GENERALIZED PRODUCT FORMULAS AND QUANTUM CONTROL

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Quantum Control

Given a system whose evolution is generated by some generic Hamiltonian H, we are interested in achieving a controlled evolution of the system characterized by superselection sectors among which transitions are avoided.,

The controlled evolution

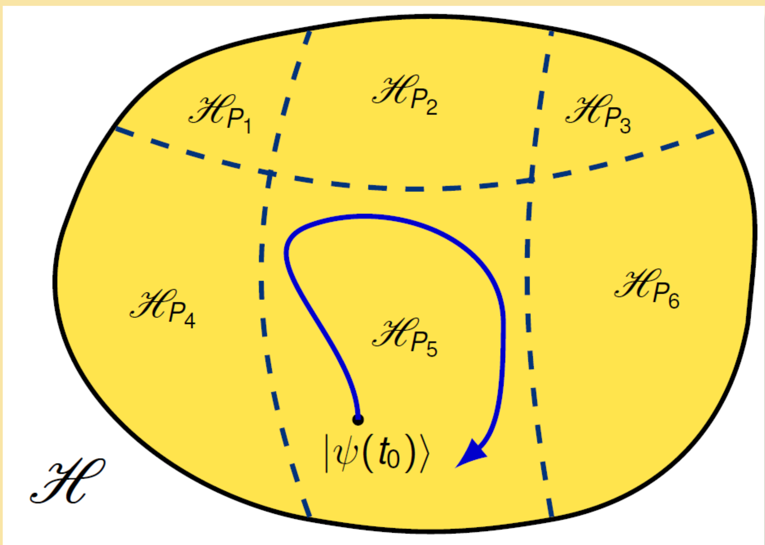
$$U_Z = e^{-itH_Z},$$

is generated by the Zeno Hamiltonian:

$$H_Z = \sum_{\mu=1}^{m} P_\mu H P_\mu\,,$$

whose structure determines a partitioning of the Hilbert space into Zeno subspaces:

$$\mathcal{H} = \bigoplus_{\mu=1}^{m} \mathcal{H}_\mu \qquad \mathcal{H}_\mu = P_\mu \mathcal{H}$$



Two control procedures

Strong Continuous Coupling

The controlled evolution can be achieved through a continuous coupling with a control potential:

$$V = \sum_{\mu=1}^{m} \lambda_\mu P_\mu,$$

with a strong coupling constant, i.e:

$$e^{-it(KV+H)} = e^{-itKV} e^{-itH_Z} + \mathcal{O}\left(\frac{1}{K}\right)$$

$$(K \to \infty)$$

Pulsed evolution

The controlled evolution can be achieved by kicking the system with a unitary:

$$U_{kick} = \sum_{\mu=1}^{m} e^{-i\phi_\mu} P_\mu,$$

at high frequency, i.e:

$$\left(U_{kick} e^{-i\frac{t}{n}H}\right)^n = U_{kick}^n e^{-itH_Z} + \mathcal{O}\left(\frac{1}{n}\right)$$

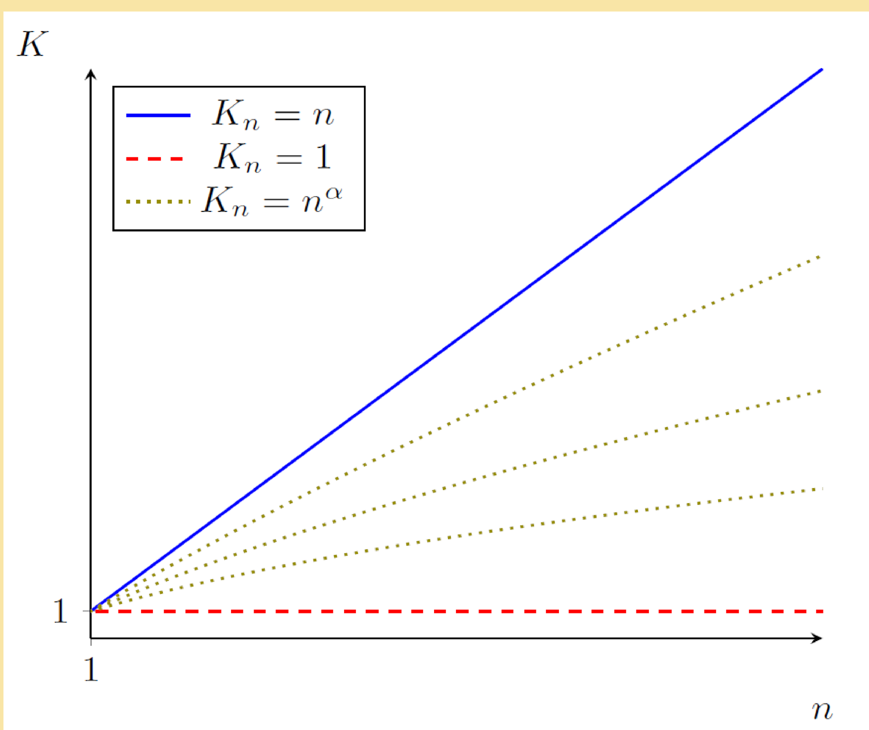
$$(n \to \infty)$$

Intermediate Situation

Two completely different procedures give rise to the same evolution in their respective limits. To compare them, we can study an intermediate situation:

$$U_{n,K}(t) = \left(e^{-i\frac{t}{n}KV} e^{-i\frac{t}{n}H}\right)^n$$

1. $K = K_0 \Rightarrow$ Continuous coupling
2. $K = nK_0 \Rightarrow$ Pulsed evolution
3. $K = n^\alpha K_0, \quad 0 \le \alpha \le 1$



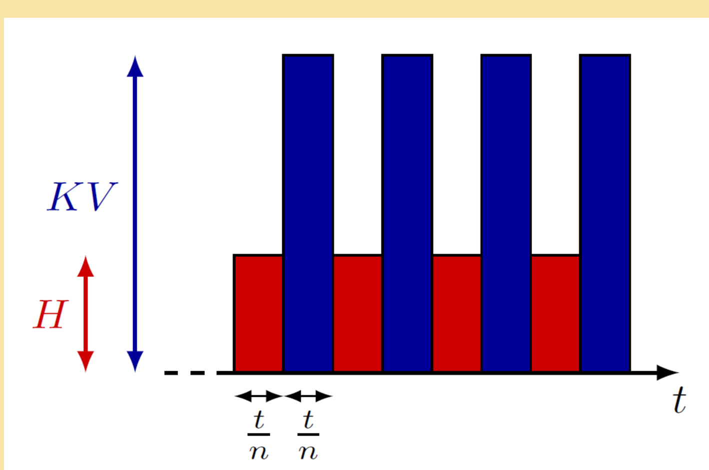
For $\alpha = 0$ we recover the case 1. There is continuous coupling without strong coupling limit, hence no control. For $\alpha = 1$ we recover the pulsed evolution, which gives control.
We investigate the intermediate case $0 < \alpha \le 1$.

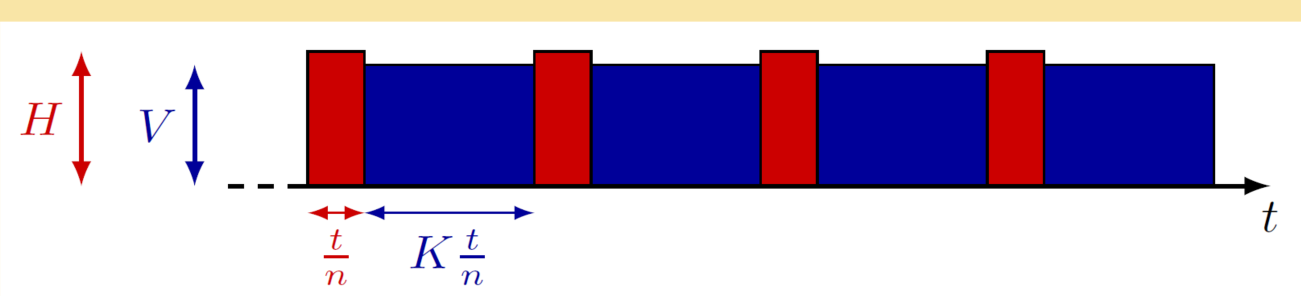
Scaling time vs Scaling strength

$$U_{n,K}(t) = \left(e^{-i\frac{t}{n}KV} e^{-i\frac{t}{n}H}\right)^n$$

Two possible interpretations:

1. *Alternating Hamiltonians with different strength with the same duration (figure on the right)*
2. *Alternating Hamiltonians with same strength and different duration (figure below)*



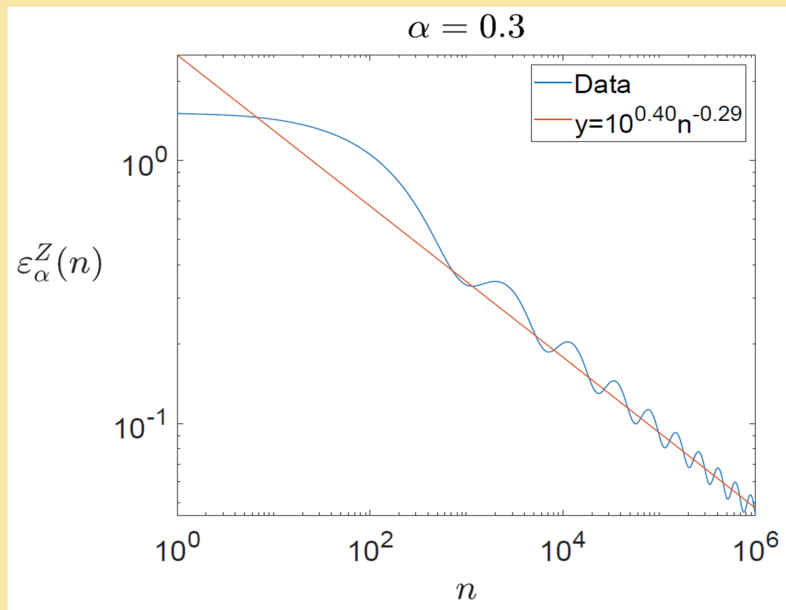


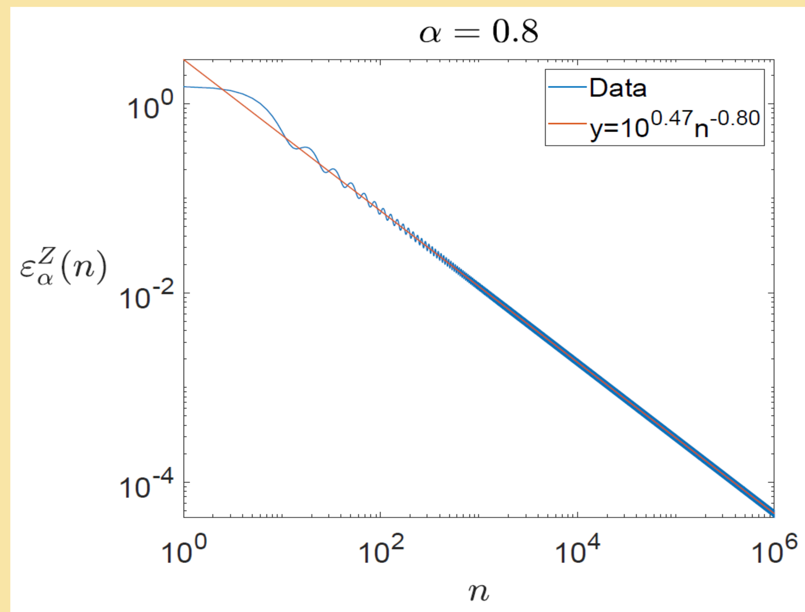
Generalized Pulsed Control

Our analytical result, valid for $n \to \infty,\ K_n \to \infty,\ K_n = o(n)$:

$$\left(e^{-i\frac{t}{n}K_n V} e^{-i\frac{t}{n}H}\right)^n - e^{-itK_n V} e^{-itH_Z} = \mathcal{O}\left(\frac{1}{K_n}\right)$$

Numerics for $K_n = n^\alpha$ shows that our bound is tight ($\varepsilon_\alpha^Z(n)$ is the error):



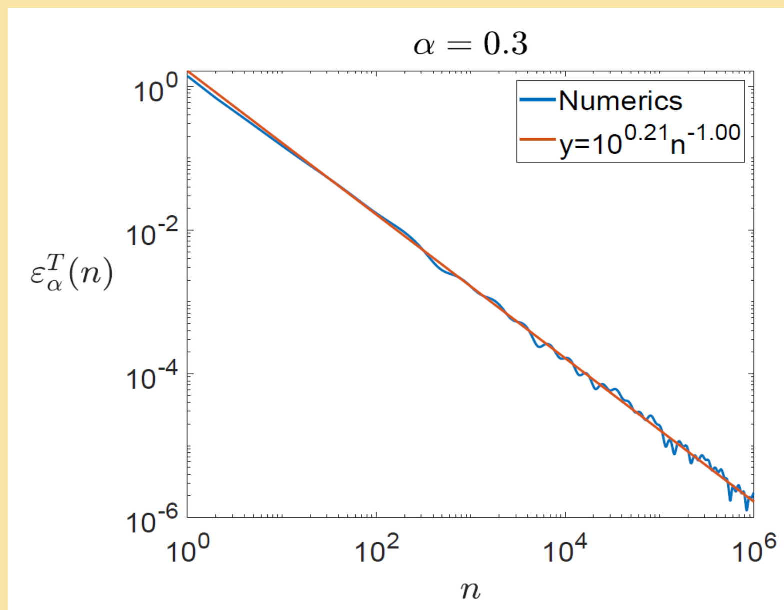


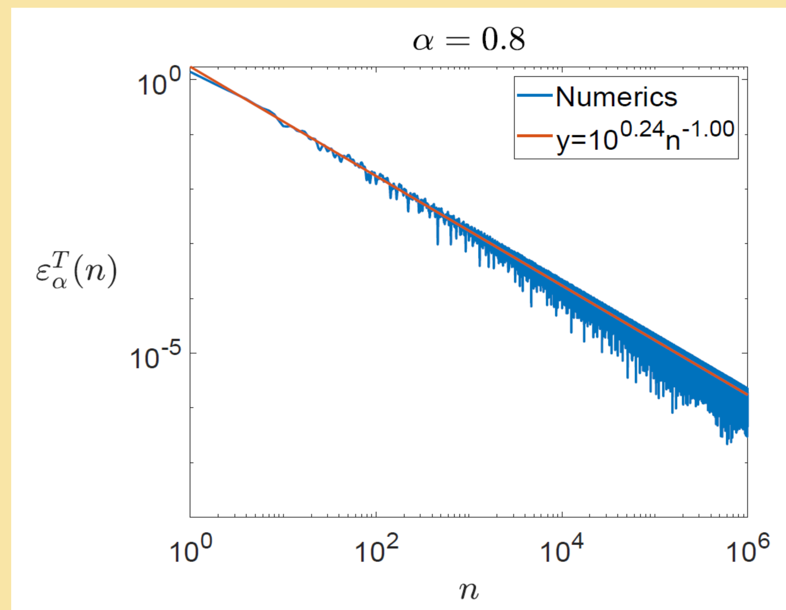
Generalized Trotter Formula

Our analytic result, valid for $n \to \infty,\ K_n = o(n)$:

$$\left(e^{-i\frac{t}{n}K_n V} e^{-i\frac{t}{n}H}\right)^n - e^{-itK_n V + H} = \min\left\{\mathcal{O}\left(\frac{1}{K_n}\right), \mathcal{O}\left(\frac{K_n}{n}\right)\right\}$$

Numerics for $K_n = n^\alpha$ shows that the error $\varepsilon_\alpha^T(n)$ is $\mathcal{O}(1/n)$ for every $0 \le \alpha \le 1$





Qubit Example: a tight bound

We show a particular analytical example where we prove the strict bound $\mathcal{O}(1/n)$ considering a qubit and the Pauli matrices as alternating Hamiltonians. In particular, we compare the two evolutions:

$$U_n = \left(e^{-in^\alpha Z/n} e^{-iX/n}\right)^n = e^{-in\theta_n \vec{u}_n \cdot \vec{\sigma}_n} \qquad V_n = e^{-i(n^\alpha Z + X)} = e^{-i\phi_n \vec{v}_n \cdot \vec{\sigma}_n}$$

For large n the asymptotic for the differences between the angles and the directions yields:

$$\phi_n - n\theta_n \sim \frac{1}{6}\frac{n^\alpha}{n^2} \qquad \vec{u}_n - \vec{v}_n \sim \left(-\frac{n^\alpha}{3n^2}, \frac{1}{n}, -\frac{1}{6n^2}\right)$$

The leading error comes therefore from the difference between the directions $\vec{u}_n$ and $\vec{v}_n$. We obtain therefore the explicit asymptotic behaviour:

$$U_n - V_n \sim -\frac{\mathrm{i}}{\mathrm{n}} \sin\phi_n \mathrm{Y}$$

References

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2. P. Facchi and S. Pascazio, Physical Review Letters 89, 080401 (2002)
3. P. Facchi, D. Lidar, and S. Pascazio, Physical Review A 69, 032314 (2004).
4. D. Burgarth, P. Facchi, G. Gramegna, S. Pascazio, arXiv:1906.04498 (2019)