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Is ESIC stronger than Realignment Criterion?

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Definition

In $\mathcal{C}^d$ a SIC-POVM is a family of $\{\Pi_k\}_{k=1}^{d^2}$ such that $\Pi_k = |\psi_k\rangle\langle\psi_k| / d \quad \forall k$,

$$|\langle\psi_k|\psi_l\rangle|^2 = \frac{d\delta_{kl}+1}{d+1} \quad \text{thus} \sum_{k=1}^{d^2} \Pi_k = \boldsymbol{I} \tag{1}$$

One can decompose a state $\rho \in \mathcal{B}\left(\mathbb{C}^d\right)$

$$\rho = d\left(d+1\right)\sum_{k=1}^{d^2} p_k \Pi_k - \boldsymbol{I}_d, \qquad p_k = \mathrm{Tr}\rho\Pi_k, \qquad \sum_{k=1}^{d^2} p_k = 1. \tag{2}$$

$$\mathrm{Tr}\rho^2 = d\left(d+1\right)\sum_{k=1}^{d^2} p_k^2 - 1 \Rightarrow \sum_{k=1}^{d^2} p_k^2 = \frac{1+\mathrm{Tr}\rho^2}{d\left(d+1\right)} \leqslant \frac{2}{d\left(d+1\right)} \tag{3}$$

ESIC criterion

Proposition

Measuring in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ *with* $\{\Pi^{A,B}\}_{\alpha=1}^{d^2}$ *SIC-POVM in* $\mathbb{C}^{d_{A,B}}$ *the outcomes are*

$$\mathcal{P}_{\alpha\beta} = \mathrm{Tr}\rho\Pi_\alpha^{A\,T} \otimes \Pi_\beta^B. \qquad (4)$$

Theorem

If $\rho \in \mathcal{B}\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\right)$ *is separable, one has*

$$||\mathcal{P}||_{tr} = \mathrm{Tr}\sqrt{\mathcal{P}^\dagger\mathcal{P}} \leqslant \frac{2}{\sqrt{d_A\left(d_A+1\right)d_B\left(d_B+1\right)}}. \qquad (5)$$

Proof of ESIC Criterion

For product states $\rho = \sigma^A \otimes \sigma^B$, we use "singular value decomposition"

$$\mathcal{P}_{\alpha\beta} = p^A_\alpha p^B_\beta \to \mathcal{P} = |p^A\rangle\langle p^B| = \underbrace{|p^A||p^B|}_{||\mathcal{P}||_{\text{tr}}} |\tilde{p}^A\rangle\langle\tilde{p}^B| \,, \tag{6}$$

and Eq.(4). Therefore

$$||\mathcal{P}(\sigma^A \otimes \sigma^B)||_{\text{tr}} = |p^A||p^B| \leqslant \sqrt{\frac{2}{d_A(d_A+1)}}\sqrt{\frac{2}{d_B(d_B+1)}}. \tag{7}$$

For any separable state

$$||\mathcal{P}\Big(\sum_i p_i \sigma_i^A \otimes \sigma_i^B\Big)||_{\text{tr}} \leqslant \sum_i p_i ||\mathcal{P}(\sigma_i^A \otimes \sigma_i^B)||_{\text{tr}} \leqslant \underbrace{\sum_i p_i}_{1} \sqrt{\frac{2}{d_A(d_A+1)}}\sqrt{\frac{2}{d_B(d_B+1)}} \quad \square. \tag{8}$$

Realignment criterion (CCNR)

Proposition

A bipartite density state in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ *with* $\{G^{A,B}\}_{\alpha=1}^{d^2}$ *BON in* $\mathbb{C}^{d_{A,B}}$ *is*

$$\rho = \sum_{\alpha=1}^{d^2} \sum_{\beta=1}^{d^2} \mathcal{C}_{\alpha\beta} G_\alpha^A \otimes G_\beta^B, \qquad \mathcal{C}_{\alpha\beta} = \mathrm{Tr}\rho {G_\alpha^A}^T \otimes G_\beta^B. \tag{9}$$

Theorem

If $\rho \in \mathcal{B}\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\right)$ *is separable, one has*

$$||\mathcal{C}||_{tr} = \sum_{k=1}^{d^2} \lambda_k \leqslant 1, \qquad \lambda_k' s \text{ singular values of } \mathcal{C}. \tag{10}$$

Proof of the trace norm of C

Proof.

Comparing the decomposition by covariance matrix $\mathcal{C}$ and by Schmidt decomposition

$$\rho = \sum_{\alpha=1}^{d^2} \sum_{\beta=1}^{d^2} \mathcal{C}_{\alpha\beta} G_\alpha^A \otimes G_\beta^B , \qquad \rho = \sum_{k=1}^{d^2} \lambda_k G_k'^A \otimes G_k'^B \tag{11}$$

writing $\mathcal{C} = U \Lambda V^T$, with U, V unitary and $\Lambda = \mathrm{diag}\{\lambda_k\}_{i=1}^{d^2}$ yields

$$||\mathcal{C}||_{\mathrm{tr}} = \max_{W \in \mathcal{O}(d^2)} \langle \mathcal{C} | W \rangle_{HS} = \max_{W \in \mathcal{O}(d^2)} \left\langle \Lambda | \underbrace{V^T W U}_{I} \right\rangle_{HS} = \sum_{k=1}^{d^2} \lambda_k \tag{12}$$

□

Comparison CCNR and ESIC by Entanglement witnesses EWs

Comparing by EWs with $O \in \mathcal{O}(d^2)$ for $(d_A = d = d_B)$

$$W_O = I_d \otimes I_d + \sum_{\alpha,\beta=1}^{d^2} O^{\alpha\beta} G_\alpha^A \otimes G_\beta^B, \qquad \text{CCNR EWs} \tag{13}$$

$$W_O = I_d \otimes I_d + \frac{d(d+1)}{2} \sum_{\alpha,\beta=1}^{d^2} O^{\alpha\beta} \Pi_\alpha^A \otimes \Pi_\beta^B, \qquad \text{ESIC EWs} \tag{14}$$

Requiring $\langle G_\alpha | G_\beta \rangle_{HS} = \delta_{\alpha\beta}$ with Eq. (1) yield d^2 equations

$$G = \sqrt{d(d+1)}\Pi + \frac{-\sqrt{d+1} \pm 1}{\sqrt{d^3}} I. \tag{15}$$

Criterion by EWs (Example on CCNR)

For any separable state

$$\max_{O\in\mathcal{O}(d^2)} \langle \mathcal{C}|O\rangle_{HS} = ||\mathcal{C}||_{\mathrm{tr}} \leqslant 1, \tag{16}$$

$\forall O \in \mathcal{O}(d^2)$ we have

$$\sum_{\alpha,\beta=1}^{d^2} \mathcal{C}_{\alpha\beta} O^{\alpha\beta} = \sum_{\alpha,\beta=1}^{d^2} \mathrm{Tr}\left(\rho G^A_\alpha \otimes G^B_\beta\right) O^{\alpha\beta} \leqslant \mathrm{Tr}\rho I. \tag{17}$$

ρ is separable: $\forall O \quad \mathrm{Tr}\rho\left(I + \sum_{\alpha,\beta=1}^{d^2} G^A_\alpha \otimes G^B_\beta O^{\alpha\beta}\right) = \mathrm{Tr}\rho W_O \geqslant 0. \quad (18)$

EWs criteria

From Eq. (19)

$$\mathrm{Tr}\rho W_O = 1 + \frac{1}{2}\sum_{\alpha,\beta=1}^{d^2} O^{\alpha\beta} X_{\alpha\beta}, \tag{19}$$

with $\alpha = (\sqrt{d+1} - 1)$ and $|\tilde{1}\rangle$ normalized ones,

$$X = (I + \alpha|\tilde{1}\rangle\langle\tilde{1}|)\mathcal{C}(I + \alpha|\tilde{1}\rangle\langle\tilde{1}|). \tag{20}$$

ESIC and CCNR criterion are rewritten respectively as

$$\rho \text{ separable: } \mathrm{Tr} W_O\rho \geqslant 0 \Longrightarrow \|X\|_{\mathrm{tr}} \leqslant 2 \wedge \|\mathcal{C}\|_{\mathrm{tr}} \leqslant 1. \tag{21}$$

ESIC $\Rightarrow$ CCNR ?

Question: is ESIC stronger than CCNR?

$$\forall \rho \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) : 2 \geqslant \|X\|_{\text{tr}} \implies 1 \geqslant \|\mathcal{C}\|_{\text{tr}} ?$$

$$2 \geqslant || U^T (I + \alpha |\tilde{1}\rangle\langle\tilde{1}|) \underbrace{\mathcal{C}}_{U\Lambda V^T} (I + \alpha |\tilde{1}\rangle\langle\tilde{1}|) V ||_{\text{tr}}$$

$$= || (I + \alpha |a\rangle\langle a|) \Lambda (I + \alpha |b\rangle\langle b|) ||_{\text{tr}}$$

$$\geqslant ||\mathcal{C}||_{tr} + \alpha \sum_i \lambda_i (a_i^2 + b_i^2) + \alpha^2 (\boldsymbol{b} \cdot \boldsymbol{a}) \sum_i \lambda_i a_i b_i$$

where we used $||(\cdot)||_{\text{tr}} \geqslant |\mathrm{Tr}(\cdot)| \geqslant \mathrm{Tr}(\cdot)$, $\quad |a\rangle = U^T |\tilde{1}\rangle$, $|b\rangle = V^T |\tilde{1}\rangle$.

Calculations

✓ $\mathcal{C}$-symm and positive, $||(\cdot)||_{\mathrm{tr}} = \mathrm{Tr}(\cdot)$ and $a_i = b_i$. Then $2 \geqslant ||\mathcal{C}||_{tr} + 1$ □.
✗ $\mathcal{C}$-symm, $a_i = \pm b_i$.

$$2 \geqslant \cdots = ||\mathcal{C}||_{tr} + \alpha \sum_{a_i \neq b_i} \lambda_i (a_i - b_i)^2 + \alpha(2 + \alpha \boldsymbol{b} \cdot \boldsymbol{a})/d$$

$$= ||\mathcal{C}||_{tr} + 1 + 4\alpha \underbrace{\sum_{a_i \neq b_i} a_i^2 \overbrace{\left(\lambda_i - \frac{\sqrt{d+1} - 1}{2d}\right)}^{\geqslant 0? \to \text{gap!}, \lambda_i = -l_i}}_{\geqslant 0? \to \text{No!}}$$

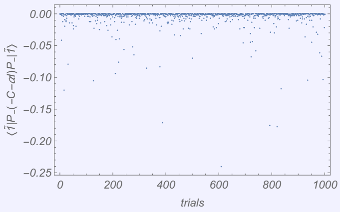
eigenvalues
$l_i \notin [-\frac{\sqrt{d+1}-1}{2d}, 0]$
sounds strange!

Numerical counterexample

We checked for $d = 2$,

$$\sum_{a_i \neq b_i} a_i^2 \left(\lambda_i - \frac{\sqrt{d+1} - 1}{2d} \right) = \left\langle \tilde{1} \middle| P_-(-\mathcal{C} - \alpha' I) P_- \middle| \tilde{1} \right\rangle \geqslant 0? \qquad (22)$$

with P_- is the projector onto negative eigenvalues of $\mathcal{C}$ and $\alpha' = \alpha/2d$.



ANSWER: False.
The maximum value is 0.001, therefore we lose too much here $||(\cdot)||_{\text{tr}} \geqslant |\text{Tr}(\cdot)|$.

Numerical outcomes $||X||_{\text{tr}} - ||\mathcal{C}||_{\text{tr}}$ of the stronger conjecture

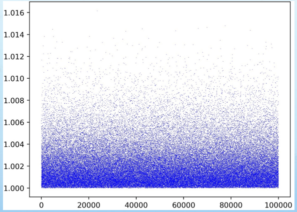


Figure: d=2

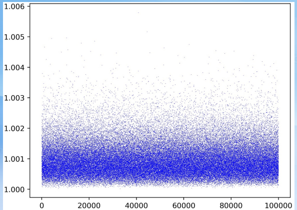


Figure: d=3

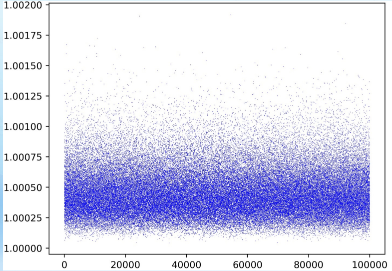
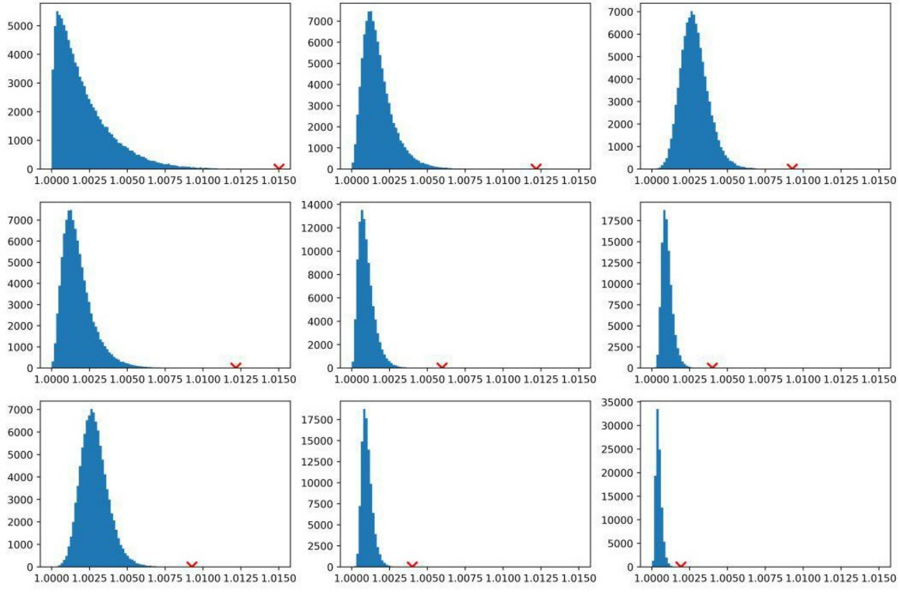


Figure: d=4

Stronger conjecture: $\forall \rho$
$1+\epsilon(d_A, d_B) \geqslant ||X||_{\text{tr}} - ||\mathcal{C}||_{\text{tr}} \geqslant 1$



0
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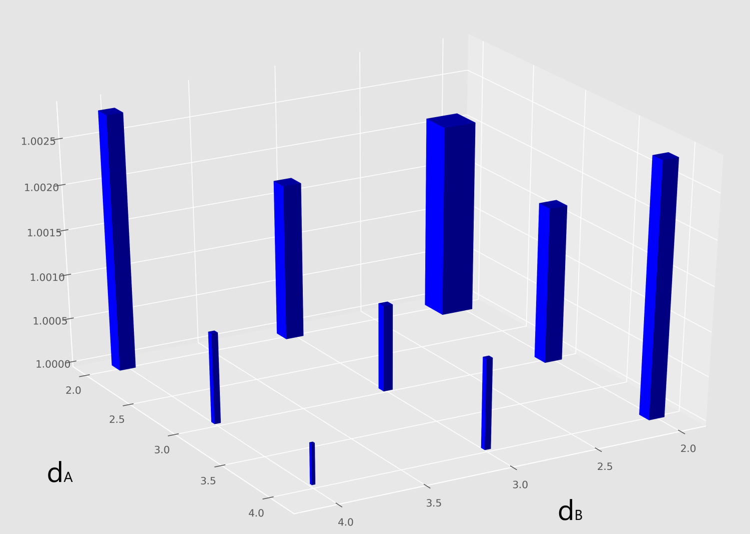
Numerical outcomes $\langle ||X||_{\mathrm{tr}} - ||\mathcal{C}||_{\mathrm{tr}} \rangle$ of the stronger conjecture



$\mathcal{C}-$ symmetric and positive

We solved the problem for symmetric and positive $\mathcal{C}$, but we observe

Conjecture

$\mathcal{C}$ symmetric and positive $\Longrightarrow$ ρ is PPT.

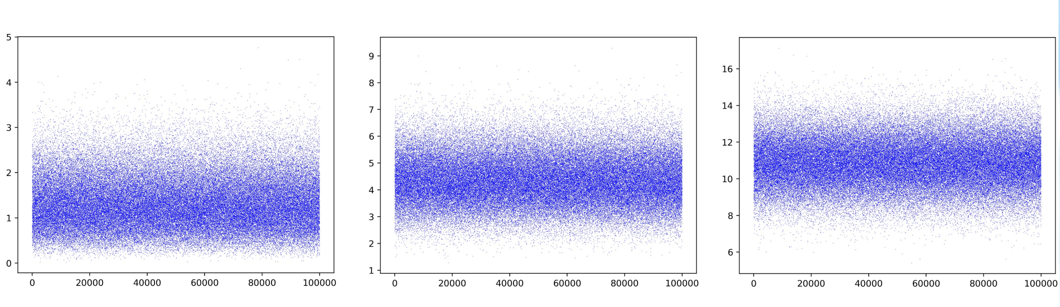


Figure: d=2,3,4. We never find entanglement by PPT criterion.

ρ bosonic and $\mathcal{C} \leqslant 0$

Question

What does mean $\mathcal{C}-$ symmmetric and positive?

Example

$\rho-$ bosonic and separable $\rho = \frac{1}{2}(\sigma \otimes \eta + \eta \otimes \sigma)$.

$$\mathcal{C}_{\alpha\beta} = \mathrm{Tr}\rho G_\alpha \otimes G_\beta = \frac{1}{2}(\mathrm{Tr}G_\alpha\sigma\mathrm{Tr}G_\beta\eta + \mathrm{Tr}G_\alpha\eta\mathrm{Tr}G_\beta\sigma) = \frac{1}{2}(x_\alpha y_\beta + y_\alpha x_\beta) \tag{23}$$

$$x = \begin{bmatrix} a \\ b \end{bmatrix}, y = \begin{bmatrix} b \\ a \end{bmatrix}, \mathcal{C} = \begin{bmatrix} ab & \frac{(a^2+b^2)}{2} \\ \frac{(a^2+b^2)}{2} & ab \end{bmatrix}, \ \det\mathcal{C} = -\frac{(a^2-b^2)^2}{4} \leqslant 0. \tag{24}$$

Reference: *Enhanced entanglement criterion via SIC measurements*, O.Gühne et al. arXiv:1805.03955v3,(2018).

Question

Is ESIC stronger than Realignment criterion?
Probabily yes, but just a little bit.

END. . . (for today)

PS: tomorrow I hope to switch the "e" into "a". . .

GS.