'} W_{nn'}(\bullet) W_{nn'}^*,$$

where $W_{nn'}$ is given by

$$W_{nn'} = \left| x_n \right\rangle \left\langle x_n \right| \otimes \left| x_{n'} \right\rangle \left\langle x_{n'} \right|$$

satisfying $\sum_{n,n'} W_{nn'}^* W_{nn'} = I \otimes I.$

5. Compound States

Theorem [NW] One can also define a linear CP channel $\tilde{\Xi}^*$ depending on the Schatten decomposition of ρ from $\omega_E = \sum_n \lambda_n E_n \otimes E_n$ to $\Phi_E = \left[\sum_k \sqrt{\lambda_k} |x_k\rangle \otimes |x_k\rangle\right]\left[\sum_{k'} \sqrt{\lambda_{k'}} \langle x_{k'}| \otimes \langle x_{k'}|\right]$ as follows:

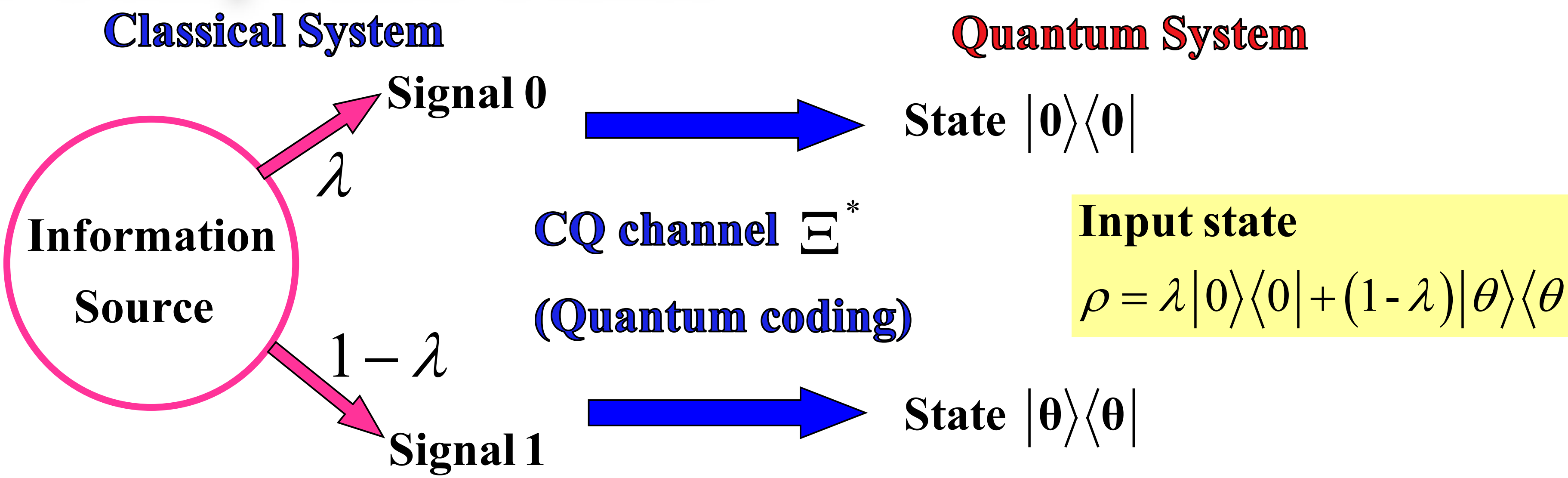
$$\tilde{\Xi}^*(\bullet) = \sum_{n,n'} w_{nn'}(\bullet) w_{nn'}^*,$$

where $w_{nn'}$ is given by

$$w_{nn'} = \left(\sum_k \sqrt{\lambda_k} |x_k\rangle \otimes |x_k\rangle\right)\langle x_n| \otimes \langle x_{n'}|$$

satisfying $\sum_{n,n'} w_{nn'}^* w_{nn'} = I \otimes I.$

5. Compound States



5. Compound States

Quantum System

Schatten decompotition of ρ

$$\rho = \|\rho\|\left(a|0\rangle + b|\theta\rangle\right)\left(\bar{a}\langle 0| + \bar{b}\langle\theta|\right)$$
$$+\left(1-\|\rho\|\right)\left(c|0\rangle + d|\theta\rangle\right)\left(\bar{c}\langle 0| + \bar{d}\langle\theta|\right)$$

Input state

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$$

Attenuation channel Λ_{α}^{*}

Output state

$$\Lambda^{*}\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\alpha\theta\rangle\langle\alpha\theta|$$

Schatten decompotition of $\Lambda^{*}\rho$

$$\Lambda^{*}\rho = \|\Lambda^{*}\rho\|\left(a'|0\rangle + b'|\alpha\theta\rangle\right)\left(\overline{a'}\langle 0| + \overline{b'}\langle\alpha\theta|\right)$$
$$+\left(1-\|\Lambda^{*}\rho\|\right)\left(c'|0\rangle + d'|\alpha\theta\rangle\right)\left(\overline{c'}\langle 0| + \overline{d'}\langle\alpha\theta|\right)$$

5. Compound States

Quantum System

Schatten decompotition of input state ρ

$$\rho = \|\rho\| \left(a|0\rangle + b|\theta\rangle \right)\left(\bar{a}\langle 0| + \bar{b}\langle \theta| \right) + \left(1 - \|\rho\|\right)\left(c|0\rangle + d|\theta\rangle \right)\left(\bar{c}\langle 0| + \bar{d}\langle \theta| \right)$$

$tr_{\mathcal{H}_2}\Phi$

Compound state Φ with respect to ρ and $\Lambda^*\rho$

$$\Phi = \left(\sqrt{\|\rho\|}\left(a|0\rangle + b|\theta\rangle \right) \otimes \sqrt{\|\Lambda^*\rho\|}\left(a'|0\rangle + b'|\alpha\theta\rangle \right) + \sqrt{1-\|\rho\|}\left(c|0\rangle + d|\theta\rangle \right) \otimes \sqrt{1-\|\Lambda^*\rho\|}\left(c'|0\rangle + d'|\alpha\theta\rangle \right) \right)$$

$$\left(\sqrt{\|\rho\|}\left(\bar{a}\langle 0| + \bar{b}\langle \theta| \right) \otimes \sqrt{1-\|\Lambda^*\rho\|}\left(\bar{c}'\langle 0| + \bar{d}'\langle \alpha\theta| \right) + \sqrt{1-\|\rho\|}\left(\bar{c}\langle 0| + \bar{d}\langle \theta| \right) \otimes \sqrt{\|\Lambda^*\rho\|}\left(\bar{a}'\langle 0| + \bar{b}'\langle \alpha\theta| \right) \right)$$

$$+ \left(\sqrt{\|\rho\|}\left(a|0\rangle + b|\theta\rangle \right) \otimes \sqrt{1-\|\Lambda^*\rho\|}\left(c'|0\rangle + d'|\alpha\theta\rangle \right) + \sqrt{1-\|\rho\|}\left(c|0\rangle + d|\theta\rangle \right) \otimes \sqrt{\|\Lambda^*\rho\|}\left(a'|0\rangle + b'|\alpha\theta\rangle \right) \right)$$

Attenuation channel Λ_α^*

$tr_{\mathcal{H}_1}\Phi$

Schatten decompotition of output state $\Lambda^*\rho$

$$\Lambda^*\rho = \|\Lambda^*\rho\| \left(a'|0\rangle + b'|\alpha\theta\rangle \right)\left(\bar{a}'\langle 0| + \bar{b}'\langle \alpha\theta| \right) + \left(1 - \|\Lambda^*\rho\|\right)\left(c'|0\rangle + d'|\alpha\theta\rangle \right)\left(\bar{c}'\langle 0| + \bar{d}'\langle \alpha\theta| \right)$$

Communication Process

Quantum System

Schatten decompotition of input state ρ

$$\rho = \|\rho\|\left(a|0\rangle + b|\theta\rangle\right)\left(\overline{a}\langle 0| + \overline{b}\langle\theta|\right)$$
$$+\left(1-\|\rho\|\right)\left(c|0\rangle + d|\theta\rangle\right)\left(\overline{c}\langle 0| + \overline{d}\langle\theta|\right)$$

$$S\left(\Phi, \rho\otimes\Lambda_\alpha^*\rho\right) = S\left(\Lambda_\alpha^*\rho\right) + S(\rho) - \sum_{k=1}^{2}\eta(\mu_k)$$

$$\mu_k = \frac{1}{2}\left(1+(-1)^k\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\theta|^2\right)\right)}\sqrt{1-4\lambda(1-\lambda)\left(1-\exp\left(-|\alpha|^2|\theta|^2\right)\right)}\right)$$

$$\eta(t) = -t\log t \quad (0\le t\le 1)$$

Attenuation channel Λ_α^*

Schatten decompotition of output state $\Lambda^*\rho$

$$\Lambda^*\rho = \|\Lambda^*\rho\|\left(a'|0\rangle + b'|\alpha\theta\rangle\right)\left(\overline{a'}\langle 0| + \overline{b'}\langle\alpha\theta|\right)$$
$$+\left(1-\|\Lambda^*\rho\|\right)\left(c'|0\rangle + d'|\alpha\theta\rangle\right)\left(\overline{c'}\langle 0| + \overline{d'}\langle\alpha\theta|\right)$$

5. Compound States

Quantum System

Schatten decompotition of input state ρ

$$\rho = \|\rho\| \left(a|0\rangle + b|\theta\rangle \right)\left(\bar{a}\langle 0| + \bar{b}\langle \theta| \right) + \left(1 - \|\rho\|\right)\left(c|0\rangle + d|\theta\rangle \right)\left(\bar{c}\langle 0| + \bar{d}\langle \theta| \right)$$

$$I_{LN}\left(\rho;\Lambda_\alpha^*\right) \equiv S\left(\Lambda_\alpha^*\rho\right) + S(\rho) - S_e\left(\rho,\Lambda_\alpha^*\right)$$

$$= S\left(\Lambda_\alpha^*\rho\right) + S(\rho) - \sum_{k=1}^{2} \eta(\nu_k)$$

$$\nu_k = \frac{1}{2}\left(1 + (-1)^k \sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-\left(1 - |\alpha|^2\right)|\theta|^2\right)\right)}\right)$$

Attenuation channel Λ_α^*

Schatten decompotition of output state $\Lambda^*\rho$

$$\Lambda^*\rho = \|\Lambda^*\rho\| \left(a'|0\rangle + b'|\alpha\theta\rangle \right)\left(\overline{a'}\langle 0| + \overline{b'}\langle \alpha\theta| \right) + \left(1 - \|\Lambda^*\rho\|\right)\left(c'|0\rangle + d'|\alpha\theta\rangle \right)\left(\overline{c'}\langle 0| + \overline{d'}\langle \alpha\theta| \right)$$

5. Compound States

Case 1) $|\alpha| = |\beta| = \frac{\sqrt{2}}{2}$

$$S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right) = S\left(\Lambda_\alpha^* \rho\right) + S(\rho) - \sum_{k=1}^{2} \eta(\mu_k)$$

$$\mu_k = \frac{1}{2}\left(1 + (-1)^k \sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\theta|^2\right)\right)} \sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\alpha|^2 |\theta|^2\right)\right)}\right)$$

there exists a positive number $\omega_{\Lambda_\alpha^*} = 2\log\left(\sqrt{2} + 1\right)$

(1) if the input state ρ satifying $\omega_{\Lambda_\alpha^*} < |\theta|^2$ then

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) > S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda_1 < \lambda < \lambda_2$

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) = S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda = \lambda_1$ or λ_2

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) < S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $0 < \lambda < \lambda_1$ or $\lambda_2 < \lambda < 1$

where $\lambda_k = \frac{1}{2}\left(1 - (-1)^k \sqrt{\frac{\exp\left(-\frac{3}{2}|\theta|^2\right) + \exp\left(-\frac{1}{2}|\theta|^2\right)}{\left(1 - \exp\left(-|\theta|^2\right)\right)\left(1 - \exp\left(-\frac{1}{2}|\theta|^2\right)\right)}}\right)$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ_{α}^{*} *and the input state*

$$\rho = \lambda |0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if Λ_{α}^{*} is given by $0 \le |\alpha| \le 1$ then there exists a positive number $\omega_{\Lambda_{\alpha}^{*}}$.

(2) if the input state ρ satifying $\omega_{\Lambda_{\alpha}^{*}} = |\theta|^2$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) = S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } \lambda = 1 \text{ or } 0$$

(3) if the input state ρ satifying $0 \le |\theta|^2 < \omega_{\Lambda_{\alpha}^{*}}$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) > S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } 0 \le \lambda \le 1$$

5. Compound States

Case 2) $|\alpha|^2 = \frac{1}{3}, \quad |\beta|^2 = \frac{2}{3}$

$$S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right) = S\left(\Lambda_\alpha^* \rho\right) + S(\rho) - \sum_{k=1}^{2} \eta(\mu_k)$$

$$\mu_k = \frac{1}{2}\left(1 + (-1)^k \sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\theta|^2\right)\right)}\sqrt{1 - 4\lambda(1-\lambda)\left(1 - \exp\left(-|\alpha|^2|\theta|^2\right)\right)}\right)$$

there exists a positive number $\omega_{\Lambda_\alpha^*} = 3\log\left(\frac{\sqrt{33}-5}{2}\sqrt[3]{\frac{\sqrt{33}}{9} + \frac{27}{17}} + \frac{3\left(7-\sqrt{33}\right)}{4}\sqrt[3]{\frac{34\sqrt{33}}{243} + \frac{586}{729}} + \frac{1}{3}\right)$

(1) if the input state ρ satifying $\omega_{\Lambda_\alpha^*} < |\theta|^2$ then

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) > S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda_1 < \lambda < \lambda_2$

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) = S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda = \lambda_1$ or λ_2

$I_{LN}\left(\rho; \Lambda_\alpha^*\right) < S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $0<\lambda<\lambda_1$ or $\lambda_2 < \lambda < 1$

where $\lambda_k = \frac{1}{2}\left(1 - (-1)^k \sqrt{\frac{\exp\left(-\frac{4}{3}|\theta|^2\right) + \exp\left(-\frac{2}{3}|\theta|^2\right)}{\left(1 - \exp\left(-|\theta|^2\right)\right)\left(1 - \exp\left(-2|\theta|^2\right)\right)}}\right)$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ_{α}^{*} *and the input state*

$$\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if Λ_{α}^{*} is given by $0 \le |\alpha| \le 1$ then there exists a positive number $\omega_{\Lambda_{\alpha}^{*}}$.

(2) if the input state ρ satifying $\omega_{\Lambda_{\alpha}^{*}} = |\theta|^2$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) = S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } \lambda = 1 \text{ or } 0$$

(3) if the input state ρ satifying $0 \le |\theta|^2 < \omega_{\Lambda_{\alpha}^{*}}$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) > S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } 0 \le \lambda \le 1$$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ_α^* *and the input state*

$$\rho = \lambda |0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if Λ_α^* is given by $0 \le |\alpha| \le 1$ then there exists a positive number $\omega_{\Lambda_\alpha^*}$.

(1) if the input state ρ satifying $\omega_{\Lambda_\alpha^*} < |\theta|^2$ then

$I_{LN}\left(\rho;\Lambda_\alpha^*\right) > S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda_1 < \lambda < \lambda_2$

$I_{LN}\left(\rho;\Lambda_\alpha^*\right) = S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $\lambda = \lambda_1$ or λ_2

$I_{LN}\left(\rho;\Lambda_\alpha^*\right) < S\left(\Phi, \rho \otimes \Lambda_\alpha^* \rho\right)$ for $0<\lambda<\lambda_1$ or $\lambda_2 < \lambda < 1$

where $\lambda_k = \frac{1}{2}\left(1-(-1)^k \sqrt{\frac{\exp\left(-\left(1+|\alpha|^2\right)|\theta|^2\right)+\exp\left(-\left(1-|\alpha|^2\right)|\theta|^2\right)}{\left(1-\exp\left(-|\theta|^2\right)\right)\left(1-\exp\left(-|\alpha|^2|\theta|^2\right)\right)}}\right)$

5. Compound States

Theorem [NW] *For the attenuation channel* Λ_{α}^{*} *and the input state*

$$\rho = \lambda |0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|,$$

if Λ_{α}^{*} is given by $0 \le |\alpha| \le 1$ then there exists a positive number $\omega_{\Lambda_{\alpha}^{*}}$.

(2) if the input state ρ satifying $\omega_{\Lambda_{\alpha}^{*}} = |\theta|^2$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) = S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } \lambda = 1 \text{ or } 0$$

(3) if the input state ρ satifying $0 \le |\theta|^2 < \omega_{\Lambda_{\alpha}^{*}}$ then

$$I_{LN}\left(\rho;\Lambda_{\alpha}^{*}\right) > S\left(\Phi, \rho \otimes \Lambda_{\alpha}^{*}\rho\right) \text{ for } 0 \le \lambda \le 1$$

Conclusion

1) We explained the quantum channels associated with the open system dynamics and the quantum communication processes.
2) The Ohya mutual entropy is treated for purely quantum systems, and semi-quantum mutual entropy are special case of the Ohya mutual entropy.
3) The Ohya mutual entrpy is one of the most suitable transmitted complexity for discussing the efficiency of information transmission in quantum communication systems
4) We briefly reviewed the mean entropy and the mean mutual entropy for general quantum systems.
5) The lower bound of the mean entropy for the open system dynamics is obtained. For a given assumption, the mean entropy and the mean mutual entropy for the open system dynamics are calculated.
6) We discuss how to construct the compound states. We will show the relation between the Lindblad entropy and the relative entropy with respect to the entangled compound state and trivial compound state consisted of the input state and the output state.

Thank you very much!

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