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1. Abstract

Monitoring photon quadratures and free masses are useful tools to detect small disturbances such as gravitational waves. Here we report a large class of states for photon quadratures and free masses potentially useful for this purpose: (1) "generic coherent states" (GCS) of photons, whose width is independent of time and uncertainty product $\sigma(x)\sigma(p)$ is arbitrarily large (a generalization of the minimum uncertainty Schrödinger coherent states [Sc26]) and (2) "squeezed generic contractive states" (SGCS) for photons and free masses (a generalization of the Yuen states [Yu83]) whose width decreases with time, uncertainty product is arbitrarily large, and the covariance squared $<\Delta\hat{x}, \Delta\hat{p}>^2$ has an arbitrary value within the allowed range $(0, 4\sigma^2(x)\sigma^2(p) - 1)$.

2. Standard Quantum Limit (SQL)

The spreading of wave packets occupies a particularly important position in discussions of quantum foundations. Heuristic arguments proposed that the accuracy of monitoring position of a free mass m is limited by the standard quantum limit (SQL) [BV74, CTDSZ80]:

$$\sigma^2(X(t)) \geq \sigma^2(X(0)) + \frac{t^2}{m^2}\sigma^2(P(0)) \geq 2\frac{t}{m}\sigma(X(0))\sigma(P(0)) \geq \frac{\hbar t}{m} \quad (1)$$

where $\sigma^2(X(t))$ and $\sigma^2(P(t))$ denote variances of the Heisenberg representation position and momentum operators at time t .

For the free mass, the inequality 1 is particularly visual for Gaussian states:

$$\langle p | \psi(t) \rangle = (\pi\alpha)^{-1/4} \exp\left(-\frac{(p-\beta)^2}{2\alpha} - it\frac{p^2}{2m}\right)$$

$$\sigma^2(P(t)) = \frac{\alpha}{2}, \text{ and } \sigma^2(X(t)) = \hbar^2 \left(\frac{1+(at/(m\hbar))^2}{2\alpha}\right)$$

3. Rigorous Quantum Limit (RQL), for free mass

For free masses, Yuen discovered in 1983 [Yu83] a class of states called 'twisted coherent states' which are 'contractive states', i.e. states whose position uncertainty decreases with time for a certain duration. The SQL is incorrect for these states. However, one can give rigorous quantum limits (RQL) [Ro18], valid for all states including contractive states. For any observable with Schrödinger operator A any Hamiltonian H , the Heisenberg operator $A(t)$ at time t and its variance $\sigma^2(A(t))$ are defined by:

$$A(t) := \exp\left(\frac{iHt}{\hbar}\right)A\exp\left(-\frac{iHt}{\hbar}\right), \langle A(t) \rangle := \langle \psi(0) | A(t) | \psi(0) \rangle,$$

$$\Delta A(t) := A(t) - \langle A(t) \rangle, \sigma^2(A(t)) := \langle \psi(0) | (\Delta A(t))^2 | \psi(0) \rangle,$$

where $|\psi(0)\rangle$ is the initial state. For a free mass, $H = \frac{P^2}{2m}$; the Heisenberg equation yields, $\Delta X(t) = \Delta X(0) + (t/m)\Delta P(0)$, and hence,

$$\sigma^2(X(t)) = \sigma^2(X(0)) + \frac{t^2}{m^2}\sigma^2(P(0)) + \frac{t}{m}\langle \psi(0) | \{\Delta X(0), \Delta P(0)\} | \psi(0) \rangle \quad (2)$$

One obtains the SQL (1) if one assumes that the covariance $\langle \{\Delta X(0), \Delta P(0)\} \rangle$ is non-negative. The covariance is negative for certain states, in particular, the Yuen States. Rigorous quantum limits (RQL) can be obtained on the covariance, and hence on $\sigma^2(X(t))$. Using the Schrödinger-Robertson uncertainty inequality and the Cauchy Inequality on equation 2, the RQL on expansion and contraction of wave packets [Ro18] is:

$$|\sigma^2(X(t)) - \sigma^2(X(0)) - \frac{t^2}{m^2}\sigma^2(P(0))| \leq \frac{t}{m}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2} \quad (3)$$

valid for arbitrary states. The only states saturating the inequalities are those which obey:

$$\Delta P(0) | \psi(0) \rangle = i\lambda \Delta X(0) | \psi(0) \rangle, \langle X' | \psi(0) \rangle = \left(\frac{Re\lambda}{\hbar}\right)^{1/4} \exp\left(\frac{i(P(0)X'}{\hbar} - \frac{\lambda(X'-\langle X(0) \rangle)^2}{2\hbar}\right) \quad (4)$$

with $Re\lambda > 0$, $|Im\lambda| = \frac{1}{2\sigma^2(X(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}$, $\sigma^2(X(0)) = \frac{\hbar}{2Re\lambda}$, $\sigma^2(P(0)) = \frac{\hbar|\lambda|^2}{2Re\lambda}$, and, $\langle \psi(0) | \{\Delta X(0), \Delta P(0)\} | \psi(0) \rangle = \mp\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}$, if $Im\lambda = \pm|Im\lambda|$

The positive sign of $Im\lambda$ corresponds to **maximally contractive** (Yuen states [Yu83]), and the negative sign of $Im\lambda$ to **maximally expanding** wave packets. The initial state 4 with positive $Im\lambda$, the state at time t remains contractive upto $t = \frac{t_M}{2}$, where, $t_M = \frac{m}{\sigma^2(P(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}$, and, for a given uncertainty product, by choosing $\frac{t}{m}\sigma^2(P(0)) = \sigma(X(0))\sigma(P(0))$, $\sigma^2(X(t))$ can be made $\approx \frac{\hbar^2}{4m\sigma(X(0))\sigma(P(0))}$ for a large uncertainty product, and can be much smaller than the heuristic standard quantum limit $\frac{\hbar t}{m}$.

4. Rigorous Quantum Limit (RQL) on Monitoring Photon Quadratures

For the single mode photon Hamiltonian $H = \hbar\omega a^\dagger a = \frac{1}{2}\hbar\omega(p^2 + x^2 - 1)$, where the quadrature operators x, p are given by $a = \frac{1}{\sqrt{2}}(x + ip)$; $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$, the Heisenberg equations of motion yield:

$$\sigma^2(x(t)) = \cos^2(\omega t)\sigma^2(x(0)) + \sin^2(\omega t)\sigma^2(p(0)) + \frac{1}{2}\sin(2\omega t)\langle \psi(0) | \{\Delta x(0), \Delta p(0)\} | \psi(0) \rangle$$

$$|\sigma^2(x(t)) - \cos^2(\omega t)\sigma^2(x(0)) - \sin^2(\omega t)\sigma^2(p(0))| \leq \frac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1} \quad (5)$$

The extremal states saturating these RQL are complex Gaussians corresponding to 4; both the maximally contractive and maximally expanding states can be designated as 'twisted coherent states' [Yu83] or 'squeezed coherent states' (SCS)

$$(b - \beta) | \psi(0) \rangle = 0, b = \mu a + \nu a^\dagger, \alpha := \langle \psi(0) | a | \psi(0) \rangle, \beta := \mu\alpha + \nu\alpha^*$$

$$\mu = \cosh r, \nu = e^{i\theta} \sinh r, | \psi(0) \rangle = | \alpha, r \exp(i\theta) \rangle := D(\alpha, a)S(\xi) | 0 \rangle \quad (6)$$

where $r > 0$ is the squeezing parameter, θ is real and $|0\rangle$ denotes the vacuum state; here the unitary displacement operator $D(\alpha, a)$ and squeeze operator $S(\xi)$ are, $D(\beta, b) = D(\alpha, a) = \exp(\alpha a^\dagger - \alpha^* a)$, $S(\xi) = \exp\left(\frac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2})\right)$, $\xi \equiv r \exp(i\theta)$. Explicit values for the standard deviations and covariance in the SCS 6 are then easily derived,

$$\sigma^2(x(0)) = \frac{1}{2}(\cosh(2r) - \cos(\theta)\sinh(2r)), \sigma^2(p(0)) = \frac{1}{2}(\cosh(2r) + \cos(\theta)\sinh(2r))$$

$$\langle \psi(0) | \{\Delta x(0), \Delta p(0)\} | \psi(0) \rangle = -\sin(\theta)\sinh(2r) = -\text{sgn}(\sin(\theta))\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1} \quad (7)$$

For $r > 0$, the state is squeezed, i.e. $\sigma^2(x(0)) < \sigma^2(p(0))$, if $\cos(\theta) > 0$, and the state is contractive for small positive t if $\sin(\theta) > 0$.

5. Generic Coherent States

Roy and Singh [RS82] noted that the property of time-independent width of the wave packets also holds for the generalised coherent states,

$$| \psi(\alpha, n) \rangle = D(\alpha, a) | n \rangle, \sigma(x(t)) = \sigma(p(t)) = \sqrt{n + \frac{1}{2}}, a^\dagger a | n \rangle = n | n \rangle \quad (8)$$

$n = 0$ gives the Schrödinger states. The property of time-independent width of the wave packet holds for a class of states much larger than these displaced oscillator eigenstates. We call this new class, "Generic coherent states" (GCS); they have arbitrarily large continuous values of the uncertainty product. From 5, $\sigma(x(t))$ is time-independent iff, the "GCS condition" is satisfied:

$$\langle \psi | (\Delta a)^2 | \psi \rangle = 0 \quad (9)$$

The GCS include the usual coherent states $\Delta a | \psi \rangle = 0$ as a special case.

Theorem: If $|\phi\rangle$ is a normalized state obeying $\langle \phi | a | \phi \rangle = 0$, and $\langle \phi | a^2 | \phi \rangle = 0$, and $|\psi(\alpha, \phi)\rangle \equiv D(\alpha, a) |\phi\rangle$, where α is an arbitrary complex parameter, then $|\psi(\alpha, \phi)\rangle$ is a generic coherent state (GCS).

When $|\phi\rangle = |n\rangle$, we get the Roy-Singh 8 states; but the possible states $|\phi\rangle$ form a much larger set allowing arbitrarily large continuous values of the uncertainty product: $\sigma^2(x(0)) = \sigma^2(p(0)) = \sigma(x(0))\sigma(p(0)) = \bar{n} + 1/2$; $\langle \psi(\alpha, \phi) | \{\Delta x(0), \Delta p(0)\} | \psi(\alpha, \phi) \rangle = 0$, $\bar{n} \equiv \langle \phi | a^\dagger a | \phi \rangle$. It remains only to show that states $|\phi\rangle$ giving arbitrary non-negative values of \bar{n} exist. Let $|\phi\rangle = \sum_{m=n}^N c_m |m\rangle$ and $\langle \phi | \phi \rangle = 1$. It can be shown that, if $|\phi\rangle = \sum_{r=0}^s c_{3r} |3r\rangle$, $\sum_{r=0}^s |c_{3r}|^2 = 1$, then, 9 is obeyed, and $\bar{n} = \sum_{r=0}^s 3r |c_{3r}|^2 \in [0, 3s]$, which can equal any value in the continuous interval $[0, 3s]$.

6. Squeezed Generic Coherent States

We define: $|\psi(\alpha, \xi, \phi)\rangle = D(\alpha, a)S(\xi) |\phi\rangle$, (by replacing $|0\rangle$ in the SCS by the GCS state $|\phi\rangle$). These states obey the SGCS conditions, which are obvious generalisations of the SCS 6 and GCS 9 conditions:

$$\langle \psi(\alpha, \xi, \phi) | b - \beta | \psi(\alpha, \xi, \phi) \rangle = 0, \langle \psi(\alpha, \xi, \phi) | (b - \beta)^2 | \psi(\alpha, \xi, \phi) \rangle = 0$$

$$\langle \Delta b \rangle = 0, \langle (\Delta b)^2 \rangle = 0$$

Unlike the SCS wave functions, the SGCS wave functions are not complex Gaussians. E.g., when $|\phi\rangle = |n\rangle$, we get the displaced and scaled oscillator eigenfunctions:

$$\langle x | S(\xi) | n \rangle = \frac{1}{\sqrt{|\mu - \nu| h_n}} H_n \left(\frac{x}{|\mu - \nu|} \right) \exp\left(-\frac{1}{2}\lambda x^2\right)$$

$$\langle x | \psi(\alpha, \xi, \phi) \rangle = \left\langle x - \sqrt{2}\alpha_1 \middle| S(\xi) \middle| n \right\rangle \exp(i\sqrt{2}\alpha_2(x - \frac{\alpha_1}{\sqrt{2}}))$$

where $h_n = \sqrt{\pi} 2^n n!$, $\lambda = \frac{\mu + \nu}{\mu - \nu}$, $\alpha_1 = Re\alpha$, $\alpha_2 = Im\alpha$

For general $|\phi\rangle$ (calculated in the above section), we obtain a generalization of the SCS expressions,

$$\sigma^2(x(0)) = (\bar{n} + 1/2)(\cosh(2r) - \cos(\theta)\sinh(2r))$$

$$\sigma^2(p(0)) = (\bar{n} + 1/2)(\cosh(2r) + \cos(\theta)\sinh(2r))$$

$\langle \psi(\alpha, \xi, \phi) | \{\Delta x(0), \Delta p(0)\} | \psi(\alpha, \xi, \phi) \rangle = -\text{sgn}(\sin(\theta))\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - (2\bar{n} + 1)^2}$
 Time development of these generic contractive or expanding wave packets follows from 5 using $|\psi(\alpha, \xi, \phi)\rangle$ as the initial state.

For a free mass m , using $X = x\sqrt{\hbar/(m\omega)}$, $P = p\sqrt{m\hbar\omega}$. We then find the time development equation for a free mass,

$$\sigma^2(X(t)) = \sigma^2(X(0)) + \frac{t^2}{m^2}\sigma^2(P(0)) - \frac{\hbar t}{m}\text{sgn}(\sin(\theta))\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - (2\bar{n} + 1)^2}$$

The third term on the right-hand side, where the square root involves the dimensionless $x(0), p(0)$ of the last section, exhibits all possible rates of contraction and expansion of wave packets allowed by the uncertainty principle.

And finally, let, $S(\xi) |\phi\rangle \equiv |\psi(\xi, \phi)\rangle$. Then, $|\psi(\alpha, \xi, \phi)\rangle = D(\alpha, a) |\psi(\xi, \phi)\rangle$. The integration over α and the fact that $|\psi(\xi, \phi)\rangle$ is a normalized state yields the over-completeness relation,

$$\langle x | \int d^2\alpha \frac{1}{\pi} | \psi(\alpha, \xi, \phi) \rangle \langle \psi(\alpha, \xi, \phi) | x' \rangle = \delta(x - x') \quad (10)$$

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