Quantum trajectories and a priori evolution for two-level atom interacting with a single-photon wavepacket

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Model

- We consider a collision model with an environment given as an infinite chain of not interacting between themselves qubits, prepared initially in a single-photon state.
- The environment qubits do not interact between themselves, but they interact with $\mathcal{S}$, one after the other. A measurement is performed on each qubit just after its interaction with $\mathcal{S}$. A conditional evolution of $\mathcal{S}$ is governed by a set of filtering equations.
- We derive the set filtering equations for a continuous in time stochastic evolution of the system starting from the discrete model.
- As an example of open system we consider a two-level system.

Results

- A formula for the conditional state of the system, depending on the results of all measurements performed on the bath qubits, is given.
- A physical interpretation of the quantum trajectories for the system interacting with the environment in a single-photon state is presented.
- Analytical formulas for the whole statistics of the output photons are given.

1 Repeated interactions and measurements model

Composed system We consider a quantum system $\mathcal{S}$ coupled to the environment modeled by an infinite chain of qubits. The environment qubits do not interact between themselves, but they interact with $\mathcal{S}$, one after the other, each during a time interval of the length τ. The composed system, consisting of the environment and $\mathcal{S}$, is described by the Hilbert space

$$\mathcal{H}_{\mathcal{E}} \otimes \mathcal{H}_{\mathcal{S}},$$

where $\mathcal{H}_{\mathcal{S}}$ is the Hilbert space associated with $\mathcal{S}$ and $\mathcal{H}_{\mathcal{E}} = \bigotimes_{k=0}^{+\infty} \mathcal{H}_{\mathcal{E},k}$, where $\mathcal{H}_{\mathcal{E},k} = \mathbb{C}^2$ is the Hilbert space of the k-th qubit interacting with $\mathcal{S}$ in the interval $[k\tau, (k+1)\tau)$.

Initial state of the composed system

$$|1_\xi\rangle \otimes |\psi\rangle$$

$$|1_\xi\rangle = \sqrt{\tau} \sum_{k=0}^{+\infty} \xi_k |1_k\rangle$$

$$|1_k\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \ldots |0\rangle_{k-1} \otimes |1\rangle_k \otimes |0\rangle_{k+1} \otimes |0\rangle_{k+2} \ldots$$

$\sum_{k=0}^{+\infty} |\xi_k|^2 \tau = 1$. $|1_\xi\rangle$ is a discrete analogue of a single-photon state in the model of continuous modes.

Evolution of the composed system The evolution of the composed system is given by the sequence of unitary operators, for $j \geq 1$

$$U_{j\tau} = V_{j-1} V_{j-2} \ldots V_0, \qquad U_0 = \mathbb{1},$$
$$V_k = \exp(-i\tau H_k),$$
$$H_k = \mathbb{1}_k \otimes H_{\mathcal{S}} + \frac{i}{\sqrt{\tau}} \left(\sigma_k^+ \otimes L - \sigma_k^- \otimes L^\dagger \right),$$

where $H_{\mathcal{S}}$ is the Hamiltonian of $\mathcal{S}$, L is a bounded operator acting on $\mathcal{H}_{\mathcal{S}}$, $\sigma_k^- = |0\rangle_k \langle 1|$, $\sigma_k^+ = |1\rangle_k \langle 0|$, where $|0\rangle_k$ and $|1\rangle_k$ stand for the ground and excited states of the k-th qubit of the environment. We set the Planck constant $\hbar = 1$. To simplify the notation we omit the identity operators from $\bigotimes_{i \neq k} \mathcal{H}_{\mathcal{E},i}$.

Measurements A free evolution of the composed system is subsequently interrupted by the measurements performed on the bath qubits just after their interaction with the system $\mathcal{S}$. We consider the measurement of the observable

$$\sigma_k^+ \sigma_k^-, \qquad k = 0, 1, 2, \ldots . \tag{1}$$

2 Conditional evolution

Theorem The conditional state of $\mathcal{S}$ and the part of the environment chain which has not interacted with the system $\mathcal{S}$ up to time $j\tau$ is at the moment $j\tau$ given by

$$|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle = \frac{|\Psi_{j|\boldsymbol{\eta}_j}\rangle}{\sqrt{\langle \Psi_{j|\boldsymbol{\eta}_j} | \Psi_{j|\boldsymbol{\eta}_j} \rangle}},$$

where $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ is the unnormalized conditional vector from $\bigotimes_{k=j}^{+\infty} \mathcal{H}_{\mathcal{E},k} \otimes \mathcal{H}_{\mathcal{S}}$ having the form

$$|\Psi_{j|\boldsymbol{\eta}_j}\rangle = \sqrt{\tau} \sum_{k=j}^{+\infty} \xi_k |1_k\rangle_{[j,+\infty)} \otimes |\alpha_{j|\boldsymbol{\eta}_j}\rangle + |vac\rangle_{[j,+\infty)} \otimes |\beta_{j|\boldsymbol{\eta}_j}\rangle,$$

where

$$|1_k\rangle_{[j,+\infty)} = |0\rangle_0 \otimes |0\rangle_1 \otimes \ldots |0\rangle_{k-1} \otimes |1\rangle_k \otimes |0\rangle_{k+1} \otimes |0\rangle_{k+2} \ldots,$$

and $|vac\rangle_{[j,+\infty)} = |0\rangle_j \otimes |0\rangle_{j+1} \otimes |0\rangle_{j+2} \otimes |0\rangle_{j+3} \ldots$. Moreover, $\boldsymbol{\eta}_j$ is a stochastic binary j-vector $\boldsymbol{\eta}_j = (\eta_j, \eta_{j-1}, \ldots, \eta_1)$ with $\eta_k \in \{0, 1\}$ representing the results of all measurements of (1) up to the time $j\tau$. The vectors $|\alpha_{j|\boldsymbol{\eta}_j}\rangle, |\beta_{j|\boldsymbol{\eta}_j}\rangle$ from $\mathcal{H}_{\mathcal{S}}$ depend on the results of measurements up to $j\tau$ and they satisfy the set of recursive equations.

Interpretation of $|\tilde{\Psi}_{j|\boldsymbol{\eta}_j}\rangle$ The physical interpretation of $|\Psi_{j|\boldsymbol{\eta}_j}\rangle$ is very intuitive. The first term means the scenario: all qubits of the environment up to time $j\tau$ were prepared in the ground state and the qubit prepared in the excited state appears only in the future.

The second term represents the situation in which $\mathcal{S}$ has already interacted with the qubit prepared in the excited state and in the future it will interact with the environment being in the vacuum.

3 Filtering equations in the continuous limit

The stochastic evolution of $\mathcal{S}$ depending on the results of the measurements is given by of the set of coupled equations:

$$d\tilde{\rho}_t = \mathcal{L}\tilde{\rho}_t dt + [\tilde{\rho}_t^{01}, L^\dagger]\xi_t dt + [L, \tilde{\rho}_t^{10}]\xi_t^* dt$$
$$+ \left\{ \frac{1}{k_t} \left(L\tilde{\rho}_t L^\dagger + L\tilde{\rho}_t^{10}\xi_t + \tilde{\rho}_t^{01}L^\dagger\xi_t + \tilde{\rho}_t^{00}|\xi_t|^2 \right) - \tilde{\rho}_t \right\} (dn(t) - k_t dt),$$

$$d\tilde{\rho}_t^{01} = \mathcal{L}\tilde{\rho}_t^{01} dt + \left[L, \tilde{\rho}_t^{00} \right] \xi_t^* dt$$
$$+ \left\{ \frac{1}{k_t} \left(L\tilde{\rho}_t^{01} L^\dagger + L\tilde{\rho}_t^{00}\xi_t^* \right) - \tilde{\rho}_t^{01} \right\} (dn(t) - k_t dt),$$

$$d\tilde{\rho}_t^{00} = \mathcal{L}\tilde{\rho}_t^{00} dt + \left(\frac{1}{k_t} L\tilde{\rho}_t^{00} L^\dagger - \tilde{\rho}_t^{00} \right) (dn(t) - k_t dt),$$

where

$$\mathcal{L}\rho = -i[H_{\mathcal{S}}, \rho] - \frac{1}{2} \left\{ L^\dagger L, \rho, \right\} + L\rho L^\dagger,$$

$\tilde{\rho}_t^{10} = (\tilde{\rho}_t^{01})^\dagger$, and initially $\tilde{\rho}_0 = |\psi\rangle\langle\psi|$, $\tilde{\rho}_0^{01} = 0$, $\tilde{\rho}_0^{00} = |\psi\rangle\langle\psi|$.

The stochastic counting process $n(t)$ with the increment $(dn_t)^2 = dn_t$ satisfies

$$\mathbb{E}[dn(t)|\tilde{\rho}_t] = k_t dt,$$
$$k_t = \text{Tr} \left(L^\dagger L\tilde{\rho}_t + L\tilde{\rho}_t^{10}\xi_t^* + \tilde{\rho}_t^{01}L^\dagger\xi_t + \tilde{\rho}_t^{00}|\xi_t|^2 \right).$$

$\tilde{\rho}_t$ is the *a posteriori* state of $\mathcal{S}$.

4 Master equations

The set of master equations describing the *a priori* evolution of $\mathcal{S}$ has the form

$$\dot{\tilde{\varrho}}_t = \mathcal{L}\tilde{\varrho}_t + [\tilde{\varrho}_t^{01}, L^\dagger]\xi_t + [L, \tilde{\varrho}_t^{10}]\xi_t^*,$$
$$\dot{\tilde{\varrho}}_t^{01} = \mathcal{L}\tilde{\varrho}_t^{01} + \left[L, \tilde{\varrho}_t^{00} \right] \xi_t^*,$$
$$\dot{\tilde{\varrho}}_t^{00} = \mathcal{L}\tilde{\varrho}_t^{00},$$

where $\tilde{\varrho}_t^{10} = (\tilde{\varrho}_t^{01})^\dagger$, and initially $\tilde{\varrho}_0 = \tilde{\varrho}_0^{00} = |\psi\rangle\langle\psi|$, $\tilde{\varrho}_0^{01} = \tilde{\varrho}_0^{10} = 0$. $\tilde{\varrho}_t$ is the reduced state of $\mathcal{S}$.

The general solution to this set can be written as

$$\tilde{\varrho}_t = e^{\mathcal{L}t}\rho_0 + e^{\mathcal{L}t} \int_0^t ds \, e^{-\mathcal{L}s} \left([\tilde{\varrho}_s^{01}, L^\dagger]\xi_s + [L, \tilde{\varrho}_s^{10}]\xi_s^* \right)$$

where

$$\tilde{\varrho}_t^{01} = e^{\mathcal{L}t} \int_0^t ds \, e^{-\mathcal{L}s} [L, \tilde{\varrho}_s^{00}]\xi_s^*,$$
$$\tilde{\varrho}_t^{00} = e^{\mathcal{L}t}\rho_0,$$

and ρ_0 is the initial state of $\mathcal{S}$.

5 From a posteriori to a priori solution

By its definition, the *a priori* state of $\mathcal{S}$ is the mean value of the *a posteriori* states. To obtain $\tilde{\varrho}_t$ we take an average of the quantum trajectories $\tilde{\rho}_t$ over all possible realization of the output process $n(t)$,

$$\varrho_t = \sum_{m=0}^{+\infty} \int_0^t dt_m \ldots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \rho_{t|t_m, \ldots, t_2, t_1},$$

where $\rho_{t|t_m, \ldots, t_2, t_1}$ are the conditional operators of the form

$$\rho_{t|0} = |\alpha_{t|0}\rangle\langle\alpha_{t|0}| \int_t^{+\infty} \tau|\xi_s|^2 ds + |\beta_{t|0}\rangle\langle\beta_{t|0}|,$$

$$\rho_{t|t_m, \ldots, t_2, t_1} = |\alpha_{t|t_m, \ldots, t_2, t_1}\rangle\langle\alpha_{t|t_m, \ldots, t_2, t_1}| \int_t^{+\infty} \tau|\xi_s|^2 ds + |\beta_{t|t_m, \ldots, t_2, t_1}\rangle\langle\beta_{t|t_m, \ldots, t_2, t_1}|.$$

we have a sum over all possible pathways of the photon detection for the number of photons ranging from $m = 0$ to $m = \infty$ from the time 0 to t. We detect photons at $t_1, t_2, \ldots, t_m$ and no other photons from 0 to t.

References

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6 Two-level system

We consider the system $\mathcal{S}$ being a two-level atom with the eigenstates $|g\rangle$ and $|e\rangle$:

$$H_{\mathcal{S}} = -\Delta_0 \sigma_z, \quad L = \sqrt{\Gamma}\sigma_-,$$

where $\Gamma \in \mathbb{R}_+$, $\sigma_- = |g\rangle\langle e|$, $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, and $\Delta_0 = \omega_c - \omega_0$, where ω_0 is the carrier frequency of the input wave packet.

Photon statistics The probability of no photons up to time t:

$$P_0^t(0) = || |\alpha_{t|0}\rangle ||^2 \int_t^{+\infty} ds |\xi_s|^2 + || |\beta_{t|0}\rangle ||^2,$$

$$|\alpha_{t|0}\rangle = \left(e^{-i\Delta_0 t}|g\rangle\langle g| + e^{(i\Delta_0 - \frac{\Gamma}{2})t}|e\rangle\langle e| \right) |\psi_0\rangle,$$

$$|\beta_{t|0}\rangle = -\sqrt{\Gamma}e^{(i\Delta_0 - \frac{\Gamma}{2})t} \int_0^t ds \xi_s e^{(-2i\Delta_0 + \frac{\Gamma}{2})s}|e\rangle\langle g|\psi_0\rangle.$$

The probability density of observing a trajectory corresponding to one photons at t_1 ($t > t_1 > 0$) and no other photons in $[0, t)$:

$$p_0^t(t_1)dt_1 = || |\alpha_{t|t_1}\rangle ||^2 \int_t^{+\infty} ds |\xi_s|^2 + || |\beta_{t|t_1}\rangle ||^2,$$

$$|\alpha_{t|t_1}\rangle = \sqrt{\Gamma}e^{-i\Delta_0 t} e^{(2i\Delta_0 - \frac{\Gamma}{2})t_1}|g\rangle\langle e|\psi_0\rangle,$$

$$|\beta_{t|t_1}\rangle = e^{-i\Delta_0 t} \left[\left(\xi_{t_1} - \Gamma \int_0^{t_1} ds \xi_s e^{(2i\Delta_0 - \frac{\Gamma}{2})(t_1 - s)} \right) |g\rangle\langle g| \right.$$
$$\left. + e^{(2i\Delta_0 - \frac{\Gamma}{2})t} \left(\xi_{t_1} - \Gamma \int_{t_1}^t ds \xi_s e^{(2i\Delta_0 - \frac{\Gamma}{2})(t_1 - s)} \right) |e\rangle\langle e| \right] |\psi_0\rangle.$$

The probability density of observing a trajectory corresponding to two photons at $t > t_2 > t_1 > 0$ and no other photons in $[0, t)$

$$p_0^t(t_2, t_1)dt_2 dt_1 = || |\alpha_{t|t_2, t_1}\rangle ||^2 \int_t^{+\infty} ds |\xi_s|^2 + || |\beta_{t|t_2, t_1}\rangle ||^2,$$

$$|\alpha_{t|t_1, t_2}\rangle = 0,$$

$$|\beta_{t|t_1, t_2}\rangle = \sqrt{\Gamma}e^{-i\Delta_0 t} \times \left(\xi_{t_1} e^{-(2i\Delta_0 - \frac{\Gamma}{2})t_2} \right.$$
$$\left. + \xi_{t_2} e^{-(2i\Delta_0 - \frac{\Gamma}{2})t_1} - \Gamma \int_{t_1}^{t_2} ds e^{(-2i\Delta_0 + \frac{\Gamma}{2})s}\xi_s \right) |g\rangle\langle e|\psi_0\rangle.$$

POVM For a fixed time t we arrive at 3-element POVM labelled by the number of counts ($\gamma = -2i\Delta_0 + \frac{\Gamma}{2}$):

$$M_0 = \left(\int_t^{+\infty} ds |\xi_s|^2 + \Gamma e^{-\Gamma t} \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right) |g\rangle\langle g|$$
$$+ e^{-\Gamma t} \int_t^{+\infty} ds |\xi_s|^2 |e\rangle\langle e|$$

$$M_1 = \left(\int_0^t dt_1 |\xi_{t_1}|^2 - \Gamma e^{-\Gamma t} \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right) |g\rangle\langle g|$$
$$+ e^{-\Gamma t} \left(\left(e^{\Gamma t} - 1 \right) \int_t^\infty ds |\xi_s|^2 + \int_0^t ds |\xi_s|^2 + \Gamma \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right.$$
$$\left. -4\Gamma \text{Re} \int_0^t dt_1 \xi_{t_1}^* e^{\gamma^* t_1} \int_0^{t_1} ds \xi_s e^{-\gamma^* s} \right) |e\rangle\langle e|$$

$$M_2 = e^{-\Gamma t} \left(\left(e^{\Gamma t} - 1 \right) \int_t^\infty ds |\xi_s|^2 - \Gamma \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right.$$
$$\left. + 4\Gamma \text{Re} \int_0^t dt_1 \xi_{t_1}^* e^{\gamma^* t_1} \int_0^{t_1} ds \xi_s e^{-\gamma^* s} \right) |e\rangle\langle e|$$

A priori state

$$\rho_t = \left[\langle g|\rho_0|g\rangle \left(1 - \Gamma e^{-\Gamma t} \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right) \right.$$
$$+ \langle e|\rho_0|e\rangle \left(1 - e^{-\Gamma t} - \Gamma e^{-\Gamma t} \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 \right)$$
$$\left. + 4\Gamma e^{-\Gamma t} \text{Re} \int_0^t dt_1 \xi_{t_1}^* e^{\gamma^* t_1} \int_0^{t_1} ds \xi_s e^{-\gamma^* s} \right) \right] |g\rangle\langle g|$$
$$+ \langle e|\rho_0|g\rangle e^{-\gamma t} \left(1 - 2\Gamma \int_0^t dt_1 \xi_{t_1} e^{-\gamma^* t_1} \int_0^{t_1} ds \xi_s^* e^{\gamma^* s} \right) |e\rangle\langle g|$$
$$+ \langle g|\rho_0|e\rangle e^{-\gamma t} \left(1 - 2\Gamma \int_0^t dt_1 \xi_{t_1}^* e^{-\gamma t_1} \int_0^{t_1} ds \xi_s e^{\gamma s} \right) |g\rangle\langle e|$$
$$+ \left[\langle g|\rho_0|g\rangle \Gamma e^{-\Gamma t} \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 + \langle e|\rho_0|e\rangle e^{-\Gamma t} \right.$$
$$\times \left(1 + \left| \int_0^t ds \xi_s e^{\gamma s} \right|^2 - 4\Gamma \text{Re} \int_0^t dt_1 \xi_{t_1}^* e^{\gamma^* t_1} \int_0^{t_1} ds \xi_s e^{-\gamma^* s} \right) \right] |e\rangle\langle e|$$