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Physics Alliance



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Maximally entangled mixed states with fixed marginals

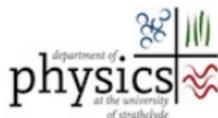
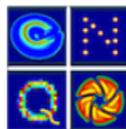
Giuseppe Baio

SUPA & University of Strathclyde, Glasgow, UK

51 Symposium of Mathematical Physics, Toruń, Poland

17th June 2019

Computational Nonlinear and Quantum Optics

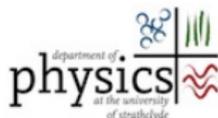
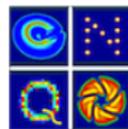


Cold Atoms, Nanophotonics, Quantum Information and Many-Body Physics, Structured Light etc. <http://cnqo.phys.strath.ac.uk>

ColOpt ITN



Collective effects and optomechanics in ultra cold matter <https://www.colopt.eu/>



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Talk based on recent paper: [Phys. Rev. A 99, 062312 \(2019\)](#)

Joint work with: **D. Chruściński**, **G. Sarbicki** (Toruń, Poland), **P. Horodecki** (Gdańsk, Poland), **A. Messina** (Palermo, Italy)

- Maximally entangled mixed states (MEMS)

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- Fixing marginals: reconstructing states from local information

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- Future directions

- **von Neumann entropy** for pure states $|\Psi_{AB}\rangle\langle\Psi_{AB}|$:

$$\mathcal{E}(\Psi_{AB}) = S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A) \quad (1)$$

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$$\text{EOF}(\rho_{AB}) = \min_{p_k, \Psi_k} \sum_k p_k \mathcal{E}(\Psi_k) \quad (2)$$

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- Several tools adopted: concurrence and **negativity**¹:

$$N(\rho_{AB}) \equiv \frac{1}{2} (\|\rho_{AB}^\tau\|_1 - 1) \quad (3)$$

Partial transpose: $\rho_{AB}^\tau = (\mathbb{I} \otimes \tau) \rho_{AB}$

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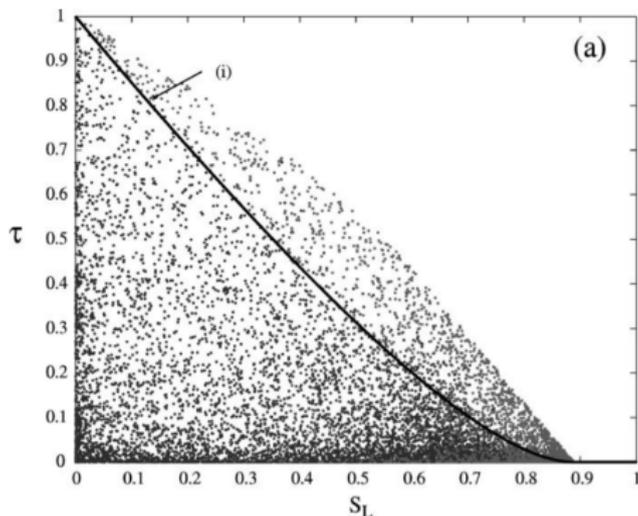
Maximally entangled mixed states (MEMS)

- Relation between entanglement and purity $\text{Tr}(\rho_{AB}^2)$:²

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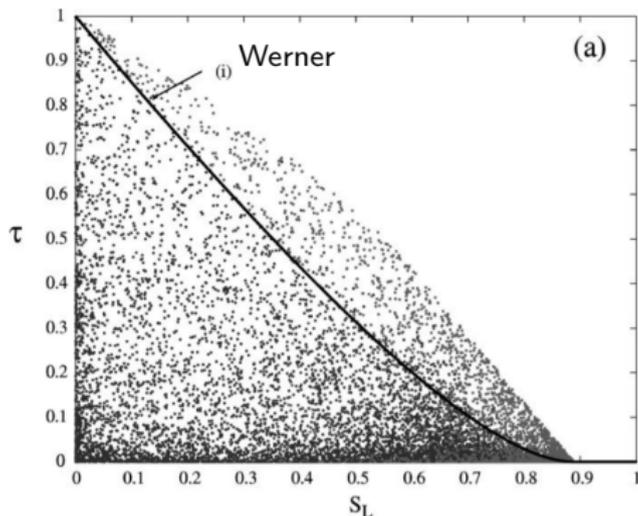
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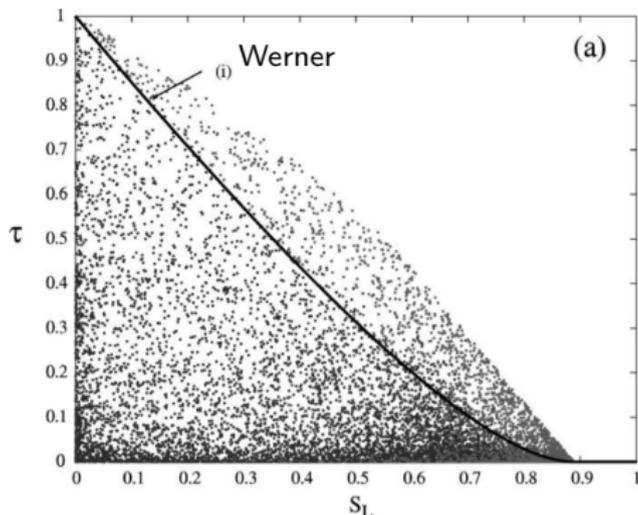
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- **MEMS**: states ρ^* such that any measure $\mathcal{E}(\rho^*) \geq \mathcal{E}(U\rho^*U^\dagger)$, $\forall U$.

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Maximally entangled mixed states (MEMS)

- Two qubit MEMS found solving the spectral constrained analogue³

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Theorem (Verstraete, 2001)

Given a state $\rho = \Phi\Lambda\Phi^\dagger$, the unitary maximising EOF and negativity is:

$$U = (U_1 \otimes U_2) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} D_\phi \Phi^\dagger \quad (4)$$

- MEMS depend on the entanglement measure considered⁴

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For negativity: $\rho_{\text{MEMS}} = \frac{1-r}{4}\mathbb{I}_2 \otimes \mathbb{I}_2 + rP_2^+$, $P_2^+ = \frac{1}{2} \sum_{i,j=1}^2 |ii\rangle\langle jj|$

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Fixing marginals: reconstructing states from local info

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- What is the upper bound $\mathcal{E}_{\max}(\rho)$ on bipartite entanglement when only marginals are known?

MEMS with respect to fixed marginals

Given ρ_A, ρ_B , find $\mathcal{E}_{\max}(\rho) : \text{Tr}_B(\rho) = \rho_A, \text{Tr}_A(\rho) = \rho_B$

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- Characterizing states from **local measurements**:
Quantum marginal constraints (Klyachko), Quantum tomography etc.

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What is known: Two qubits

- Let $\rho_A = \text{diag}\{1 - \lambda_A, \lambda_A\}$, $\rho_B = \text{diag}\{1 - \lambda_B, \lambda_B\}$ be two qubit states.
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- $\mathcal{C}(\rho_A, \rho_B)$ set of two-qubit states with fixed marginals:

$$\rho_{AB} = \rho_A \otimes \rho_B + \left(\begin{array}{cc|cc} \epsilon & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ & -\epsilon & \Delta_{23} & -\Delta_{13} \\ \hline \text{(c.c)} & & -\epsilon & -\Delta_{12} \\ & & & \epsilon \end{array} \right) \quad (5)$$

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$$\tilde{\rho}_{AB} = \left(\begin{array}{cc|cc} 1 - \lambda_A & \cdot & \cdot & \sqrt{(1 - \lambda_A)\lambda_B} \\ \cdot & 0 & \cdot & \cdot \\ \hline \cdot & \cdot & \lambda_A - \lambda_B & \cdot \\ \sqrt{(1 - \lambda_A)\lambda_B} & \cdot & \cdot & \lambda_B \end{array} \right) \quad (5)$$

- Maximal neg: $N(\tilde{\rho}_{AB}) = \frac{1}{2} \left(\lambda_A - \lambda_B - \sqrt{(\lambda_A - \lambda_B)^2 + 4\lambda_B(1 - \lambda_A)} \right)$

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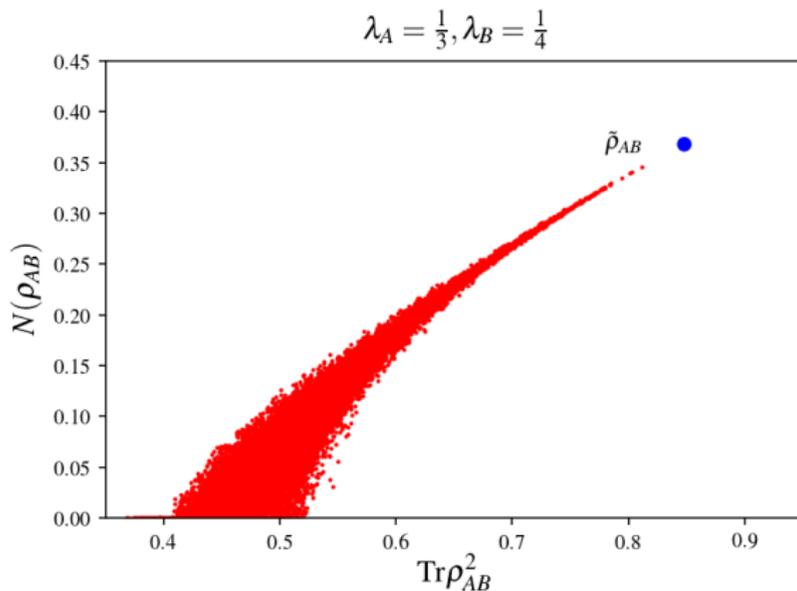
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2-qubit MEMS with respect to ρ_A and ρ_B

$$\tilde{\rho}_{AB} = (1 - \eta)|\Psi_{\text{mc}}\rangle\langle\Psi_{\text{mc}}| + \eta|10\rangle\langle 10|, \quad \sigma_{\text{mc}} = \sum_{i,j=0}^{d-1} \alpha_{ij}|ii\rangle\langle jj| \quad (6)$$

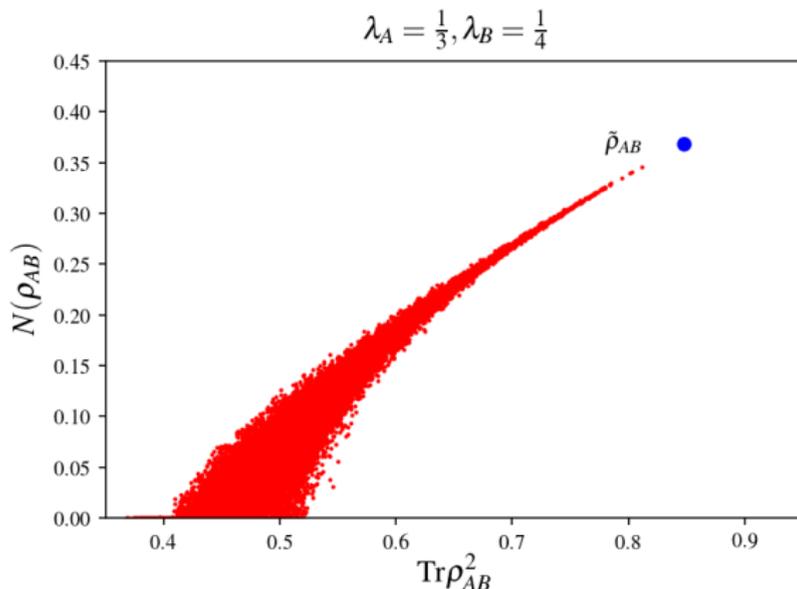
What is known: Two qubits

- Random two qubit states with fixed marginals ($N - P$ plot):



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- Both P and N are maximised by the same state $\tilde{\rho}_{AB}$

Interlude: Extremal states

- $N(\rho_{AB})$ convex function and $\mathcal{C}(\rho_A, \rho_B)$ also convex: maximised by **extremal point**. How to characterize extremals in $\mathcal{C}(\rho_A, \rho_B)$ ^{6,7}?

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Extremal points in $\mathcal{C}(\rho_A, \rho_B)$

$$\begin{aligned} \frac{1}{d} \Lambda_\rho(\mathbb{I}_d) &= \frac{1}{d} \sum_\alpha K_\alpha K_\alpha^\dagger = \rho_A \\ \frac{1}{d} \Lambda_\rho^*(\mathbb{I}_d) &= \frac{1}{d} \sum_\alpha K_\alpha^\dagger K_\alpha = \rho_B \end{aligned} \quad \{K_\alpha^\dagger K_\beta \oplus K_\beta K_\alpha^\dagger\}_{\alpha, \beta=1, \dots, d^2} \text{ L.I.} \quad (7)$$

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Higher dimensions

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- We generalize the two qubit results to $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\tilde{\rho} = (1 - \eta)\sigma_{\text{mc}} + \sum_{i \neq j} p_{ij} |ij\rangle\langle ij| \quad (8)$$

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- Example : Two qutrits

$$\tilde{\rho} = \left(\begin{array}{ccc|ccc|ccc} \rho_{11} & \cdot & \cdot & \cdot & \Delta_{15} & \cdot & \cdot & \cdot & \Delta_{19} \\ & \rho_{22} & \cdot \\ & & \rho_{33} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline & & & \rho_{44} & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \rho_{55} & \cdot & \cdot & \cdot & \Delta_{59} \\ & & & & & \rho_{66} & \cdot & \cdot & \cdot \\ \hline & & & & & & \rho_{77} & \cdot & \cdot \\ & & & & & & & \rho_{88} & \cdot \\ (c.c.) & & & & & & & & \rho_{99} \end{array} \right) \quad (9)$$

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- We identify in this family candidates for MEMS with fixed marginals

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- Marginals $\rho_A = \text{diag}\{1 - \lambda_1 - \lambda_2, \lambda_1, \lambda_2\}$, $\rho_B = \text{diag}\{1 - \mu_1 - \mu_2, \mu_1, \mu_2\}$
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Partial transpose:

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- Guideline for max negativity: lowest possible number of $\rho_{ii} \neq 0$ (2)

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valid when $\lambda_1 > \mu_1$ and $\lambda_2 > \mu_2$,

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when $\lambda_1 > \mu_1$ and $\lambda_2 < \mu_2$, and finally ($\lambda_1 < \mu_1$ and $\lambda_2 > \mu_2$)

$$\begin{aligned}\tilde{\rho}_{AB}^{(3)} &= (1 - p_{20} - p_{21}) |\Psi_{\text{mc}}^{(3)}\rangle\langle\Psi_{\text{mc}}^{(3)}| + p_{20} |20\rangle\langle 20| + p_{21} |21\rangle\langle 21| \\ p_{20} &= \lambda_1 + \lambda_1 - (\mu_1 + \mu_2), p_{21} = \mu_1 - \lambda_1\end{aligned}\quad (13)$$

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valid when $\lambda_1 > \mu_1$ and $\lambda_2 > \mu_2$,

$$\begin{aligned}\tilde{\rho}_{AB}^{(2)} &= (1 - p_{10} - p_{12}) |\Psi_{\text{mc}}^{(2)}\rangle\langle\Psi_{\text{mc}}^{(2)}| + p_{10} |10\rangle\langle 10| + p_{12} |12\rangle\langle 12| \\ p_{10} &= \lambda_1 + \lambda_1 - (\mu_1 + \mu_2), p_{12} = \mu_2 - \lambda_2\end{aligned}\quad (12)$$

when $\lambda_1 > \mu_1$ and $\lambda_2 < \mu_2$, and finally ($\lambda_1 < \mu_1$ and $\lambda_2 > \mu_2$)

$$\begin{aligned}\tilde{\rho}_{AB}^{(3)} &= (1 - p_{20} - p_{21}) |\Psi_{\text{mc}}^{(3)}\rangle\langle\Psi_{\text{mc}}^{(3)}| + p_{20} |20\rangle\langle 20| + p_{21} |21\rangle\langle 21| \\ p_{20} &= \lambda_1 + \lambda_1 - (\mu_1 + \mu_2), p_{21} = \mu_1 - \lambda_1\end{aligned}\quad (13)$$

Proposition 1

States (11), (12), (13) are extremal in $\mathcal{C}(\rho_A, \rho_B)$.

- Both the 2-qubit state and our candidates are **quasidistillable**⁹

⁹M. Horodecki, P. Horodecki & R. Horodecki, Phys. Rev. A 60, 1888 (1999).

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Def. 1: Non-collective distillation

ρ is *distillable* iff there exist A and B (filtering) such that:

$$\frac{(A \otimes B)\rho(A^\dagger \otimes B^\dagger)}{\text{Tr}[(A \otimes B)\rho(A^\dagger \otimes B^\dagger)]} = P_d^+ \quad (14)$$

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- No mixed state is non-collectively distillable!

Def. 2: Quasidistillation

ρ is *quasidistillable* iff there exist $\{A_n\}$ and $\{B_n\}$ such that:

$$\frac{(A_n \otimes B_n)\rho(A_n^\dagger \otimes B_n^\dagger)}{\text{Tr}[(A_n \otimes B_n)\rho(A_n^\dagger \otimes B_n^\dagger)]} \xrightarrow{n \rightarrow \infty} P_d^+, \quad p_n = \text{Tr}[\Lambda^{(n)}(\rho)] \rightarrow 0 \quad (15)$$

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- We characterize quasidistillable states within the family:

$$\tilde{\rho} = (1 - \eta)\sigma_{\text{mc}} + \sum_{i \neq j} p_{ij} |ij\rangle \langle ij| \quad (16)$$

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σ_{mc} is quasidistillable iff it is rank-1, i.e. $\sigma_{\text{mc}} = |\Psi_{\text{mc}}\rangle\langle\Psi_{\text{mc}}|$, and $s - \text{rank}(|\Psi_k\rangle) = d$.

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Theorem 2

ρ of the form (16) quasidistillable iff among $p_{ij} \neq 0$, $p_{ij}p_{jk} \dots p_{li} = 0$. (No looping indices)

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Theorem 2

ρ of the form (16) quasidistillable iff among $p_{ij} \neq 0$, $p_{ij}p_{jk} \dots p_{li} = 0$. (No looping indices)

- At most $\binom{d}{2}$ non-zero p_{ij}

- Our candidate states are quasidistillable:

$$\tilde{\rho}_{AB}^{(1)} = (1 - p_{10} - p_{20}) |\Psi_{\text{mc}}^{(1)}\rangle\langle\Psi_{\text{mc}}^{(1)}| + p_{10} |10\rangle\langle 10| + p_{20} |20\rangle\langle 20|$$

$$\tilde{\rho}_{AB}^{(2)} = (1 - p_{10} - p_{12}) |\Psi_{\text{mc}}^{(2)}\rangle\langle\Psi_{\text{mc}}^{(2)}| + p_{10} |10\rangle\langle 10| + p_{12} |12\rangle\langle 12|$$

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- Another two-qutrit example¹⁰:

$$\rho = \eta P_3^+ + \frac{(1 - \eta)}{3} (|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|) \quad (17)$$

¹⁰M. Horodecki, P. Horodecki & R. Horodecki, Phys. Rev. A 60, 1888 (1999).

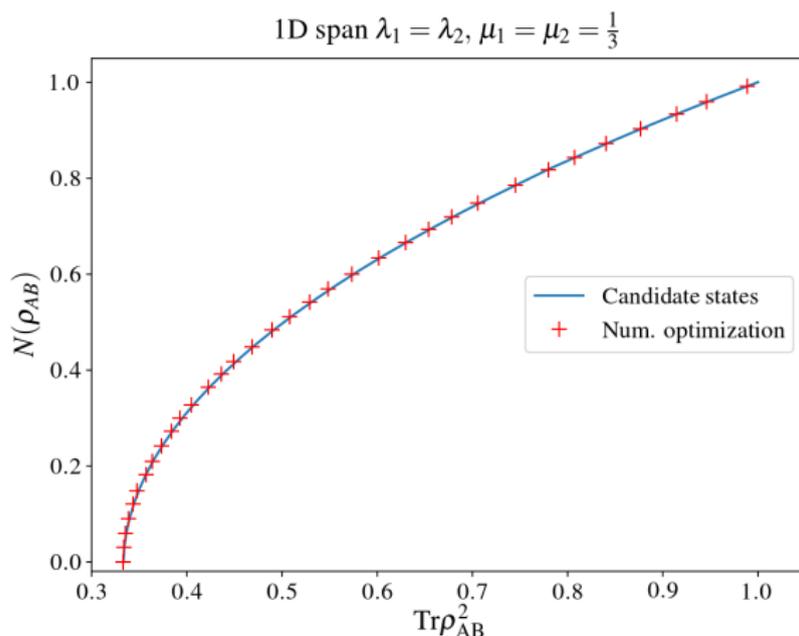
Numerical results

- Constrained optimization in $\mathcal{C}(\rho_A, \rho_B)$. Rank-4 two-qutrit states¹¹

¹¹K. R. Parthasarathy, Ann. l'Inst. H. Poincaré (B) Prob. Stat. 41, 257 (2005)

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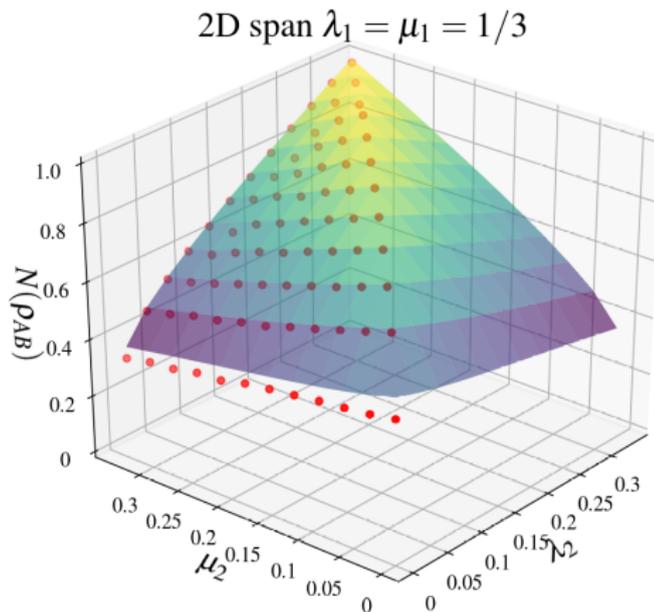
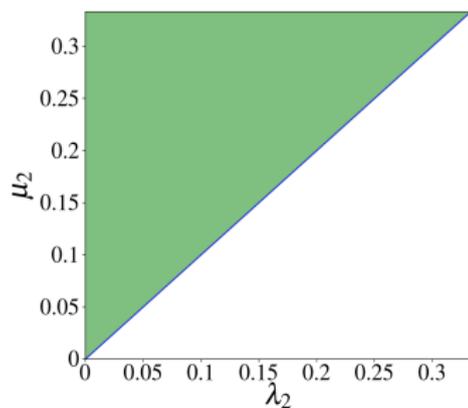
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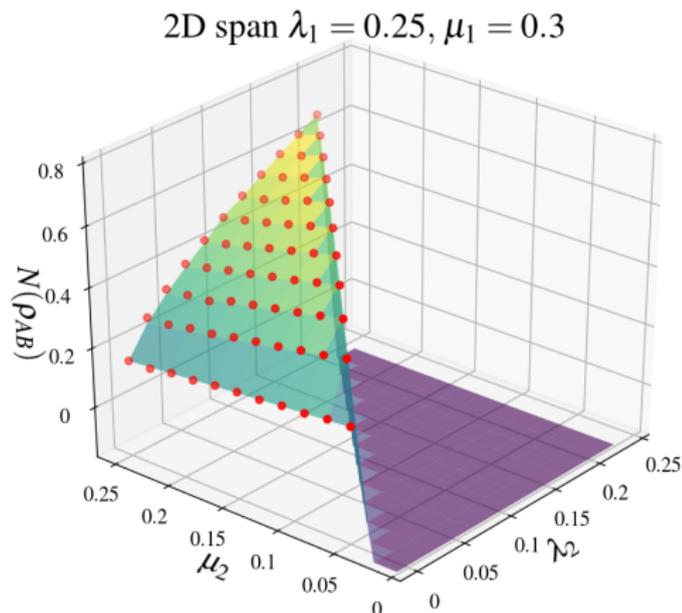
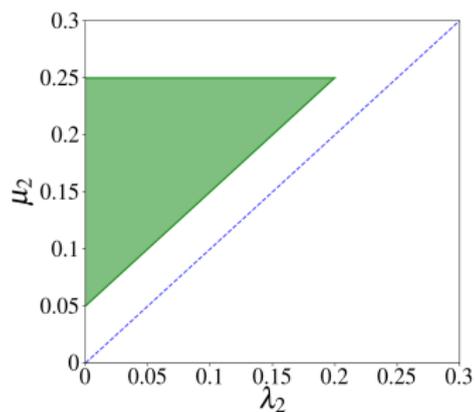
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- We identify a family of candidate states as MEMS with fixed marginals
- Generalization of 2-qubit results + extremal states
- Quasidistillation and MEMS: a deep connection
- Numerical results strengthen our conjecture

- Quasidistillation may help in finding higher dimensional MEMS
- Fixed marginal purities/other non convex sets
- Generalization to many body/multipartite entanglement
- Upper bounds on other measures of interest

Thank you for your attention

