Three-diagonal blocks of two-magnon Hamiltonians with specified wavenumbers for Heisenberg rings

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Abstract

We demonstrate the method of creating blocks of Hamiltonian matrices for Heisenberg rings with N nodes and $r = 2$ overturned spins, depending on total quasimomentum k. Initial problem of dimension $\binom{N}{2}$ reduces, approximately, N-tuply, depending on N and k numbers. We consider block Hamiltonians using a particula????? basis, called wavelet basis. Here, these blocks take three-diagonal form.

Introduction

In general:

- Bethe Ansatz: the diagonalization of the Heisenberg Hamiltonian of a ring of N nodes
- Invariance of sectors with a given number of magnons with respect to $\hat{H}$
- $\hat{H}$-invariance of the subspaces with a given wavenumber k
- Decomposition of the Hilbert space $\mathcal{H}$, as well as $\hat{H}$, with respect to r and k:

$$\mathcal{H} = \bigoplus_{r,k} \mathcal{H}_{r,k}\,; \qquad H = \bigoplus H_{r,k}$$

In details:

- two-magnon sector ($r = 2$), N nodes
- Geometry of a ring, as well as Heisenberg Hamiltonian, displays the symmetry of the cyclic group $\mathbb{C}_N$, isomorphic with $\mathbb{Z}_N$ group, and even reacher $\mathbb{D}_N$ group
- The focus: $\mathbb{C}_N$ subgroup which enables to describe the basis of orbits and the Fourier transform from the basis of orbits to the basis of wavelets
- The goal: matrices of block Hamiltonians $H_{2,k}$ incl. their parity, in the wavelet basis

Bases

A magnetic configuration:

$$\{j_1, j_2\} : j_1, j_2 \in \mathbb{Z}_N, \quad j_1 \neq j_2.$$

Then, **basis of positions**:

$$\mathbf{j} = |\{j_1, j_2\}\rangle, \quad 1 \le j_1 < j_2 \le N$$

Basis of orbits: elements labeled by the one of the number of nodes, and by the vector of relative positions of nodes $\mathbf{t}_\alpha$ (defining the orbit):

$$|j, \mathbf{t}_\alpha\rangle = |\{j, j+\alpha\}\rangle, \quad \alpha = 1, 2, \ldots, \left[\frac{N}{2}\right],$$

α - the distance between two spin deviations in a configuration, $j \in \mathbb{Z}_N$, except of N even and $\alpha = \frac{N}{2}$ - when $j = 1, \ldots, \frac{N}{2}$ (rarefied orbit)

Basis of wavelets formed by a single, discrete Fourier transform with respect to the first index of the basis of orbits j:

$$|\mathbf{t}_\alpha, k\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \omega^{-jk} |j, \mathbf{t}_\alpha\rangle, \quad \text{for } \alpha \neq \frac{N}{2},$$

and $|\mathbf{t}_\alpha, k\rangle = \sqrt{\frac{2}{N}} \sum_{j=1}^{\frac{N}{2}} \omega^{-jk} |j, \mathbf{t}_\alpha\rangle$, $\omega = e^{\frac{2\pi i}{N}}$ for $\alpha = \frac{N}{2}$, and N even, only for k even,

with admissible quasimomenta k from the range

$$k = 0, \pm 1, \ldots, \begin{cases} \pm(N/2 - 1), N/2 & \text{for } N \text{ even}, \\ \pm(N-1)/2 & \text{for } N \text{ odd} \end{cases}$$

The action of blocks of Hamiltonians in the basis of wavelets

A general form of the Hamiltonian for $r = 2$ spin deviations ($\mathbf{j}'$ - n.n. configurations of $\mathbf{j}$):

$$\hat{H}|\mathbf{j}\rangle = \sum_{\mathbf{j}'} (|\mathbf{j}'\rangle - |\mathbf{j}\rangle)$$

The Hamiltonian in the basis of wavelts (a single Fourier transform within a one orbit)

- $\alpha = 1$

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = (1 + \omega^{-k})|\mathbf{t}_{\alpha+1}, k\rangle - 2|\mathbf{t}_\alpha, k\rangle$$

- $\alpha = 2, 3, \ldots, \left[\frac{N}{2}\right] - 1$ for N odd; $\alpha = 2, 3, \ldots, \frac{N}{2} - 2$ for N even

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = \left(1 + \omega^{k}\right)|\mathbf{t}_{\alpha-1}, k\rangle + \left(1 + \omega^{-k}\right)|\mathbf{t}_{\alpha+1}, k\rangle - 4|\mathbf{t}_\alpha, k\rangle$$

- $\alpha = \left[\frac{N}{2}\right]$ for N odd

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = \left(1 + \omega^{k}\right)|\mathbf{t}_{\alpha-1}, k\rangle + \left(\omega^{k\mathbf{t}_\alpha} + \omega^{(k+1)\mathbf{t}_\alpha}\right)|\mathbf{t}_{\alpha}, k\rangle - 4|\mathbf{t}_\alpha, k\rangle$$

- $\alpha = \frac{N}{2} - 1$ for N even, k even

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = \left(1 + \omega^{k}\right)|\mathbf{t}_{\alpha-1}, k\rangle + \sqrt{2}\left(1 + \omega^{-k}\right)|\mathbf{t}_{\alpha+1}, k\rangle - 4|\mathbf{t}_\alpha, k\rangle$$

- $\alpha = \frac{N}{2}$ for N even, k even

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = \sqrt{\frac{2}{N}} \sum_{j=1}^{\frac{N}{2}} \omega^{-kj} H|j, \mathbf{t}_\alpha\rangle = \sqrt{2}\left(\omega^{k} + 1\right)|\mathbf{t}_{\alpha-1}, k\rangle - 4|\mathbf{t}_\alpha, k\rangle$$

- $\alpha = \frac{N}{2} - 1$ for N even, k odd

$$\hat{H}|\mathbf{t}_\alpha, k\rangle = \left(1 + \omega^{k}\right)|\mathbf{t}_{\alpha-1}, k\rangle - 4|\mathbf{t}_\alpha, k\rangle$$

Hamiltonian matrices

Hamiltonian matrices depending on the parity of N and k ($c = 1 + \omega^k$, $c^* = 1 + \omega^{-k}$)

- N odd

$$H = \begin{pmatrix} -2 & c_k & 0 & \ldots & \ldots & \ldots & \ldots \\ c_k^* & -4 & c_k & \ldots & \ldots & \ldots & \ldots \\ 0 & c_k^* & -4 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & c_k^* & \ddots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_k & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & -4 & c_k \\ \vdots & \vdots & \vdots & \vdots & \ldots & c_k^* & d_k - 4 \end{pmatrix}_{\left[\frac{N}{2}\right]\times\left[\frac{N}{2}\right]} \qquad d_k = (-1)^k\left(\omega^{\frac{k}{2}} + \omega^{-\frac{k}{2}}\right)$$

- N even, k even

$$H = \begin{pmatrix} -2 & c_k & 0 & \ldots & \ldots & \ldots & \ldots \\ c_k^* & -4 & c_k & \ldots & \ldots & \ldots & \ldots \\ 0 & c_k^* & -4 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & c_k^* & \ddots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_k & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & -4 & \sqrt{2}c_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \sqrt{2}c_k^* & -4 \end{pmatrix}_{\frac{N}{2}\times\frac{N}{2}}$$

- N even, k odd

$$H = \begin{pmatrix} -2 & c_k & 0 & \ldots & \ldots & \ldots & \ldots \\ c_k^* & -4 & c_k & \ldots & \ldots & \ldots & \ldots \\ 0 & c_k^* & -4 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & c_k^* & \ddots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_k & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & -4 & c_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & c_k^* & -4 \end{pmatrix}_{\left(\frac{N}{2}-1\right)\times\left(\frac{N}{2}-1\right)}$$

Remark: In the theory of three-diagonal matrices, it is a well known fact, that their determinants are expressed by means of Chebyshev polynomials. For more details please see references.

References

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