Genuinely entangled subspaces

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Outline

- Background:
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  - completely entangled subspaces (CES),
  - unextendible product bases (UPB),
- Genuinely entangled subspaces (GES),
- From UPBs to GESs,
- Entanglement of GESs and states,
- Conclusions+open questions.

Background: entanglement

Consider N parties $A_1, A_2, \ldots, A_N =: \mathrm{A}$.
A pure state $|\psi\rangle_{A_1 \ldots A_N}$ is:

- *fully product* if $|\psi\rangle_{A_1 \cdots A_N} = |\varphi\rangle_{A_1} \otimes \cdots \otimes |\xi\rangle_{A_N}$,
- *entangled* if $|\psi\rangle_{A_1 \cdots A_N} \neq |\varphi\rangle_{A_1} \otimes \cdots \otimes |\xi\rangle_{A_N}$ (e.g., $|0\rangle_A \otimes |0\rangle_B \otimes |\psi_-\rangle_{CD}$),
- *biproduct* if $|\psi\rangle_{A_1 \cdots A_N} = |\varphi\rangle_S \otimes |\phi\rangle_{\bar{S}}$ (e.g., $|\psi_-\rangle_{AB} \otimes |\psi_-\rangle_{CD}$),
- *genuinely multiparty entangled (GME)* if

$$|\psi\rangle_{A_1 \cdots A_N} \neq |\varphi\rangle_S \otimes |\phi\rangle_{\bar{S}}, \quad S \subset \mathrm{A}, \quad \bar{S} = \mathrm{A} \setminus S,$$

  e.g.,

$$|\mathrm{GHZ}_N\rangle = 1/\sqrt{2} \left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N} \right).$$

A state ρ_A is GME if it is not *biseparable*, i.e.,

$$\rho_\mathrm{A} \neq \sum_{S|\bar{S}} p_{S|\bar{S}} \sum_i q^i_{S|\bar{S}} \varrho^i_S \otimes \sigma^i_{\bar{S}}.$$

Background: completely entangled subspaces (CES)

Definition (CES) [Parthasarathy 2004, Bhat 2006]

A subspace $\mathcal{C} \subset \mathcal{H}_{d_1,\ldots,d_N}$ is called a completely entangled subspace (CES) if all $|\psi\rangle \in \mathcal{C}$ are entangled. In other words, CES is a subspace void of *fully* product vectors.

Why cosider CESs? A state ϱ with $\mathrm{supp}(\varrho) \subset \mathcal{C}$ is entangled.

The maximal size of a CES in $\mathcal{H}_{d_1\ldots d_N}$ is:

$$\prod_{i=1}^{N} d_i - \sum_{i=1}^{N} d_i + N - 1.$$

Qubits: $2^N - N - 1$. For $N = 3$: 4.

Background: unextendible product bases (UPB)

Definition (UPB) [Bennett et al. 1999]

An unextendible product basis (UPB) U is a set of fully product vectors

$$U = \{|\psi_i\rangle \equiv |\varphi_i\rangle_{A_1} \otimes \ldots \otimes |\xi_i\rangle_{A_N}\}_{i=1}^{u} ,$$

$|\psi_i\rangle \in \mathcal{H}_{d_1,\ldots,d_N}$, with the property that it spans a proper subspace of $\mathcal{H}_{d_1,\ldots,d_N}$, i.e., $u < \dim \mathcal{H}_{d_1,\ldots,d_N}$, and no *fully* product vector exists in the complement of its span.

- $|\psi_i\rangle$'s orthogonal $\rightarrow$ *orthogonal* unextendible product basis (oUPB),
- otherwise $\rightarrow$ *non–orthogonal* unextendible product basis (nUPB).

Background: unextendible product bases (UPB) [cont'd]

Example 1. (oUPB) Consider $(\mathbb{C}^2)^{\otimes 3}$ and two different orthonormal bases in $\mathbb{C}^2$: $\{|0\rangle, |1\rangle\}$ and $\{|e\rangle, |\overline{e}\rangle\}$. The following set is an oUPB:

$$U = \{|000\rangle, |1\overline{e}e\rangle, |e1\overline{e}\rangle, |\overline{e}e1\rangle\}.$$

Example 2. (nUPB) Consider $\mathbb{C}^d \otimes \mathbb{C}^d$ and the set of vectors

$$U' = \{|e\rangle \otimes |e\rangle \,||e\rangle \in \mathbb{C}^d\}.$$

$\mathrm{span}\ U' = \mathrm{Symm}(\mathcal{H}_{d,d})$, $(\mathrm{span}\ U')^{\perp} = \mathrm{Antisymm}(\mathcal{H}_{d,d}) \rightarrow U'$ is unextedible (not yet a basis). Select $\binom{d+1}{2}$ linearly independent vectors $\rightarrow U'$ becomes an nUPB.

Qubit case $(d = 2)$; $\dim \mathrm{span}\ U' = \binom{3}{2} = 3$. (i) take $\{|0\rangle|0\rangle, |1\rangle|1\rangle, |+\rangle|+\rangle\}$, (ii) orthogonalize $\rightarrow$ new basis $\{|00\rangle, |11\rangle, |01\rangle + |10\rangle\}$, which is not product.

Important fact: No oUPB at all in $\mathbb{C}^2 \otimes \mathbb{C}^2$ (even more generally, in $\mathbb{C}^2 \otimes \mathbb{C}^d$) [Bennett et al. 1999].

Background: connection between UPB and CES

Observation

Orthogonal complement of a subspace spanned by a UPB, whether its members are mutually orthogonal or not, is a CES,

$$(\text{span UPB})^{\perp} = \text{CES}.$$

Not true in the opposite direction: the orthocomplement of a CES does not necessarily admit a UPB (neither orthogonal nor non–orthogonal).

Even more: it can be $\text{CES}^{\perp} = \text{CES}$ [Walgate&Scott 2008, Skowronek 2011].

Genuinely entangled subspaces: definition, examples

No *fully* product states in a CES, but there still might be present other *biproduct* states. Why not consider CESs only with GME states?

Definition (GES) [MD&Augusiak 2018, Cubitt et al. 2008]

A subspace $\mathcal{G} \subset \mathcal{H}_{d_1,\ldots,d_N}$ is called a genuinely entangled subspace (GES) of $\mathcal{H}_{d_1,\ldots,d_N}$ if all $|\psi\rangle \in \mathcal{G}$ are genuinely multiparty entangled (GME).

A state ϱ with $\mathrm{supp}(\varrho) \subset \mathcal{G}$ is GME

Example 1. Antisymmetric subspace. Dimension $\binom{d}{N}$, empty for $N > d$.
Example 2. Subspace spanned by $|W\rangle$ and $|\bar{W}\rangle = \sigma_x^{\otimes N}|W\rangle$ [Kaszlikowski et al. 2008].

The maximal dimension of a GES ($2 \leq d_i \leq d_{i+1}$) [Cubitt et al. 2008]:

$$\prod_{i=1}^{N} d_i - (d_1 + d_2 \cdot d_3 \cdot \ldots \cdot d_N) + 1, \quad (d^{N-1} - 1)(d-1) \quad \text{(equal dimensions)}$$

Qubits: $2^{N-1} - 1$. For $N = 3$: 3.

Genuinely entangled subspaces: how to construct

How to construct a GES?

(i) choose randomly a not too large number of vectors (any number below the maximal dimension is allowable),

(ii) build a multipartite UPB with the property $(\mathrm{span\,UPB})^{\perp} = \mathrm{GES}$.

Observation

A multipartite UPB has a GES in the orthocomplement of its span if and only if it is a bipartite UPB across any of the possible cuts in the parties, i.e., cannot be extended with biproduct vectors.

$\Longrightarrow$ tools from the bipartite case are useful,

$\Longrightarrow$ applicability of oUPBs is limited, e.g., no oUPB with a qubit subsystem can lead to a GES and oUPBs do not exist with all cardinalities.

Idea: Use nUPBs to have a general construction.

Genuinely entangled subspaces: construction – preliminaries

Crucial lemma [a version of Bennett et al 1999]

Let there be given a set of product vectors $B = \{|\varphi_x\rangle \otimes |\phi_x\rangle\}_x$ from $\mathbb{C}^m \otimes \mathbb{C}^n$ with cardinality $|B| \geq m + n - 1$. If any m–tuple of vectors $|\varphi_x\rangle$ spans $\mathbb{C}^m$ and any n–tuple of $|\phi_x\rangle$'s spans $\mathbb{C}^n$, then there is no product vector in the orthocomplement of $\mathrm{span} B$, i.e., B is unextendible.
We say that $|\varphi_x\rangle$'s and $|\phi_x\rangle$'s possess the spanning property.

Looking for a product vector:

B_1	B_2
$|\varphi_1\rangle \otimes |\phi_1\rangle$	$|\varphi_{s+1}\rangle \otimes |\phi_{s+1}\rangle$
$|\varphi_2\rangle \otimes |\phi_2\rangle$	$\vdots$
$\vdots$	$|\varphi_{|B|}\rangle \otimes |\phi_{|B|}\rangle$
$|\varphi_s\rangle \otimes |\phi_s\rangle$	

$|f\rangle \perp \mathrm{span}\{|\varphi_1\rangle, \ldots, |\varphi_s\rangle\}$, $|g\rangle \perp \mathrm{span}\{|\phi_{s+1}\rangle, \ldots, |\phi_{|B|}\rangle\} \rightarrow |f\rangle \otimes |g\rangle \perp \mathrm{span} B$.
Not possible if the vectors possess the *spanning property*.

Genuinely entangled subspaces: construction – preliminaries (cont'd)

It is easy to construct sets of vectors with the spanning property: use Vandermonde vectors [Bhat 2006]: $|v_p(a)\rangle = (1, a, a^2, a^3, \ldots, a^{p-1}) \in \mathbb{C}^p$.

(i) Take $|\varphi_i\rangle = |v_m(\lambda_i)\rangle$, $|\phi_i\rangle = |v_n(\lambda_i)\rangle$, with arbitrary λ_i's, $\lambda_i \neq \lambda_j$ for $i \neq j$, and construct the set $B = \{|\varphi_i\rangle \otimes |\phi_i\rangle\}_{i=1}^{s}$, $s \geq m+n-1$. The subspace orthogonal to $\mathrm{span}B$ is a CES.

(ii) "Works" also in the multiparty case: $\{|\psi_i^{(1)}\rangle \otimes \cdots \otimes |\psi_i^{(N)}\rangle\}_{i=1}^{s}$, $|\psi_i^{(j)}\rangle = |v_{d_j}(\lambda_i)\rangle$, $s \geq \sum_{j=1}^{N} d_j + N - 1$;
the orthocomplement is a CES, but not a GES:
$(1, a)_{A_1} \otimes (1, a)_{A_2} \otimes \cdots = (1, a, a, a^2)_{A_1 A_2} \otimes \cdots \perp (0, 1, -1, 0) \otimes \cdots$
$\rightarrow$ we need different sets of vectors with the spanning property.

From nUPB to GES

The approach:

1. Consider $\mathcal{B} = \{|\Psi(\alpha)\rangle \equiv \bigotimes_{k=1}^{N} |\psi_k(\alpha)\rangle_{A_k} | \alpha \in \mathbb{C}\}$,
2. Coordinates (monomials of polynomials) of $|\psi_k(\alpha)\rangle \in \mathbb{C}^d$ are such that the coordinates of $\bigotimes_{k\in I} |\psi_k(\alpha)\rangle_{A_k}$, $I \subset \{1, 2, \ldots, N\}$, are lin. indep. functions of α for any $I \implies$ locally, for any partition, the vectors span corresponding whole spaces on subsystems,
3. Let $u \equiv \dim \operatorname{span} \mathcal{B}$. Choose u values of α to construct $\bar{\mathcal{B}} = \{|\Psi_i\rangle \equiv |\Psi(\alpha_i)\rangle\}_{i=1}^{u}$, such that (i) $\operatorname{span}\bar{\mathcal{B}} = \operatorname{span}\mathcal{B}$, and (ii) $\bar{\mathcal{B}}$ locally has the spanning property for any bipartite cut.
4. Due to Crucial lemma, there is no biproduct vector in the orthocomplement of $\operatorname{span} \bar{\mathcal{B}} \implies \bar{\mathcal{B}}$ is a UPB giving rise to a GES.

From nUPB to GES (cont'd)

The constructions:

- For $k = 2, 3, \ldots, N$: $|\psi_k(\alpha)\rangle = (1, \alpha^{d^{N-k}}, \alpha^{2d^{N-k}}, \ldots, \alpha^{(d-1)d^{N-k}})$,
- It holds

$$\begin{aligned} \bigotimes_{k=2}^{N} |\psi_k(\alpha)\rangle_{A_k} &= (1, \alpha, \alpha^2, \alpha^3, \cdots, \alpha^{d^{N-1}-1})_{A_2 A_3 \cdots A_N} \\ &= |v_{d^{N-1}}(\alpha)\rangle_{A_2 A_3 \cdots A_N}, \end{aligned}$$

- carefully choose $|\psi_1(\alpha)\rangle$.

From nUPB to GES (cont'd)

	$\lvert\psi_1^{(m)}(\alpha)\rangle$	dim GES
V_1	$\left(1,\alpha^{\widetilde{d}},\alpha^{2\widetilde{d}}\ldots,\alpha^{(d-1)\widetilde{d}}\right)$, $\widetilde{d}:=\sum_{k=2}^{N-1}(d-1)d^{N-k}+1$	$(d-1)^2$
V_2	$\left(1,\alpha^{p_1},\alpha^{p_2},\ldots,\alpha^{p_{d-1}}\right)$, $p_i:=\sum_{k=2}^{N} i d^{N-k}$	$d^N-(2d^{N-1}-1)$
V_3	$\left(1,P_1(\alpha),P_2(\alpha),\ldots,P_{d-1}(\alpha)\right)$, $P_i(\alpha):=\sum_{k=2}^{N}\alpha^{i d^{N-k}}$	$d^{N-2}(d-1)^2$

Finding dim GES: V_1, V_2 – counting different monomials, V_3 – counting linearly independent polynomials.

Fact

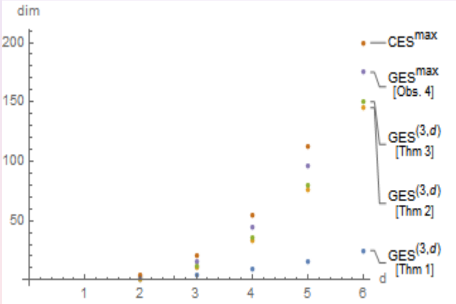
Construction V_3 is optimal.

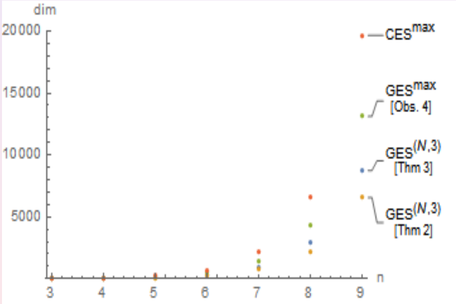
Example (V_3):

$$\Psi(\alpha)=(1,\alpha+\alpha^2)_A\otimes(1,\alpha^2)_B\otimes(1,\alpha)_C=(1,\alpha+\alpha^2)_A\otimes(1,\alpha,\alpha^2,\alpha^3)_{BC}=$$
$$=(1,\alpha^2,\alpha+\alpha^2,\alpha^3+\alpha^4)_{AB}\otimes(1,\alpha)_C=(1,\alpha,\alpha+\alpha^2,\alpha^2+\alpha^3)_{AC}\otimes(1,\alpha^2)_B=$$
$$(1,\alpha,\alpha^2,\alpha^3,\alpha+\alpha^2,\alpha^2+\alpha^3,\alpha^3+\alpha^4,\alpha^4+\alpha^5)_{ABC}$$
$$\Psi(\alpha)\perp(0,1,1,0,-1,0,0,0),(0,0,1,1,0,-1,0,0).$$

From nUPB to GES (cont'd)

Comparison of dimensions.





Genuinely entangled subspaces: entanglement

I. Entanglement of a GES → ent. of a subspace $\mathcal{S}$ [Gour&Wallach 2007]:

$$E(\mathcal{S}) = \min_{|\psi\rangle \in \mathcal{S}} E(|\psi\rangle).$$

II. Entanglement of states:

$$\varrho_{\mathcal{G}}(p) = (1-p)\frac{\mathcal{P}_{\mathcal{G}}}{d_{\mathcal{G}}} + p\frac{\mathbb{1}_D}{D}, \quad D = \Pi_i d_i,$$

(i) $E(\varrho_{\mathcal{G}}(0) \equiv \frac{\mathcal{P}_{\mathcal{G}}}{d_{\mathcal{G}}})$,

(ii) $p*$ – the white noise tolerance: $\varrho_{\mathcal{G}}(p > p*)$ are not entangled/GME

Measures: (generalized) geometric measure of entanglement:

$$E_{(G)GM}(|\psi\rangle) = 1 - \max_{|\psi_{(bi)prod}\rangle} |\langle\psi_{(bi)prod}|\psi\rangle|^2.$$

For mixed states:

$$E_{(G)GM}(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i E_{(G)GM}(|\psi_i\rangle).$$

Entanglement of a GES – methods of computation

Key fact [Branciard et al. 2010]:

$$E_{(G)GM}(\mathcal{S}) = 1 - \max_{K|\bar{K}} \max_{|\varphi_K\rangle\otimes|\bar{\varphi}_{\bar{K}}\rangle} \langle\varphi_K| \otimes \langle\bar{\varphi}_{\bar{K}}|\mathcal{P}_{\mathcal{S}}|\varphi_K\rangle \otimes |\bar{\varphi}_{\bar{K}}\rangle.$$

Approach:
Define

$$\mathfrak{S}_{\bar{K}} := \langle\bar{\varphi}_{\bar{K}}|\mathcal{P}_{\mathcal{S}}|\bar{\varphi}_{\bar{K}}\rangle$$

We then can write:

$$E_{GGM}(\mathcal{S}) = 1 - \max_{K|\bar{K}} \lambda_{\max}(\mathfrak{S}_{\bar{K}}),$$

(Similarly for GM.)

Entanglement of a GES – application

Subspace $\mathcal{S}^{\theta}_{2\times d^{N-1}}$ spanned by $(d-1)^{N-1}$ vectors ($i_k = 0, 1, \ldots, d-2$):

$$|\Phi_{i_2\ldots i_N}\rangle = \cos(\theta/2)|0\rangle_{A_1}|i_2\rangle_{A_2}\ldots|i_N\rangle_{A_N} + \mathrm{e}^{\mathrm{i}\xi}\sin(\theta/2)|1\rangle_{A_1}|i_2+1\rangle_{A_2}\ldots|i_N+1\rangle_{A_N}.$$

It holds: $\boxed{E_{GGM}(\mathcal{S}^{\theta}_{2\times d^{N-1}}) = \frac{1}{2}\left(1-\sqrt{1-\sin^2\theta\sin^2\left(\frac{\pi}{d}\right)}\right).}$ Moreover,

entanglement of $\mathcal{S}^{\theta}_{2\times d^{N-1}}$ is the same across any bipartite cut .

Main ingredient: Eigenvalues of [Yueh 2005]:

$$\begin{pmatrix} \alpha & g & 0 & \cdots & 0 & 0 \\ g^* & \alpha+\beta & g & \cdots & 0 & 0 \\ 0 & g^* & \alpha+\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \alpha+\beta & g & 0 \\ 0 & 0 & \cdots & g^* & \alpha+\beta & g \\ 0 & 0 & \cdots & 0 & g^* & \beta \end{pmatrix}, \quad \lambda_k = \alpha+\beta+2|g|\cos\frac{k\pi}{d}$$

Entanglement of a GES – computable bounds

Computation of the minimum might not be a simple problem → one can use SDP to obtain bounds:

$$\begin{aligned}\min_{|\psi_{prod}\rangle} \langle \psi_{prod}| P_{\mathcal{S}^\perp} |\psi_{prod}\rangle &= \min_{|\psi_{prod}\rangle} \mathrm{tr}[P_{\mathcal{S}^\perp} |\psi_{prod}\rangle\langle\psi_{prod}|] \\ &\geq \min_{\substack{\rho\geq 0 \\ \forall_i \rho^{T_i}\geq 0}} \mathrm{tr}[P_{\mathcal{S}^\perp}\rho],\end{aligned}$$

$$\begin{aligned}\min_{|\psi_{biprod}\rangle} \langle \psi_{biprod}| P_{\mathcal{S}^\perp} |\psi_{biprod}\rangle &= \min_{|\psi_{biprod}\rangle} \mathrm{tr}[P_{\mathcal{S}^\perp} |\psi_{biprod}\rangle\langle\psi_{biprod}|] \\ &\geq \min_{\text{all biparitions } S|\bar{S}} \left\{ \min_{\substack{\rho\geq 0 \\ \rho^{T_S}\geq 0}} \mathrm{tr}[P_{\mathcal{S}^\perp}\rho] \right\}.\end{aligned}$$

Entanglement of a GES – results

d	$E_{GM}(\mathcal{S}^{\pi/2}_{2\times d^2})$	$E^{SDP}_{GM}(\mathcal{S}^{\pi/2}_{2\times d^2})$	$E_{GGM}(\mathcal{S}^{\pi/2}_{2\times d^2})$	$E^{SDP}_{GGM}(\mathcal{S}^{\pi/2}_{2\times d^2})$
3	0.42857	0.41416	0.25000	0.25000
4	0.26543	0.26543	0.14645	0.14645
5	0.17837	0.17837	0.09549	0.09549
6	0.12742	0.12742	0.06699	0.06699
7	0.09530	0.09530	0.04952	0.04952
8	0.07384	0.07384	0.03806	0.03806

d	N	$\dim \mathcal{V}_2$	$E^{SDP}_{GM}(\mathcal{V}_2)$	$E^{SDP}_{GGM}(\mathcal{V}_2)$
3	3	10	0.19022 (0.19036)	0.025078 (0.030844)
4	3	33	0.03696	0.000976 (0.001144)
5	3	76	0.00629	0.000016 (0.000024)

d	N	$\dim \mathcal{V}_3$	$E^{SDP}_{GM}(\mathcal{V}_3)$	$E^{SDP}_{GGM}(\mathcal{V}_3)$
3	3	12	0.05856	$4.8023 \cdot 10^{-3}$ $(4.8184 \cdot 10^{-3})$
4	3	36	0.00753	$1.2579 \cdot 10^{-4}$ $(1.2649 \cdot 10^{-4})$
5	3	80	0.00124	$2.2147 \cdot 10^{-6}$ $(2.2727 \cdot 10^{-6})$

Entanglement of states – methods

1. Bounds [Zhang *et al.* 2019]:

$$E_{GM}(\rho) = 1 - \max_{\sigma \text{ fully sep.}} F^2(\rho,\sigma) \geq 1 - \max_{\substack{\sigma\geq 0\\ \forall_K \sigma^{T_K}\geq 0}} F^2(\rho,\sigma) =: E^F_{GM}(\rho),$$

$$E_{GGM}(\rho) = 1 - \max_{K|\bar{K}} \max_{\sigma \text{ sep. on } K|\bar{K}} F^2(\rho,\sigma) \geq 1 - \max_{K|\bar{K}} \max_{\substack{\sigma\geq 0\\ \sigma^{T_K}\geq 0}} F^2(\rho,\sigma) =: E^F_{GGM}(\rho).$$

2. "Exact" — numerical value of (G)GM [Streltsov *et al.* 2010].

Entanglement of states – results

Entanglement E	d	$\mathcal{S}^{\pi/2}_{2\times d^2}$	$\mathcal{V}_2$	$\mathcal{V}_3$
$E_{GGM}(\mathcal{G})/E_{GM}(\mathcal{G})$	3	0.2500 / 0.42857	0.030844 / 0.19036	0.0048184 / 0.05856
	4	0.1465 / 0.26543	0.001144 / 0.03696	0.0001265 / 0.00753
	5	0.0955 / 0.17837	0.000024 / 0.00629	0.0000023 / 0.00124
	6	0.0670 / 0.12742	—	—
	7	0.0495 / 0.09530	—	—
	8	0.0381 / 0.07384	—	—
E_{ppt}/E^{fully}_{ppt}	3	0.3008 / 0.2253	0.09514 / 0.07735	0.05999 / 0.02525
	4	0.1905 / 0.1361	0.03750 / 0.02190	0.02457 / 0.00865
	5	0.1347 / 0.0902	—	—
	6	0.1012 / 0.0641	—	—
	7	— / 0.0479	—	—
	8	— / 0.0372	—	—
E^F_{GGM}/E^F_{GM}	3	0.2500 / 0.4150	0.06645 / 0.229 718	0.04439 / 0.13799
	4	0.1667 / 0.3056	0.02434 / 0.14100	0.02221 / 0.08589
	5	0.1250 / 0.2344	—	—
	6	0.1000 / 0.1900	—	—
	7	0.0833 / 0.1597	—	—
$E^{algor.}_{GGM}/E^{algor.}_{GM}$	3	0.2500 / 0.4375	0.08156 / 0.229 720	0.04787 / 0.15238
	4	0.1667 / 0.3056	0.03475 / 0.14908	0.02449 / 0.09801
	5	0.1250 / 0.2344	0.01809 / 0.10654	0.01481 / 0.07124
	6	0.1000 / 0.1900	—	—
	7	0.0833 / 0.1597	—	—

Entanglement of states – results

Noise tolerance p^*	d	$\mathcal{S}^{\pi/2}_{2\times d^2}$	$\mathcal{V}_2$	$\mathcal{V}_3$
$p^{*witn.}_{gme}$ / $p^{*witn.}_{ent.}$	3	0.321 / 0.551	0.0490 / 0.302	0.0087 / 0.105
	4	0.204 / 0.369	0.0024 / 0.076	0.00029 / 0.017
	5	0.140 / 0.262	$6 \cdot 10^{-5}$ / 0.016	$6.3 \cdot 10^{-6}$ / 0.0034
	6	0.103 / 0.195	—	—
p^{*ppt}_{gme}	3	0.410	0.225	0.129
	4	0.301	0.127	0.077
	5	0.244	—	—
	6	0.213	—	—
$p^{*F}_{gme}/p^{*F}_{ent.}$	3	0.582 / 0.693	0.474 / 0.654	0.468 / 0.583
	4	0.524 / 0.640	0.375 / 0.614	0.464 / 0.578
	5	0.492 / 0.610	—	—
	6	0.471 / 0.591	—	—
$p^{*algor.}_{gme}/p^{*algor.}_{ent.}$	3	0.582 / 0.693	0.474 / 0.654	0.468 / 0.583
	4	0.524 / 0.640	0.375 / 0.614	0.464 / 0.578
	5	0.492 / 0.610	—	—
	6	0.471 / 0.591	—	—

Conlusions:

- ✓ constructions of nUPBs leading to GESs,
- ✓ construction of GME states of arbitrary dimensions,
- ✓ subspaces (states?) with computable entanglement measure(s).

Open problems:

- examples of structured nUPBs giving rise to maximal GESs,
- $(\mathrm{oUPB})^{\perp} = \mathrm{GES}$?,
- other ways to build GESs (work in progress),
- orthogonal unextendible biproduct basis leading to GES,
- analytical methods of computing entanglement measures,
- when are the SDP bounds exact?

References

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