Quantum operations
acessible by Lindblad semigroups

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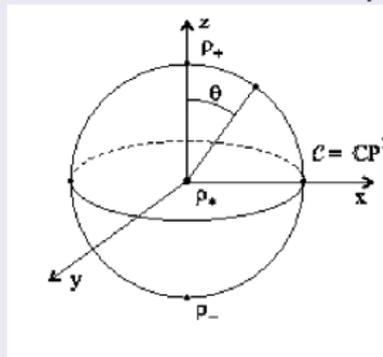**Symposium on Mathematical Physics, Toruń June 17, 2019**

Mixed Quantum States

Set $\mathcal{M}_N$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

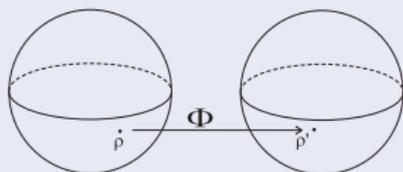example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary

The set $\mathcal{M}_N$ is compact and convex:

$\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i|$ where $a_i \geq 0$ and $\sum_i a_i = 1$.

The set $\mathcal{M}_N$ of mixed states has $N^2 - 1$ real dimensions, $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$.

Quantum maps: evolution in discrete time steps

Quantum operation: linear, completely positive trace preserving map

$\Phi : \mathcal{M}_2 \to \mathcal{M}_2$

positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, \ldots$

Enviromental form (interacting quantum system !)

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\,U^\dagger] \,.$$

where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

Kraus form

$\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \quad$ where the Kraus operators satisfy
$\sum_i A_i^\dagger A_i = \mathbb{1}$, which implies that the trace is preserved.

Classical probabilistic dynamics & Markov chains

Stochastic matrices

Classical states: N-point probability distribution, $\mathbf{p} = \{p_1, \ldots p_N\}$,
where $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$

Discrete dynamics: $p_i' = S_{ij} p_j$, where S is a **stochastic matrix** of size N and maps the simplex of classical states into itself, $S: \ \Delta_{N-1} \to \Delta_{N-1}$.

Frobenius–Perron theorem

Let S be a **stochastic matrix**:
   a) $S_{ij} \geq 0$ for $i, j = 1, \ldots, N$,
   b) $\sum_{i=1}^{N} S_{ij} = 1$ for all $j = 1, \ldots, N$.
Then
i) the spectrum $\{z_i\}_{i=1}^{N}$ of S belongs to the unit disk,
ii) the leading eigenvalue equals unity, $z_1 = 1$,
iii) the corresponding eigenstate $\mathbf{p}_{\mathrm{inv}}$ is invariant, $S\mathbf{p}_{\mathrm{inv}} = \mathbf{p}_{\mathrm{inv}}$.

Quantum stochastic maps (trace preserving, CP)

Superoperator $\Phi: \quad \mathcal{M}_N \to \mathcal{M}_N$

A *quantum operation* can be described by a matrix Φ of size N^2,

$$\rho' = \Phi\rho \qquad \text{or} \qquad \rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \, \rho_{n\nu} \, .$$

The superoperator Φ can be expressed in terms of the Kraus operators A_i,
$$\Phi = \sum_i A_i \otimes \bar{A}_i \, .$$

Dynamical Matrix D: Sudarshan et al. (1961)

obtained by *reshuffling* of a 4–index matrix Φ is Hermitian,

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{m\mu \\ n\nu}} \, , \quad \text{so} \quad \text{that} \quad D_\Phi = D_\Phi^\dagger =: \Phi^R \, .$$

Theorem of Choi (1975). A map Φ is **completely positive** (CP) if and only if the dynamical matrix D_Φ is **positive**, $D \geq 0$.

Spectral properties of a superoperator Φ

Quantum analogue of the Frobenious-Perron theorem

Let Φ represent a stochastic quantum map, i.e.
a') $\Phi^R \geq 0$; **(Choi theorem)**
b') $\mathrm{Tr}_A \Phi^R = \mathbb{1} \Leftrightarrow \sum_k \Phi_{\substack{kk \\ ij}} = \delta_{ij}$. **(trace preserving condition)**

Then
i') the spectrum $\{z_i\}_{i=1}^{N^2}$ of Φ belongs to the unit disk,
ii') the leading eigenvalue equals unity, $z_1 = 1$,
iii') the corresponding eigenstate (with N^2 components) forms a matrix ω of size N, which is positive, $\omega \geq 0$, normalized, $\mathrm{Tr}\omega = 1$, and is invariant under the action of the map, $\Phi(\omega) = \omega$.

Classical case

In the case of a **diagonal dynamical matrix**, $D_{ij} = d_i \delta_{ij}$ reshaping its diagonal $\{d_i\}$ of length N^2 one obtains a matrix of size N, where $S_{ij} = D_{\substack{ii \\ jj}}$,
of size N which is **stochastic** and recovers the standard F–P theorem.

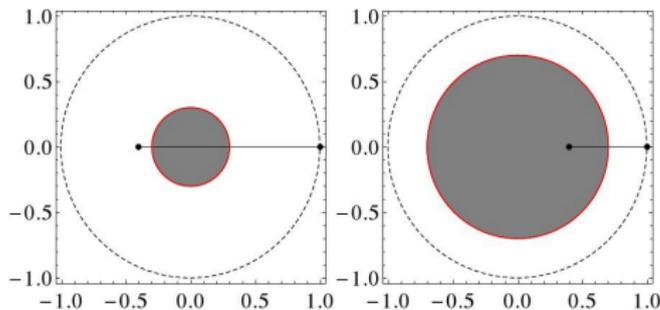# Spectra of One–Qubit Bistochastic Maps

Consider one–qubit bistochastic map (Pauli channel + unitary evolution) in the **Kraus form**,

$$\rho' = \Phi(\rho) = \sum_{i=1}^{4} A_i \rho A_i^\dagger$$

The evolution operator $\Phi = \sum_{i=1}^{4} A_i \otimes A_i^\dagger$ has spectrum with zero or two complex eigenvalues. In the latter case, $\{1, x, z, \bar{z}\}$ with real $x \in [-1, 1]$ the following bound holds

$$|z| \leq \frac{1+x}{2}$$

Rudnicki, Puchała, Życzkowski, *Quantum* **2**, (2018).

a) $x = -0.4$ b) $x = 0.4$

Exemplary constraints for the position of complex eigenvalues z and $\bar{z}$ of a bistochastic map Φ with real eigenvalue x (**black dot**).

Decoherence for quantum states and quantum maps

Quantum states $\rightarrow$ classical states = diagonal matrices

Decoherence of a state: $\rho \rightarrow \tilde{\rho} = \mathrm{diag}(\rho)$

Quantum maps $\rightarrow$ classical maps = stochastic matrices

Decoherence of a map: The **Choi matrix** becomes diagonal, $D \rightarrow \tilde{D} = \mathrm{diag}(D)$ so that the map $\Phi = D^R \rightarrow \tilde{D}^R \rightarrow S$ where for any Kraus decomposition defining $\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$ the corresponding **classical map** S is given by the **Hadamard product**,

$$S = \sum_i A_i \odot \bar{A}_i$$

If a **quantum map** Φ is trace preserving, $\sum_i A_i^\dagger A_i = \mathbb{1}$
   then the **classical map** S is **stochastic**, $\sum_j S_{ij} = 1$.
If additionally a **quantum map** Φ is unital, $\sum_i A_i A_i^\dagger = \mathbb{1}$
   then the **classical map** S is **bistochastic**, $\sum_j S_{ij} = \sum_i S_{ij} = 1$.

Unistochastic Maps

defined by an interaction of the ancilla of the same dimension initially in the maximally mixed state,

$$\rho' = \Phi_U(\rho) = \text{Tr}_{\text{env}}\Big[U(\rho \otimes \frac{1}{N} \mathbb{1}_N) \, U^\dagger \Big]$$

Unistochastic maps are unital, $\Phi_U(\mathbb{1}) = \mathbb{1}$, hence **bistochastic.**
    Is every bistochastic map **unistochastic ? No !**

One–qubit bistochastic maps = Pauli channels, $\rho' = \Phi_p(\rho)$

$$\Phi_p(\rho) \; = \; \sum_{i=0}^{3} p_i \, \sigma_i \rho \sigma_i,$$

where $\sum p_i = 1$ while $\sigma_0 = \mathbb{1}$ and remaining three σ_i denote **Pauli** matrices. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be the *damping vector* containing three axis of the ellipsoid - the image of the Bloch ball by a bistochastic map. Set $\mathcal{B}_2$ of one-qubit bistochastic maps = **regular tetrahedron** with corners at $\lambda = (1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$ corresponding to σ_i with $i = 0, \ldots, 3$.

One Qubit Unistochastic Maps

Unistochastic maps do satisfy following restrictions for the damping vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$

$$\lambda_1 \lambda_2 \leq \lambda_3, \qquad \lambda_2 \lambda_3 \leq \lambda_1, \qquad \lambda_3 \lambda_1 \leq \lambda_2. \quad (*)$$

The set $\mathcal{U}_2$ forms a (non-convex!) subset of the tetrahedron $\mathcal{B}_2$ of bistochastic maps, **Musz, Kuś, K. Ż.**, *Phys. Rev. A* **2012**

(a)
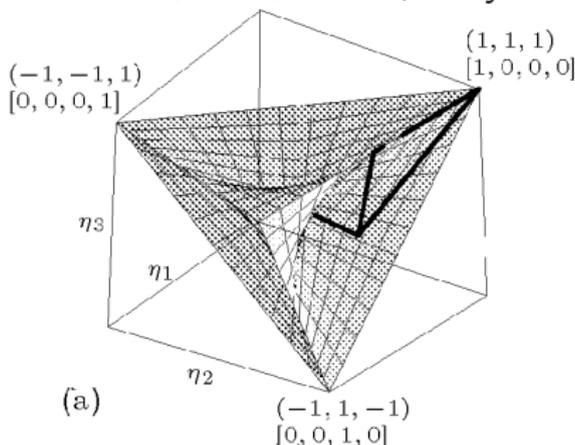
Example: The following Pauli channel $\Phi_p(\rho) = \sum_{i=1}^{3} \frac{1}{3} \sigma_i \rho \sigma_i$, of rank three is **not unistochastic**.

Lindblad dynamics in continuous time

a) closed system: **von Neumann equation**

$\frac{d\rho}{dt} = -i[H, \rho]$ leads to **unitary dynamics** (= reversible, rigid rotation):

$$\rho' = U\rho U^\dagger = e^{-iHt}\rho e^{iHt}$$

b) open system (of size N)- **interaction with environment**

described by **Gorrini - Lindblad - Kossakowski - Sudarshan** equation (1976) in terms of *jump operators L_j*,

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \sum_{j=1}^{N^2-1} \left(L_j\rho L_j^\dagger - \frac{1}{2}L_j^\dagger L_j\rho - \frac{1}{2}\rho L_j^\dagger L_j \right),$$

leads to **nonunitary** Lindblad dynamics (= irreversible contraction)

$$\rho(t) = e^{\mathcal{L}t}[\rho(0)] = \Lambda_t[\rho(0)]$$

and generates a **dynamical semigroup**, $\Lambda_s\Lambda_t = \Lambda_{t+s}$.

Lindblad operator

some matrix algebra:

A product of three matrices, $Y = ABC$, can also be written as

$Y = \Psi B$, where superoperator reads $\Psi = A \otimes C^T$

i) discrete time: superoperator Φ

corresponding to an operation in Kraus form $\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$ reads

$$\Phi = \sum_i A_i \otimes \bar{A}_i \, ,$$

ii) continuous time: Lindblad operator

reads

$$\mathcal{L} = \sum_j L_j \otimes \overline{L}_j - \frac{1}{2} \sum_j L_j^\dagger L_j \otimes \mathbb{I} - \frac{1}{2} \sum_j \mathbb{I} \otimes L_j^T \overline{L}_j.$$

For a given operation Φ corresponding to $\{A_j\}$ one can take $L_j = A_j$ to get $\mathcal{L} = \Phi - \mathbb{I}$, but in general this semigroup does not lead to the map Φ.

key **issue** :

Fixing a certain set of quantum operations of size N

find these **operations** Φ for which there exists a **quantum semigroup** $\Lambda_t = e^{t\mathcal{L}}$ such that $\Phi = \Lambda_1 = e^{\mathcal{L}}$.

In other words we look for a **logarithm** $\log \Phi = \mathcal{L}$ such that the entire trajectory $\Lambda_t = e^{t \log \Phi}$ gives a proper quantum channel.

related problems

a1) classical analogue: for which **stochastic** matrix $S \in \mathcal{S}_N$ there exists a semigroup: define $\mathcal{L}_c = \log S$ and check if the entire trajectory $\Lambda_t = e^{t\mathcal{L}_c}$ belongs to the set $\mathcal{S}_N$ of stochastic matrices of order N

a2) which **stochastic** matrix S has a **stochastic square root**,
$$S = S_2^2, \text{ where } S_2 \in \mathcal{S}_N.$$
a3) ... has a **stochastic** root of order k: $S = S_k^k$ where $S_k \in \mathcal{S}_N$

b1) which channel Φ is *divisible* so that there exist Ψ_1 and $\Psi_2 \neq \mathbb{I}$,
so that $\Phi = \Psi_2 \Psi_1$, **Wolf, Cirac** (2008)

b2) which stochastic transition matrix S is *divisible*, $S = S_2 S_1$.

Which Pauli channels belong to a semigroup ?

The set of **Pauli channels**, $\Phi_p(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i$, forms a regular tetrahedron - probability simplex $p \in \Delta_3$.

The superoperator $\Phi_p = \sum_{i=0}^{3} p_i \sigma_i \otimes \bar{\sigma}_i = \begin{pmatrix} p_0+p_3 & 0 & 0 & p_1+p_2 \\ 0 & p_0-p_3 & p_1-p_2 & 0 \\ 0 & p_1-p_2 & p_0-p_3 & 0 \\ p_1+p_2 & 0 & 0 & p_0+p_3 \end{pmatrix}$,

can be diagonalized, $E = O_4 \Phi_p O_4^{\top} = \mathrm{diag}(1, \lambda_1, \lambda_2, \lambda_3)$, where
$\lambda_1 = 1 - 2(p_2 + p_3)$, $\lambda_2 = 1 - 2(p_1 + p_3)$, $\lambda_3 = 1 - 2(p_1 + p_2)$.

Assuming that $\lambda_i > 0$ (which holds if $p_0 \geq \max\{p_1, p_2, p_3\}$) $\log E$ exists and leads to the **Lindblad generator** $\mathcal{L} = O_4^{\top} \log E \, O_4$
and the **dynamical semigroup**, $\Lambda_t = e^{\mathcal{L}t}$.

The corresponding **Choi matrix** reads

$$D_{\Lambda_t} = \Lambda_t^R = \frac{1}{2} \begin{pmatrix} 1+\lambda_3^t & 0 & 0 & \lambda_1^t+\lambda_2^t \\ 0 & 1-\lambda_3^t & \lambda_1^t-\lambda_2^t & 0 \\ 0 & \lambda_1^t-\lambda_2^t & 1-\lambda_3^t & 0 \\ \lambda_1^t+\lambda_2^t & 0 & 0 & 1+\lambda_3^t \end{pmatrix}.$$

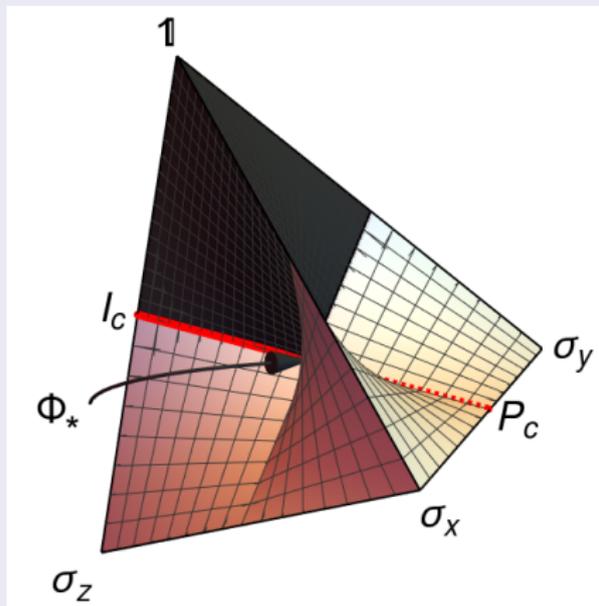Its positivity, $D_{\Lambda_t} \geq 0$ for any $t \geq 0$ implies that:
$$\lambda_3 \geq \lambda_1 \lambda_2, \quad \lambda_2 \geq \lambda_1 \lambda_3, \quad \lambda_1 \geq \lambda_2 \lambda_3. \quad (**)$$

solution: Pauli channel Φ_p belongs to a semigroup

if the probability vector p satisfies conditions equivalent to (**)
$$p_0 p_3 \geq p_1 p_2, \quad p_0 p_2 \geq p_1 p_3, \quad p_0 p_1 \geq p_2 p_3.$$

Geometric picture in the tetrahedron

The boundary, consisting of **product vectors**, e.g. $p_0 p_3 = p_1 p_2$, formes the hyperboloid (ruled surface), which determines the set of **unistochastic channels**.

Since we assumed that the identity component dominates, $p_0 \geq \max\{p_1, p_2, p_3\}$, the **set** $\mathcal{S}_2$ of maps belonging to a **semigroup** forms the **black** quarter of the set of unistochastic channels !

classical semigroup $= [l_c, \Phi_*]$.

Continuous dynamics & semigroup

Not all trajectories of the form e^{Gt} leads to a semigroup: the generator G has to impose that the map $\Phi = e^{Gt}$ is a quantum channel for any $t \geq 0$.

Geometric picture inside the tetrahedron
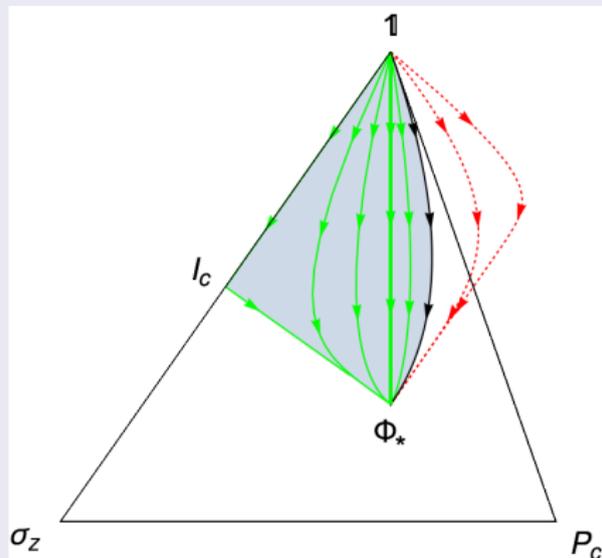
Section of the simplex of Pauli channels, $P_c = (\Phi_x + \Phi_y)/2 = \mathrm{diag}(0, 1, 1, 0)$. Gray **set** $\mathcal{S}$ of channels belonging to a semigroup is bounded by the product relation, $p_0 p_3 = p_1 p_2$.

solid arrowed lines – valid semigroups leading from $\mathbb{1}$ to Φ_*,

dashed lines do not correspond to a semigroup, as they leaves the simplex of CP maps for some $t > 0$.

Results on the set $\mathcal{S}_2$ of Pauli maps from a semigroup

- **Set $\mathcal{S}_2$** is a **non-convex** subset of the probability simplex Δ_3
- Boundaries of $\mathcal{S}_2$ are are formed by hyperboloids,
  corresponding to **product states**, $p = (1, 1 - a) \times (b, 1 - b)$.
- Let $\Lambda_s^z = \exp(\mathcal{L}_z s)$ denote the semigroup associated to $L_1 = \sigma_z$ and $\Lambda_t^x = \exp(\mathcal{L}_x t)$ to σ_x. Then the **double** composition, $\Lambda_s^z \Lambda_t^x$ describes a point at the boundary $\partial \mathcal{S}_2$.
- Any point from the **interior** of $\mathcal{S}_2$ can be acessed by a **triple** composition $\Lambda_s^z \Lambda_t^x \Lambda_u^y$.
- Set $\mathcal{S}_2$ is a **star-shaped** with respect to any point from the interval $[\mathbb{1}, \Phi_*]$, where Φ_* stands for the completely depolarising channel, $\Phi_*(\rho) = \mathbb{1}/2$.
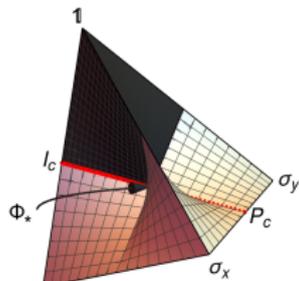
Z. Puchała, Ł. Rudnicki, K. Ż. *Phys. Lett. A* 2019

generalization of Pauli channels:

Consider the set of **mixed unitary** channels $\Psi_p(\rho) = \sum_{i=0}^{N^2-1} p_i U_i \rho U_i$, where unitary matrices U_i of order N form an **orthogonal basis** in the space of matrices of size N, while vector p belongs to probability simplex, $p \in \Delta_{N^2-1} \subset \mathbb{R}^{N^2-1}$.

We work in the **Heisenberg-Weyl** basis of **unitary** matrices of order N, $U_\mu = U_{kl} := X^k Z^l$, where $\mu = 0, \ldots, N^2 - 1$, $k, l = 0, \ldots, N - 1$, while $X|i\rangle = |i \oplus 1\rangle$ and $Z = \mathrm{diag}\{1, \omega, \omega^2, \ldots, \omega^{N-1}\}$ with $\omega = e^{2\pi i/N}$. Then
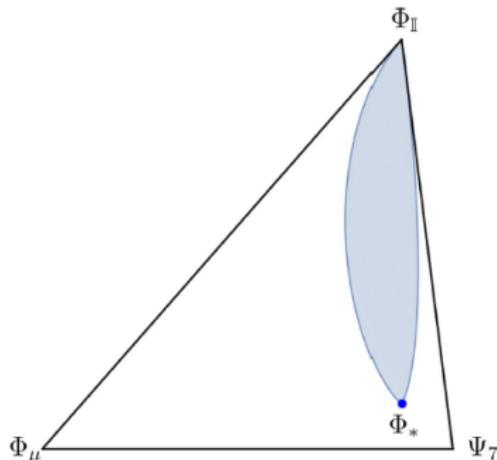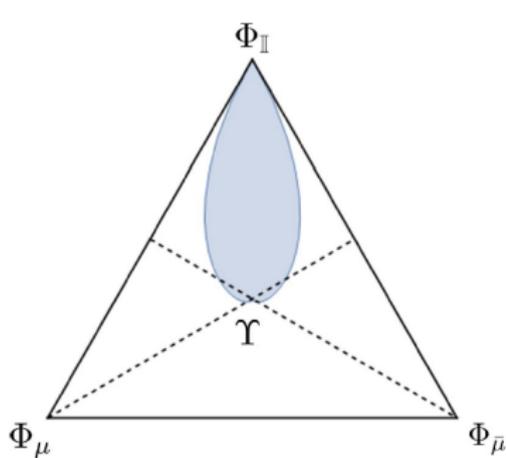
- **Lindblad generators** $\mathcal{L}_\mu = U_\mu \otimes \bar{U}_\mu - \mathbb{1}$ are commutative, so the corresponding semigroups **do commute**, $e^{t\mathcal{L}_\mu} e^{s\mathcal{L}_\nu} = e^{s\mathcal{L}_\nu} e^{t\mathcal{L}_\mu}$.
- Any composition of $N^2 - 2$ such semigroups belongs to the **boundary** of the set $\mathcal{S}_N \subset \Delta_{N^2-1}$ of channels accessible by a semigroup.
- **Set** $\mathcal{S}_N$ is a **star-shaped** with respect to the completely depolarising channel, $\Phi_*(\rho) = \mathbb{1}/N$.

Example: $N = 3$, *Fereshte Shahbeigi 2019*

$8D$ simplex of **mixed unitary** channels $\Psi_p(\rho) = \sum_{i=0}^{8} p_i U_i \rho U_i$, where $p \in \Delta_8$ and 9 unitary matrices U_i form an **orthonormal basis** in $U(3)$. Cross-section of the simplex Δ_8 and the **set $\mathcal{S}_3$ of accessible maps**:
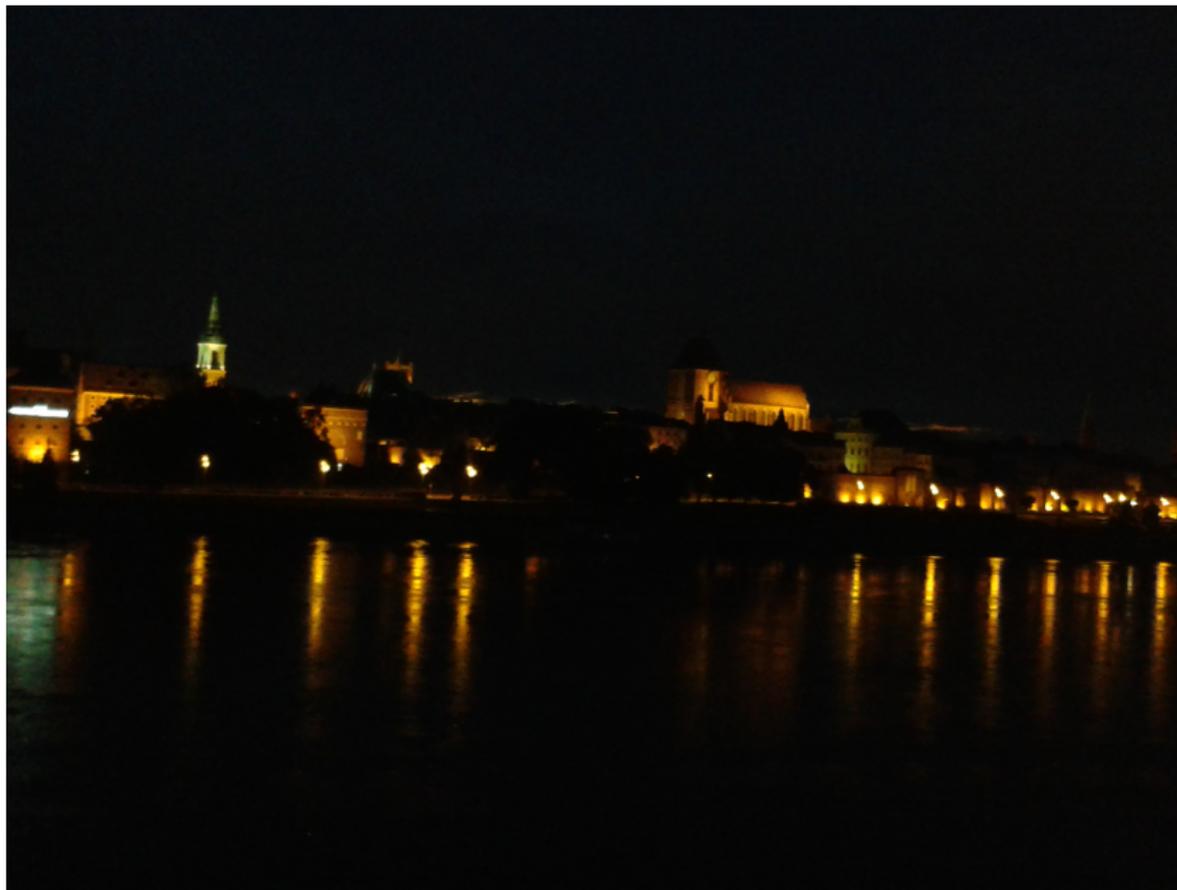
(non-convex, but **star shaped** !)

where $\Phi_\mu = U_\mu \otimes \bar{U}_\mu$, $\Phi_{\bar{\mu}} = U_\mu^\dagger \otimes \bar{U}_\mu^T$, $\qquad \Phi_7 = \frac{1}{7} \sum_{\beta=1}^{7} U_\beta \otimes \bar{U}_\beta$,

$\Upsilon = (\Phi_{\mathbb{I}} + \Phi_\mu + \Phi_{\bar{\mu}})/3$, and $\Phi_* = \frac{1}{9} \sum_{\mu=0}^{8} U_\mu \otimes \bar{U}_\mu$

Concluding   Remarks

- Stroboscopic quantum evolution (discrete time) is described by **quantum operations**: **completely positive**, trace preserving map Φ.

- Spectral properties of a superoperator Φ determine long time dynamics: spectral gap assures exponential convergence to invariant state.

- Quantum evolution in continuous time is governed by the **GKLS equation** and determined by a **Lindblad generator**, $\rho(t) = e^{\mathcal{L}t}[\rho(0)]$.

- Not every quantum operation is accesed by a dynamical semigroup. We characterized the set $\mathcal{S}_2$ of **Pauli channels** (acting on a qubit) accessible by a semigroup and described its geometry.
  complementary work: **Davalos, Ziman, Pineda**, *QUANTUM* 2019

- For the class of **mixed unitary** channels acting on a quNit we described the set $\mathcal{S}_N$ of channels accessible by a semigroup for any given dimension $N \geq 2$.