No purification in all discrete theories and the power of the complete extension

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Abstract

The concept of a purification of a state of a quantum system is one of the central notions regarding Quantum Information Theory. **We develop a counterpart of the purification in general probabilistic theories (GPTs)**, where the state of a multipartite system is described by a so-called non-signaling box. We study properties of the purified state, concerning its extremality in the set of boxes and access to statistical ensembles. We show, that the famous **Popescu-Rohrlich box is a purification of a maximally mixed box**, and interestingly the only extremal one among purifications of binary input-output boxes. Moreover, we come up with a **no-go theorem** stating that in all discrete theories most of the states does not have a purification.

Purification in quantum mechanics

Given state ρ_A of system A, its purification to system B called here ψ_{AB} satisfies the following three equivalent properties:

- It is an **extremal** state in the set of states of system AB.
- Provides **full access** to ensembles of ρ_A, generated via appropriate measurement on system B solely.
- ψ_{AB} can be **turned into** any **extension** of ρ_A via appropriate quantum channel on system B.

We ask if analogous requirements can be satisfied by some counterpart of purification of a box.

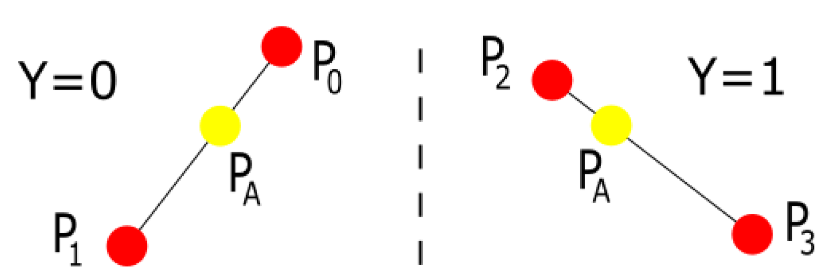

Figure 1. Setting different inputs results in choice of distinct statistical ensembles.

The realm of non-signaling boxes

The so called non-signaling boxes - normalized conditional probability distributions satisfying constraints of the form:

$$\forall_{b,y}\forall_{x,x'} \sum_a P_{AB}\left(A=a,B=b|X=x,Y=y\right) = \sum_a P_{AB}\left(A=a,B=b|X=x',Y=y\right).$$

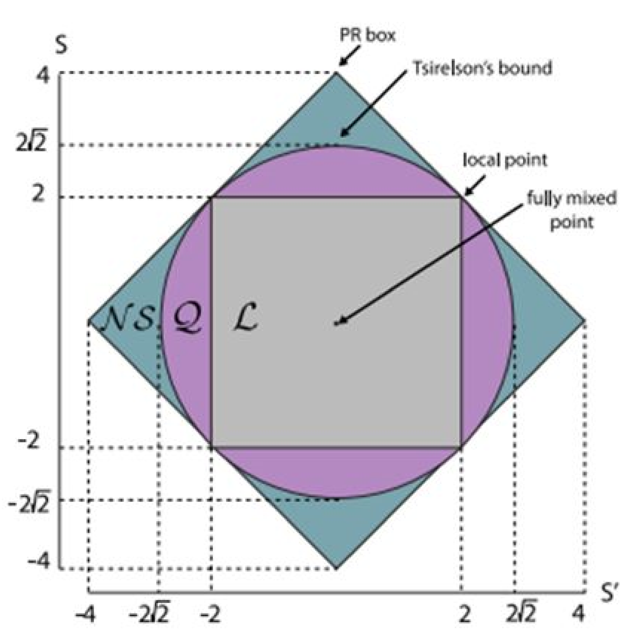

Figure 2. Illustration of a difference between possible correlation in local theory, quantum mechanics and non-signaling GPTs [5].

The complete extension

Definition. *A pure members ensemble of a box P is **minimal** if when devoid of one member is no longer an ensemble of P.*

Definition. *Given box P_A, we say that a box P_{AB} is its complete extension to system B if for any y and b there holds*

$$P_{AB}(A,B=b|X,Y=y) = P_A^{b,y}(A|X)p(B=b|Y=y),$$

such that $\{p(B=b|Y=y), P_A^{b,y}(A|X)\}$ is a minimal ensemble of the box P_A, and corresponding to each minimal ensemble with the property that it is an ensemble of box P_A, there is exactly one y which generates it.

The main result

Theorem 1. *The extending system of the complete extension gives access to any possible (even mixed) ensemble of the extended box.*

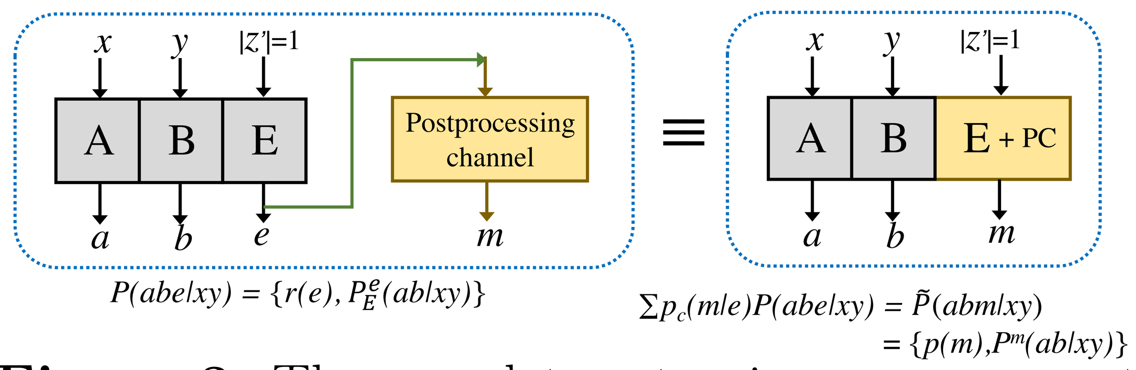

Figure 3. The complete extension upon access to arbitrary randomness can access any ensemble.

Theorem 2. *The complete extension gives access to any arbitrary extension iff it has access to all possible ensembles.*

Example: Popescu-Rohrlich Box [1] derivation

Any single binary input single binary output box is a convex combination of four deterministic, extremal boxes $P(A|X) = \sum_i q_i P_i^E(A|X)$. Let us consider a maximally mixed box $P_A^m \equiv P_A^m(A|X)$

$$\mathrm{P}_0^E = \frac{1|1}{0|0},\quad \mathrm{P}_1^E = \frac{0|0}{1|1},\quad \mathrm{P}_2^E = \frac{1|0}{0|1},\quad \mathrm{P}_3^E = \frac{0|1}{1|0}.$$

$$P_A^m = \frac{1/2|1/2}{1/2|1/2},$$

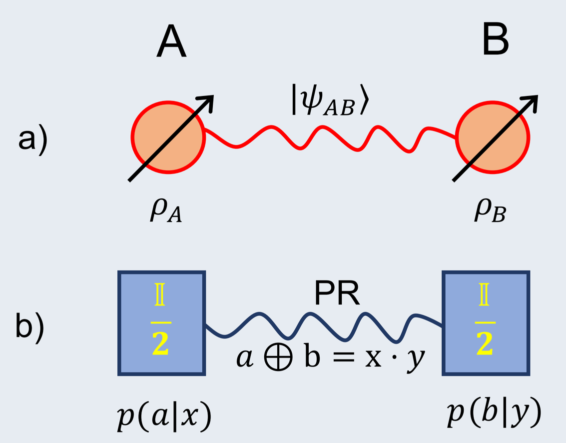

Figure 4. Extending maximally mixed state in a) quantum b) non-signaling theory.

The pairs $\{P_0, P_1\}$, $\{P_2, P_3\}$ are minimal ensembles of P_A^m. For extending system we associate ensembles with 0 and 1 input value. The link between corresponding output boxes and the setting of the input in extending system, reveals bipartite box satisfying condition $\Pr\left[A \oplus B = X \cdot Y\right] = 1$.

The no-go theorem

Theorem 3. *For any discrete theory $\mathcal{T}$, only finite number of its states can be purified.*

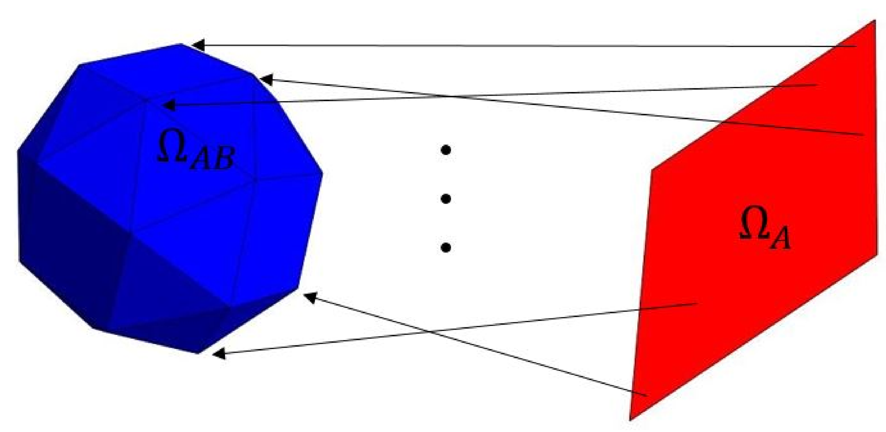

Figure 5. Main idea of the proof: state space of any discrete theory has not enough vertices to purify all states from any other theory.

Discrete theories

Theory is discrete iff its state space Ω_A is a **convex hull of finite** but possibly large number of **points** [3].

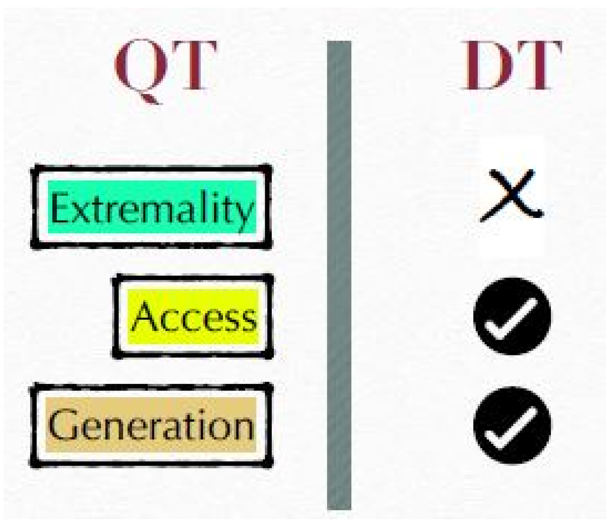

Figure 6. Diagram comparing features of quantum mechanical (**QM**) purification and the complete extension in any discrete theory (**DT**).

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