Time-independent formalism for CP-divisible quantum dynamical maps governed by periodic Lindbladians in finite dimension

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Introduction

We adapt Howland's *time-independent formalism* describing evolution of finite dimensional open quantum system governed by Markovian Master Equation (MME) of a form $\dot{\rho}_t = L_t(\rho_t)$, where L_t is a *periodic*, time-local Lindbladian in Weak Coupling Limit regime and of standard (Kossakowski-Lindblad-Gorini-Sudarshan) form. Such approach is known for some time and applied successfully in case of unitary dynamics and Schrödinger equation, and originates from Howland [3], as well as Shirley [1] and Sambe [2]. We show that under some certain assumptions, one can replace the time-dependent Cauchy problem imposed by the MME with a „static" eigenproblem of time-independent *generalized Lindbladian* $\mathcal{L}$, which is unbounded and densely defined on the *enlarged space of states*. The $\mathcal{L}$ may be understood as a generator of strongly continuous contraction semigroup of completely positive (CP) and trace preserving (TP) maps, which can be effectively used to express solutions of time-dependent MME. Finally, we mention some possible applications of the formalism, including some approximate methods such as the van Vleck block diagonalisation, as well as its possible extension onto a large and important class of *quasiperiodic* Lindbladians, governing MMEs under assumption of Lyapunov-Perron reducibility.

Periodic Lindbladians

Let $\mathbb{C}^{d\times d}$ be the C*-algebra of some open quantum system, described by time-dependent Hamiltonian $H_t = H_t^* \in \mathbb{C}^{d\times d}$ such that $t \mapsto H_t$ is periodic, $H_{t+T} = H_t$ and

$$H_t = \sum_{n\in\mathbb{Z}} H_n e^{in\Omega t}. \quad (1)$$

System is coupled to external reservoir (described by its Hilbert space $\mathscr{H}_r$) via the interaction Hamiltonian

$$H_i = \sum_{k\in\mathcal{I}} S_k \otimes R_k, \quad (2)$$

where $\mathcal{I}$ is countable, $S_k \in \mathbb{C}^{d\times d}$, $R_k \in B(\mathscr{H}_r)$. Let u_t denote the system's unitary evolution operator. From the *Floquet's theorem* there exist such periodic function $t \mapsto p_t \in \mathbb{C}^{d\times d}$ and constant, Hermitian *effective Hamiltonian* $\bar{H} \in \mathbb{C}^{d\times d}$, that

$$u_t = p_t e^{-it\bar{H}}, \quad p_{t+T} = p_t, \quad p_t^* = p_t^{-1}. \quad (3)$$

Define a set of *Bohr quasifrequencies*

$$\{\omega\} = \{\epsilon_i - \epsilon_j : \epsilon_i, \epsilon_j \in \sigma(\bar{H})\}. \quad (4)$$

In references [4, 5] it was shown that in the regime of Weak Coupling Limit and under secular approximation, the time-periodic Lindbladian L_t may be expressed as

$$L_t = -i[H(t), \cdot] + P_t \tilde{L} P_t^{-1}, \quad (5)$$

where $t \mapsto P_t = p(t) \cdot p(t)^*$ is periodic (as $t \mapsto p(t)$ is periodic) and $p(t)$ is unitary for all $t \in \mathbb{R}$.
Map $\tilde{L}$ is the interaction-picture infinitesimal generator of quantum dynamical semigroup in standard form, and is found to be [5]

$$\tilde{L}(\rho) = \sum_k \sum_{q\in\mathbb{Z}} \sum_{\{\omega\}} \gamma_{kq\omega} \left(S_{kq\omega} \rho S_{kq\omega}^* - \frac{1}{2}\{S_{kq\omega}^* S_{kq\omega}, \rho\} \right). \quad (6)$$

Operators $S_{kq\omega}$ constitute a Fourier-like expansion of time-dependent operators $S_k(t) = U_t(S_k)$, such that

$$S_k(t) = \sum_{q\in\mathbb{Z}} \sum_{\{\omega\}} S_{kq\omega} e^{it(\omega+q\Omega)}, \quad (7)$$

where $\Omega = 2\pi/T$, and satisfy $[\bar{H}, S_{kq\omega}] = \omega S_{kq\omega}$. Coefficients $\gamma_{kq\omega}$ are non-negative and given via Fourier transform of reservoir autocorrelation functions.

Theorem 1.

Solution to the Cauchy differential problem (Markovian Master Equation)

$$\frac{d\rho_t}{dt} = L_t(\rho_t) = -i[H(t),\rho_t] + P_t \tilde{L} P_t^{-1}(\rho_t) \quad (8)$$

with initial value ρ_0 reads

$$\rho_t = P_t e^{tX}(\rho_0), \quad P_t(a) = p(t)ap(t)^*, \quad (9)$$

where $X = -i[\bar{H}, \cdot] + \tilde{L}$ is of standard form.

Assume X to be diagonalizable, $X(\varphi_j) = \xi_j\varphi_j$, where $\{\varphi_j\}_{j=1}^{d^2}$ is linearly independent. From Floquet's theorem, functions $\varphi_j(t) = e^{t\xi_j} P_t(\varphi_j)$ solve the original MME, i.e.

$$L_t(\varphi_j(t)) = \frac{d\varphi_j(t)}{dt}. \quad (10)$$

Time-independent formalism

The original *time-independent formalism* was developed in the case of unitary dynamics and is due to e.g. Shirley [1] and Sambe [2], later formalized by Howland [3]. Today, it finds applications in e.g. description of NMR-related phenomena [6, 7] and, most notably, it may be also employed for finding a mathematically effective description of *quasiperiodic* systems of linear ODEs. In present work, we utilize the framework of time-independent formalism and generalize it for the language of CP-divisible maps in open systems theory.

We consider a Banach *-algebra $\mathbb{C}_1^{d\times d} := (\mathbb{C}^{d\times d}, \|\cdot\|_1)$ where $\|\cdot\|_1$ stands for the trace norm, and with duality pairing $(a, b) = \operatorname{tr} ab$ for any $a, b \in \mathbb{C}^{d\times d}$; let $\{\sigma_k\}_{k=1}^{d^2}$ be its basis. Also, let $L^2(\mathbb{T})$ be the Hilbert space of measurable, square integrable functions on circle $\mathbb{T}$ with inner product $\langle f, g \rangle_{L^2} = \frac{1}{T}\int_{\mathbb{T}} \overline{f(t)}g(t)\,dt$ and denote by $\{e_n\}_{n\in\mathbb{Z}}$ its *Fourier basis*. Then, the algebraic tensor product

$$\mathbb{C}^{d\times d} \otimes L^2(\mathbb{T}) = \left\{ \sum_{|k|\leqslant n} x_k \otimes e_k : x_k \in \mathbb{C}^{d\times d} \right\} \quad (11)$$

is isometric-isomorphically identified with subspace $\mathcal{T}$ of finite Fourier sums

$$\mathcal{T} = \left\{ f(t) = \sum_{|k|\leqslant n} f_k e^{ik\Omega t} : f_k = \frac{1}{T}\int_{\mathbb{T}} f(t)e^{-ik\Omega t}dt \right\}, \quad (12)$$

in space of all functions $\mathbb{T} \mapsto \mathbb{C}^{d\times d}$. We introduce the $\mathscr{L}^2$-norm $\|\cdot\|_{\mathscr{L}^2}$ on $\mathcal{T}$ by imposing

$$\mathcal{T} \ni f \mapsto \|f\|_{\mathscr{L}^2} = \left(\int_{\mathbb{T}} \|f(t)\|_1^2 \, dt \right)^{1/2} \quad (13)$$

and then take completion of both $\mathcal{T}$ and $\mathbb{C}^{d\times d} \otimes L^2(\mathbb{T})$ with respect to this norm; such completion is called the *Bochner space* of (square integrable) functions $\mathbb{T} \mapsto (\mathbb{C}^{d\times d}, \|\cdot\|_1)$. We then choose this space as the enlarged space of states of the system:

Definition 1. (the enlarged space of states)

The Bochner space

$$\mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d}) = \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{L}^2}} \simeq \mathbb{C}^{d\times d} \bar{\otimes}_{\|\cdot\|_{\mathscr{L}^2}} L^2(\mathbb{T}) \quad (14)$$

of all (equivalence classes of) square integrable functions $\mathbb{T} \mapsto \mathbb{C}_1^{d\times d}$ will be called the **enlarged space of states**. It is complete with respect to $\mathscr{L}^2$-norm given by (13) and isometrically-isomorphic to the topological tensor product $\mathbb{C}^{d\times d}\bar{\otimes}L^2(\mathbb{T})$, complete with respect to the same norm.

Its topological dual is then simply the Bochner space $\mathscr{L}^2(\mathbb{T}, \mathbb{C}_\infty^{d\times d})$, where $\mathbb{C}_\infty^{d\times d} := (\mathbb{C}^{d\times d}, \|\cdot\|_\infty)$ is the von Neumann matrix algebra with supremum (operator) norm $\|\cdot\|_\infty$. The duality pairing is then given by $(f, g)_{\mathscr{L}^2} = \frac{1}{T}\int_{\mathbb{T}} \operatorname{tr} f(t)g(t)\,dt$ for $f \in \mathscr{L}^2(\mathbb{T}, \mathbb{C}_\infty^{d\times d})$, $g \in \mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$. *Involution* on $\mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$ is naturally defined by

$$f \mapsto f^\star(t) = f(t)^*, \quad (15)$$

and the *positive cone* and *set of states* are, respectively,

$$P^+ = \overline{\left\{ \sum_{j=1}^n g_j^\star g_j \right\}}, \qquad \mathcal{S}_1^+ = \{f \in P^+ : \|f\|_{\mathscr{L}^2} = 1\}. \quad (16)$$

Trace on $\mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$ is then given by

$$\operatorname{tr}_{\mathscr{L}^2} f = \frac{1}{T}\int_{\mathbb{T}} \operatorname{tr} f(t)\,dt. \quad (17)$$

Dynamics in enlarged space

Let $F_z, \{F_n\}_{n\in\mathbb{Z}}$ be the *Fourier operators* acting on $L^2(\mathbb{T})$ as

$$F_z(e_n) = ne_n, \qquad F_n(e_m) = e_{n+m}. \quad (18)$$

Let $\phi_{k,n}(t) = e^{in\Omega t}\phi_k(t)$, where $\phi_k(t) = P_t(\varphi_k)$, $X(\varphi_k) = \xi_k\varphi_k$. Then, it is easy to see, that linearly independent family $\{\phi_{k,n}(t)\}$ satisfies, for each $t \in \mathbb{T}$, the eigenequation

$$L_t(\phi_{k,n}(t)) - \frac{d}{dt}\phi_{k,n}(t) = \xi_{k,n}\phi_{k,n}(t), \quad (19)$$

where $\xi_{k,n} = \xi_k - in\Omega$. Let $\phi_k \in \mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$ be the corresponding element in the enlarged space of states. Then, $\phi_{k,n} = (I\otimes F_n)(\phi_k) = \mathcal{P}(\varphi_k\otimes e_n)$, where $\mathcal{P}$ stands for *-automorphism over $\mathbb{C}^{d\times d}\bar{\otimes}L^2(\mathbb{T})$,

$$\mathcal{P} = \sum_{n\in\mathbb{Z}} P_n \otimes F_n, \quad P_n = \frac{1}{T}\int_{\mathbb{T}} P_t e^{-in\Omega t}dt. \quad (20)$$

Definition 2. (Sambe-Floquet Lindbladian)

Let $L_n = \frac{1}{T}\int_{\mathbb{T}} L_t e^{-in\Omega t}dt$. We define the **Sambe-Floquet Lindbladian** as closed, unbounded linear map $\mathcal{L}$, densely defined over domain $\operatorname{Dom}(\mathcal{L}) = (\mathscr{C}^1(\mathbb{T}, \mathbb{C}_1^{d\times d}), \|\cdot\|_\infty)$ of differentiable functions $\mathbb{T} \mapsto \mathbb{C}^{d\times d}$,

$$\mathcal{L} = \sum_{n\in\mathbb{Z}} L_n \otimes F_n - i\Omega\, I \otimes F_z, \quad \mathcal{L}(f)(t) = (L_t - \partial_t)f(t). \quad (21)$$

The MME (19) yields the point spectrum of $\mathcal{L}$ to be

$$\sigma_p(\mathcal{L}) = \sigma(X) + i\Omega\mathbb{Z}, \quad \sigma(X) = \{\xi_k : k = 1\ldots d^2\}; \quad (22)$$

this spectrum may be found (at least approximately) by appropriate diagonalisation procedure such as *van Vleck block diagonalisation*. Furthermore, $\mathcal{L}$ may be represented as a closed perturbation of semigroup generator in standard form,

$$\mathcal{L}(f) = -i[\sum_{n\in\mathbb{Z}} H_n \otimes e_n, f] - i\Omega\, I \otimes F_z(f) \quad (23)$$

$$+ \sum_{k\omega q} \left(\Gamma_{k\omega q} f\, \Gamma_{k\omega q}^\star - \frac{1}{2}\{\Gamma_{k\omega q}^\star \Gamma_{k\omega q}, f\} \right),$$

where $\Gamma_{k\omega q} = \hat{p}(S_{k\omega q} \otimes e_0)\hat{p}'$ and $\hat{p} = \sum_{n\in\mathbb{Z}} p_n \otimes e_n$.

Theorem 2.

$\mathcal{L}$ generates a one-parameter contraction $\mathcal{C}_0$-semigroup of trace preserving, CP-divisible maps over $\mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$,

$$W_\tau = e^{\tau\mathcal{L}} = \mathcal{P}(e^{\tau X} \otimes e^{-i\tau\Omega F_z})\mathcal{P}', \quad (24)$$

which leaves invariant the cone of positive elements in $\mathscr{L}^2(\mathbb{T}, \mathbb{C}_1^{d\times d})$, as well as $\mathcal{S}_1^+$.

Proof. As X is of standard form, $e^{\tau X} \otimes I$ is naturally a trivial extension of contraction $\mathcal{C}_0$-semigroup of CP, TP maps. $I\otimes e^{-i\tau\Omega F_z}$ is a *right shift* operator, acting on every function f in Bochner space as $f \mapsto I\otimes e^{-i\tau\Omega F_z}(f)(t) = f(t-\tau)$, which is well known to be CP and TP; therefore, W_τ is also CP and TP as a composition. CP-divisibility is then obvious as $W_{\tau_1}W_{\tau_2}^{-1} = e^{(\tau_1-\tau_2)\mathcal{L}}$ is CP and TP for any $\tau_1 \geqslant \tau_2$. □

Theorem 3.

Every solution of the original MME (8) with periodic Lindbladian in standard form may be presented as $\rho_t = e^{t\mathcal{L}}(\rho_0 \otimes e_0)(t)$ for some initial density operator ρ_0.

Possible generalization of presented formalism include the case of *quasiperiodic Lindbladians*, satisfying

$$L_t = \tilde{L}_{\theta(t)}, \quad \theta(t) = (\Omega_1 t, \Omega_2 t, \ldots, \Omega_s t) \bmod 2\pi, \quad (25)$$

where $\theta : \mathbb{R} \to \mathbb{T}^s$ is a parametrization of a curve on the surface of s-dimensional torus $\mathbb{T}^s$ (with Ω_i incommensurate) and $\theta \to \tilde{L}_\theta$ is 2π-periodic in each variable, alongside with simplifying assumption of MME being *Lyapunov-Perron reducible*, i.e. if $\Lambda_t = \Pi_t e^{t\Sigma}$ takes a familiar Floquet-like form with Σ constant, and Π_t quasiperiodic.

Further reading & acknowledgments

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