Symmetry adapted basis in coupled spin systems

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Abstract

We propose an alternative approach for the construction of the symmetry adapted basis in coupled spin systems. Our method is based on Gelfand-Tsetlin patterns, graph theory and a technique called pattern calculus. Representation of spin system in such basis results in a change of degrees of freedom, uncovering the information hidden in non-local degrees of freedom. This information can be used, inter alia, to study the structure of entangled states, their classification and may be useful for construction of quantum algorithms.

1. Symmetry basis

IN ORDER TO introduce the symmetry basis we exploit the rich structure of the Schur-Weyl duality between the symmetric Σ_N and unitary $U(n)$ groups [1]. The space $\mathcal{H}$ is the scene of two dual actions

$$A : \Sigma_N \times \mathcal{H} \to \mathcal{H}, \text{ and } B : U(n) \times \mathcal{H} \to \mathcal{H}.$$

These actions can be demposed into irreps

$$A = \sum_{\lambda \in \mathcal{D}_W(N,n)} m(A, \Delta^\lambda)\ \Delta^\lambda, \quad B = \sum_{\lambda \in \mathcal{D}_W(N,n)} m(B, D^\lambda)\ D^\lambda$$

where

$$m(A, \Delta^\lambda) = dim\, D^\lambda, \quad m(B, D^\lambda) = dim\, \Delta^\lambda.$$

Moreover

$$[A(\sigma), B(u)] = 0, \ \ \sigma \in \Sigma_N, \ \ u \in U(n),$$

what gives irreducible basis

$$b_{irr} = \{|\lambda\, t\, y\rangle \ : \ \lambda \in \mathcal{D}_W(N,n),\ t \in SSWT(\lambda, \tilde{n}),\ y \in SYT(\lambda, \tilde{N})\}, \tag{1}$$

where $\mathcal{D}_W(N,n)$ denotes the set of all partitions of the number N into no more than n parts, $SSWT(\lambda, \tilde{n})$ is the set of all semistandard Weyl tableaux of the shape λ on the alphabet of single node spins and $SYT(\lambda, \tilde{N})$ is the set of all standard Young tableaux of the shape λ on the alphabet of nodes.

2. Symmetry states

THE action A decomposes the set $\tilde{n}^{\tilde{N}}$ of all magnetic configurations into orbits

$$\mathcal{O}_\mu = \{f \circ \sigma^{-1} |\, \sigma \in \Sigma_N\}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, $\sum_{i \in \tilde{n}} \mu_i = N, \mu_i = |\{i_j = i \,|\, j \in \tilde{N}\}|, \ \ i \in \tilde{n}.$

Restriction of the action A to the orbit $\mathcal{O}_\mu$ gives *the transitive representation* of the group Σ_N

$$\underbrace{A|_{\mathcal{O}_\mu}}_{mag.\ conf.} \equiv \underbrace{R^{\Sigma_N : \Sigma^\mu}}_{cosets}$$

which can be decomposed into irreps (the Kostka decomposition)

$$R^{\Sigma_N : \Sigma^\mu} \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu}\ \Delta^\lambda \tag{2}$$

where sum runs over all partitions λ greather or equal to μ in dominance order. Decomposition (2) can be written on the level of bases

$$|\mu\,\lambda\,t\,y\rangle = \sum_{f \in \mathcal{O}_\mu} \langle f|\mu\lambda t y\rangle\ |f\rangle, \tag{3}$$

as linear combination of magnetic configurations with coefficients $\langle f|\mu\lambda t y\rangle$.

3. Gelfand-Tsetlin patterns

The irreducible representations D^λ of a unitary group $\mathrm{U}(n)$ are classified by a partitions $\lambda \in D_W(N,n)$. Such a partition is denoted as $\lambda \equiv [m]_n = [m_{1n} \dots m_{nn}]$. The standard basis of the carrier space $V^{[m]_n}$ of the irreducible representation $D^{[m]_n}$ is denoted by $\mathrm{GT}([m]_n, \tilde{n})$, so that

$$V^{[m]_n} = lc_{\mathbb{C}}\ \mathrm{GT}([m]_n, \tilde{n}).$$

The $\mathrm{GT}([m]_n, \tilde{n})$ denotes the set of Gelfand-Tsetlin patterns of partition $[m]_n$ on the alphabet $\tilde{n}$ [2].

We recall that the state corresponding to the GT pattern $[m]$ can be presented, in a combinatorially equivalent way, by a semistandard Weyl tableau t in the alphabet $\tilde{n}$ of spins as follows. Let the i - th row of the tableau t has the form

$$\underbrace{i \dots i}_{\tau_{ii}} \underbrace{i+1 \dots i+1}_{\tau_{i,i+1}} i+2 \dots n-1 \underbrace{n \dots n}_{\tau_{in}},$$

so that $\tau_{ik}, 1 \le i \le k \le n$, is the occupation number of the letter $k \in \tilde{n}$ in the i - th row of t ($\tau_{ik} = 0$ for $i > k$ by virtue of semistandardness of t). Then clearly

$$\sum_{i \in \tilde{n}} \tau_{ik} = \mu_k, \quad k \in \tilde{n} \tag{4}$$

and

$$\sum_{k \in \tilde{n}} \tau_{ik} = \lambda_i, \quad i \in \tilde{n} \tag{5}$$

determine the weight $\mu = (\mu_1, \dots, \mu_n)$ and the shape $\lambda = (\lambda_1, \dots, \lambda_n)$ of the tableau t. The equivalence between the tableau t and the pattern (m) is given by

$$\tau_{i,k} = \begin{cases} m_{ik} - m_{i,k-1} \text{ for } 1 \le i < k, \\ m_{ii} \text{ for } i = k, \\ 0 \text{ for } i > k, \end{cases} \tag{6}$$

together with the inverse transformation

$$m_{ik} = \sum_{1 \le k' \le k} \tau_{ik'}. \tag{7}$$

4. Calculation of the coefficients $\langle f|\mu\lambda t y\rangle$

FIRST we construct a graph Γ for the coefficient $\langle f|\mu\lambda t y\rangle$ by adding the successive letters of magnetic configuration f to Gelfand patterns, adjusted to the Young tableau y, starting with the zero Gelfand pattern and ending with that pattern which correspond to the Weyl tableau t. More precisely, using the RSK algorithm we read off from the tableau y the sequence the of partitions $\lambda_1, \lambda_{12}, \dots, \lambda_{12 \dots N} = \lambda$, and we insert the consecutive letters of configuration f in such a way, that first row of Gelfand pattern, after inserting the letter i_j, should be $\lambda_{12..j}$.

From the graph Γ we can read the amplitude as

$$\langle f|\lambda t y\rangle = \sum_{\substack{\text{all paths} \\ \text{from top to bottom} \\ \text{of the graph}}} \prod_{\substack{\text{all edges } j \\ \text{of the one path} \\ \text{of the graph}}} \left\langle \begin{matrix} [m]_n + e_n(\tau_n) \\ [m]_{n-1} + e_{n-1}(\tau_{n-1}) \\ \vdots \\ [m]_k + e_k(\tau_k) \\ (m)_{k-1} \end{matrix} \right| t_{f(j),\, row(\lambda_{1..j} \setminus \lambda_{1..j-1})} \left| \begin{matrix} [m]_n \\ [m]_{n-1} \\ \vdots \\ [m]_k \\ (m)_{k-1} \end{matrix} \right\rangle \tag{8}$$

where $t_{f(j),\, row(\lambda_{1..j} \setminus \lambda_{1..j-1})}$ is the fundamental shift operator, which can be calculated by the formula [2]

$$\left\langle \begin{matrix} [m]_n + e_n(\tau_n) \\ [m]_{n-1} + e_{n-1}(\tau_{n-1}) \\ \vdots \\ [m]_k + e_k(\tau_k) \\ (m)_{k-1} \end{matrix} \right| t_{k,\,\tau_n} \left| \begin{matrix} [m]_n \\ [m]_{n-1} \\ \vdots \\ [m]_k \\ (m)_{k-1} \end{matrix} \right\rangle = \prod_{j=k+1}^{n} \mathrm{sgn}(\tau_{j-1} - \tau_j)$$

$$\sqrt{\left| \frac{\prod_{i=1\, i \ne \tau_{j-1}}^{j-1} \prod_{i=1, i \ne \tau_{j-1}}^{j-1} (p_{\tau_j, j} - p_{i,j-1}) \prod_{i=1, i \ne \tau_j}^{j} (p_{\tau_{j-1}, j-1} - p_{i,j} + 1)}{\prod_{i=1, i \ne \tau_j}^{j} (p_{\tau_j, j} - p_{i,j}) \prod_{i=1, i \ne \tau_{j-1}}^{j-1} (p_{\tau_{j-1}, j-1} - p_{i,j-1} + 1)} \right|} \tag{9}$$

$$\sqrt{\left| \frac{\prod_{i=1}^{k-1} (p_{\tau_k, k} - p_{i,k-1})}{\prod_{i=1, i \ne \tau_k}^{k} (p_{\tau_k, k} - p_{i,k})} \right|}$$

for $k \in \tilde{n}$. The partial hook $p_{ij} = m_{ij} + j - i$, $e_i(j)$ is the zero vector of the length i with 1 on the position j, $[m]_i$ represents i-th row of Gelfand pattern (m), whereas $(m)_i$ denotes rows from 1 to i of pattern (m).

5. Example

For example, the coefficient

$$\left\langle f = (1,3,2,1) \,\middle|\, \lambda = (3,1), t = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, y = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \right\rangle,$$

for the Heisenberg magnet with $N = 4$ nodes and single node spin $s = 1$ ($n = 3$), generates a graph

$$\begin{pmatrix} 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{pmatrix}$$

$$\downarrow 1$$

$$\begin{pmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{pmatrix} \qquad \lambda_1 = (1,0,0)$$

$$\downarrow 3$$

$$\begin{pmatrix} 2 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{pmatrix} \qquad \lambda_{12} = (2,0,0)$$

$$\swarrow 2 \qquad\qquad \searrow 2$$

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix} \qquad \lambda_{123} = (2,1,0)$$

$$\searrow 1 \qquad\qquad \swarrow 1$$

$$\begin{pmatrix} 3 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix} \qquad \lambda_{1234} = \lambda = (3,1,0).$$

For the graph presented above, we have

$$\left\langle (1,3,2,1) \middle| (3,1), \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, \ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \right\rangle =$$

$$\left\langle \begin{matrix} 2\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right| t_{31} \left| \begin{matrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 2\ 1\ 0 \\ 2\ 0 \\ 1 \end{matrix} \right| t_{22} \left| \begin{matrix} 2\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 3\ 1\ 0 \\ 2\ 1 \\ 2 \end{matrix} \right| t_{11} \left| \begin{matrix} 2\ 1\ 0 \\ 2\ 0 \\ 1 \end{matrix} \right\rangle +$$

$$\left\langle \begin{matrix} 2\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right| t_{31} \left| \begin{matrix} 1\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 2\ 1\ 0 \\ 1\ 1 \\ 1 \end{matrix} \right| t_{22} \left| \begin{matrix} 2\ 0\ 0 \\ 1\ 0 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 3\ 1\ 0 \\ 2\ 1 \\ 2 \end{matrix} \right| t_{11} \left| \begin{matrix} 2\ 1\ 0 \\ 1\ 1 \\ 1 \end{matrix} \right\rangle = \frac{5}{12}$$

6. Conclusions

WE HAVE SHOWN that the coefficients of the symmetry states can be constructed by consecutive joining of nodes of the spin system according to tableaux (t, y). The standard method of calculation of the coefficients from its definition, uses the summation over the symmetric group, and thus grows exponentially with N, whereas the method proposed above is well suited for numerical implementation, since it grows polynomially.

References

[1] Weyl H., The Theory of Groups and Quantum Mechanics, New York, Dover, (1950).

[2] James D. Louck, Unitary symmetry and combinatorics, World Scientific, Singapoure (2008).