

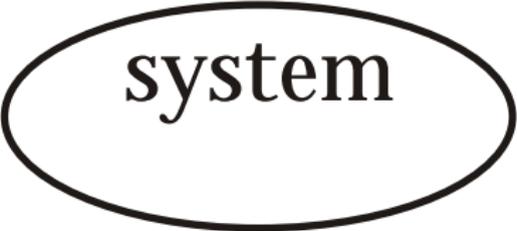
# Complete positivity and positivity of quantum dynamical maps under time deformations

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based on joint work with Dariusz Chruściński

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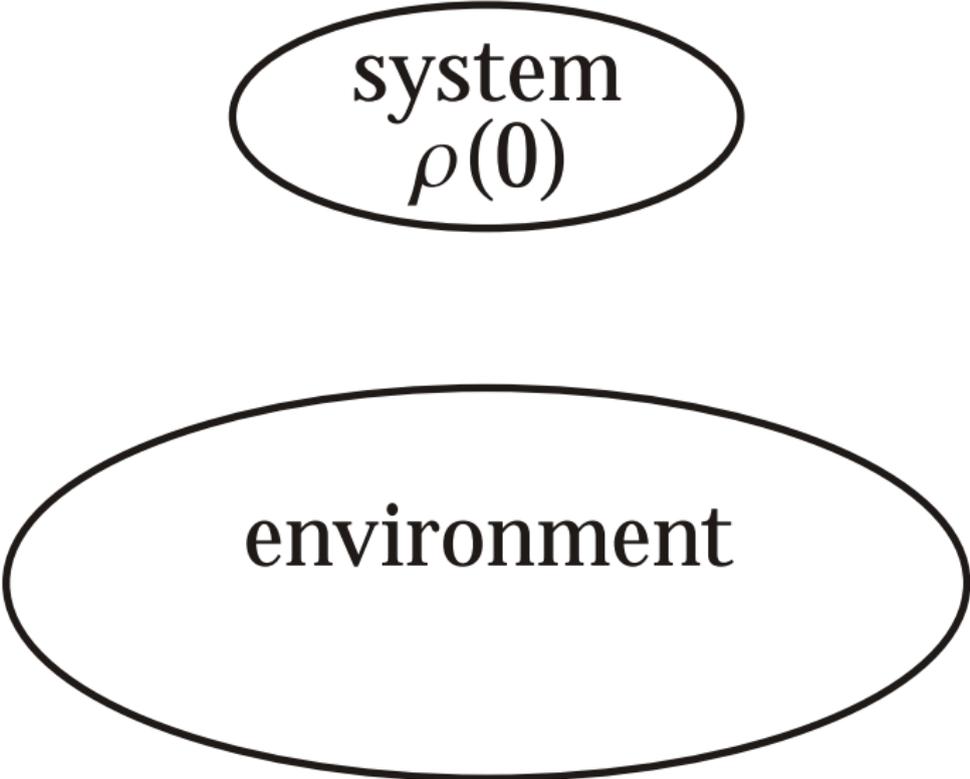


system

# Quantum dynamical map

system  
 $\rho(0)$

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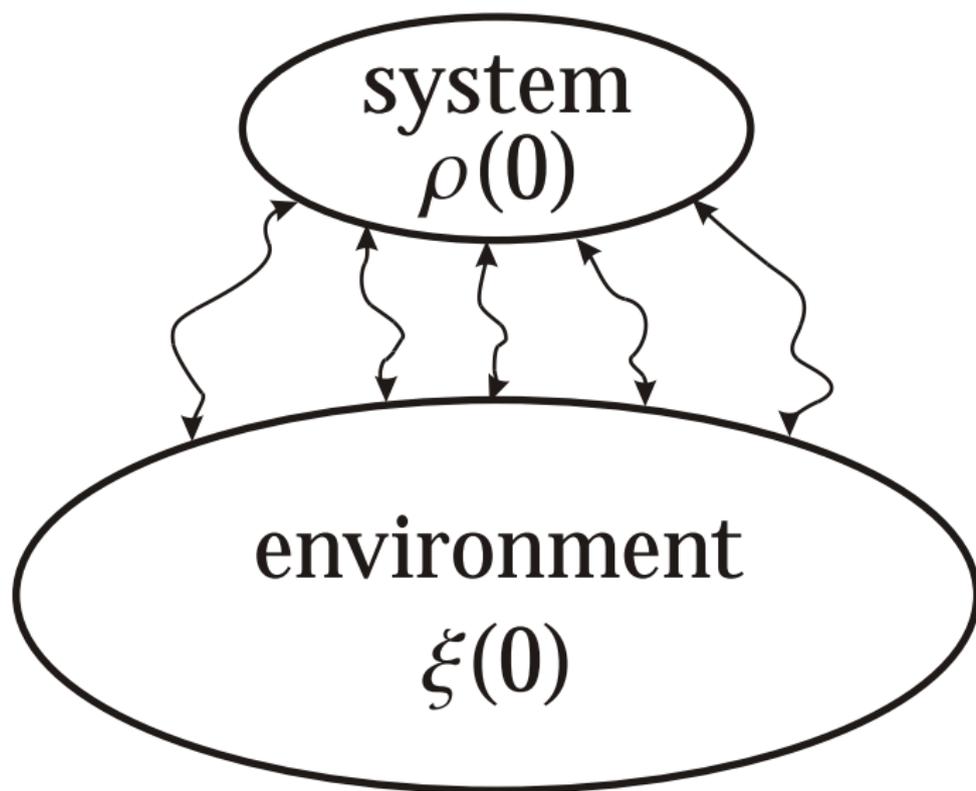
environment

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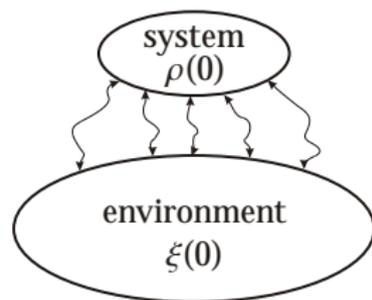
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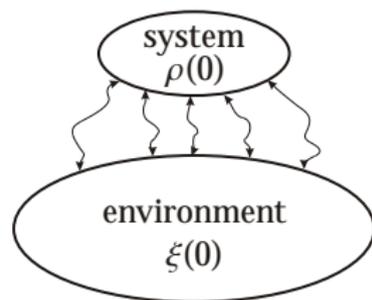


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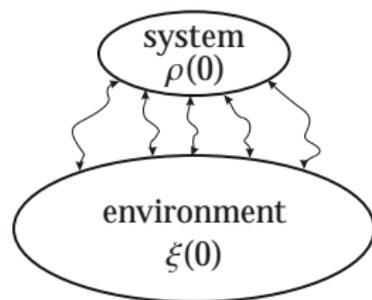
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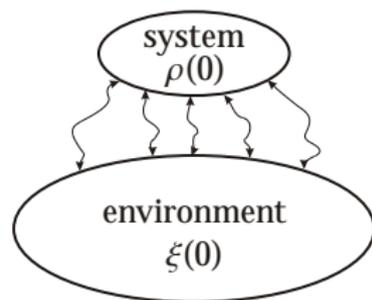


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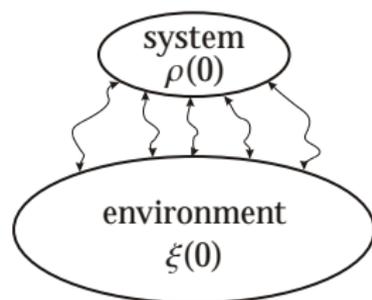
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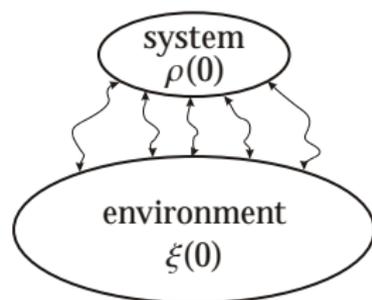
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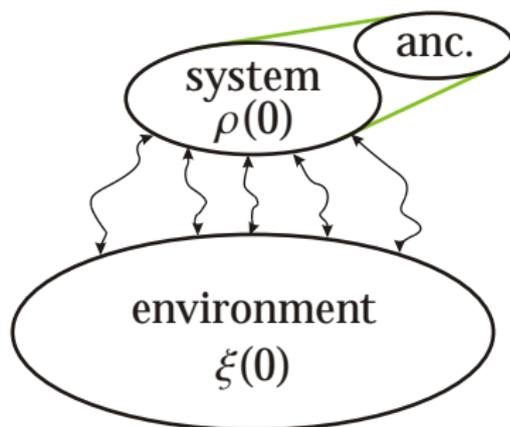
$$\Phi(0) = \text{Id}$$

# Properties of $\Phi(t)$

- 1 Linearity:  $\Phi(t)[\alpha X + \beta Y] = \alpha\Phi(t)[X] + \beta\Phi(t)[Y]$
- 2 Trace preservation:  $\text{tr}[\Phi(t)[X]] = \text{tr}[X]$
- 3 Complete positivity (CP):  $\Phi(t) \otimes \text{Id}_k$  is positive for all  $k = 1, 2, \dots$

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# GKSL master equation

$$\frac{d\rho(t)}{dt} = L[\rho(t)]$$

$$L[X] = -i[H_s, X] + \sum_k \gamma_k \left( A_k X A_k^\dagger - \frac{1}{2} \{X, A_k^\dagger A_k\} \right)$$

$$\Phi(t) = e^{Lt}$$

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The change  $t \rightarrow \alpha t$  stretches (shrinks) the timeline.

By time deformation we understand the transformation

$$\tau(t) = \int_0^t \alpha(t') dt',$$

where  $\alpha(t)$  is a positive real function quantifying the local time stretching:

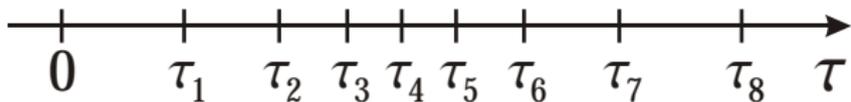
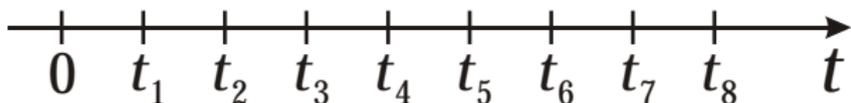
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$$\frac{d\rho(t)}{dt} = \alpha(t)L[\rho(t)]$$

$$\alpha(t) \geq 0$$

$$\tilde{\Phi}(t) = \exp\left(L \int_0^t \alpha(t') dt'\right)$$

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$\tilde{\Phi}(t)$  is not a semigroup in general, but it is still CP

## Time-convolutionless master equation

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If  $V(t_2, t_1)$  is CP for all  $t_2 > t_1 \geq 0$ , then the process  $\Phi(t)$  is called CP divisible.

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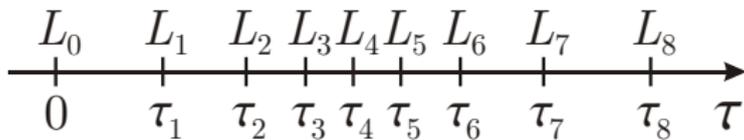
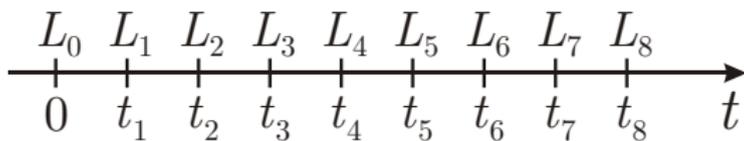
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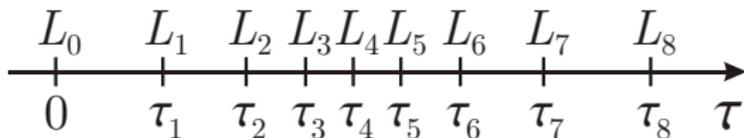
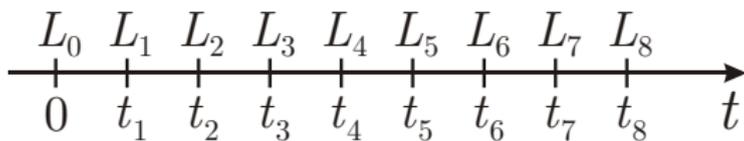
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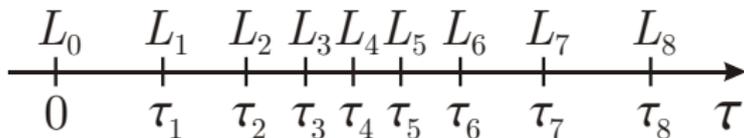
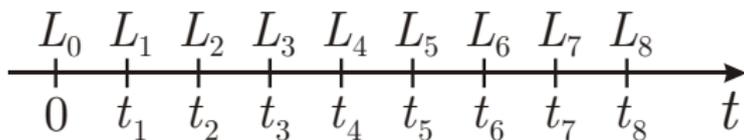
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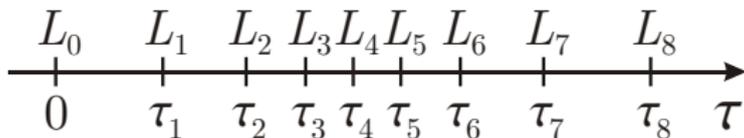
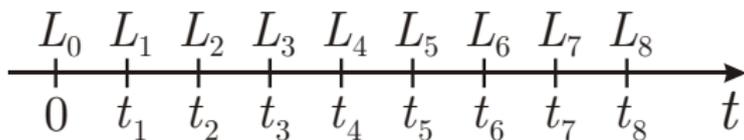
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A modified (**non-equivalent**) master equation

$$\frac{d\tilde{\varrho}(\tau(t))}{d\tau(t)} = L(t)[\tilde{\varrho}(\tau(t))]$$



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in terms of the original time  $t$  reads

$$\frac{d\varrho(t)}{dt} = \frac{d\tau}{dt} \frac{d\tilde{\varrho}}{d\tau} = \alpha(t)L(t)[\varrho(t)]$$

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CP  $\longrightarrow$  not CP in general

**Example:** CPT  $\Phi(t) \longrightarrow$  not CPT  $\tilde{\Phi}(t)$

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**Example:** CPT  $\Phi(t) \longrightarrow$  not CPT  $\tilde{\Phi}(t)$

Consider a qubit map  $\Phi(t) : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2)$  with generator<sup>1</sup>

$$L(t)[\rho] = \frac{1}{2} \sum_{i=1}^3 \gamma_i(t) (\sigma_i \rho \sigma_i - \rho),$$

where  $\sigma_1, \sigma_2, \sigma_3$  is the conventional set of Pauli operators,  
 $\gamma_1(t) = \gamma_2(t) = 1$ , and  $\gamma_3(t) = -\tanh(t)$ .

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$$\Phi(t)[X] = \frac{1}{2} \left( \text{tr}[X]I + \sum_{k=1}^3 \lambda_k(t) \text{tr}[\sigma_k X] \sigma_k \right)$$

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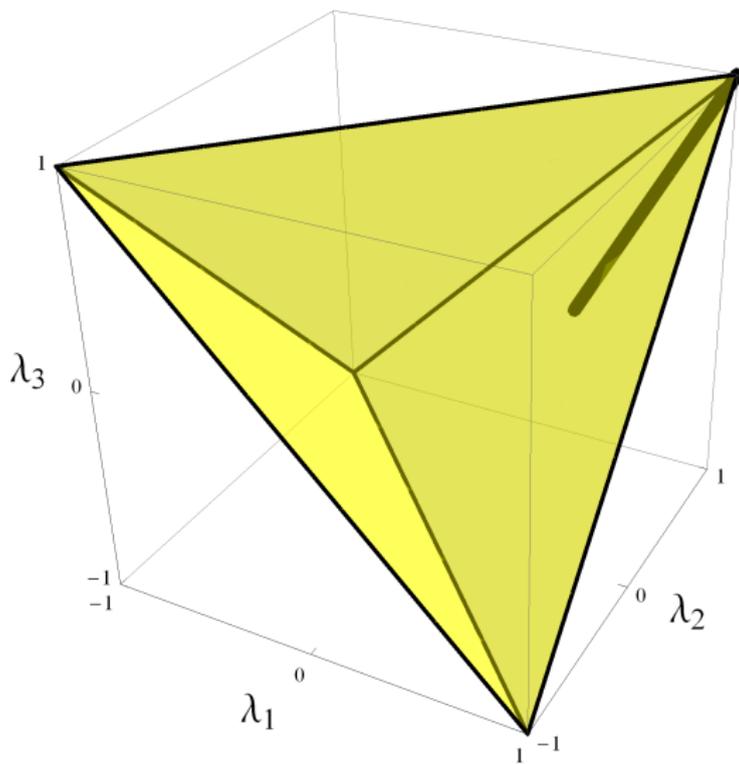
$$\lambda_1(t) = \lambda_2(t) = \frac{1}{2} (1 + e^{-2t}), \quad \lambda_3(t) = e^{-2t}$$

$\Phi(t)$  is CP because  $\lambda_1(t), \lambda_2(t), \lambda_3(t)$  satisfy the Fujiwara-Algoet condition  $|\lambda_1(t) + \lambda_2(t)| \leq 1 \pm \lambda_3(t)$ .

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Constant time deformation

$$\frac{d\rho(t)}{dt} = \alpha L(t)[\rho(t)]$$

results in the map  $\tilde{\Phi}(t)$  with parameters

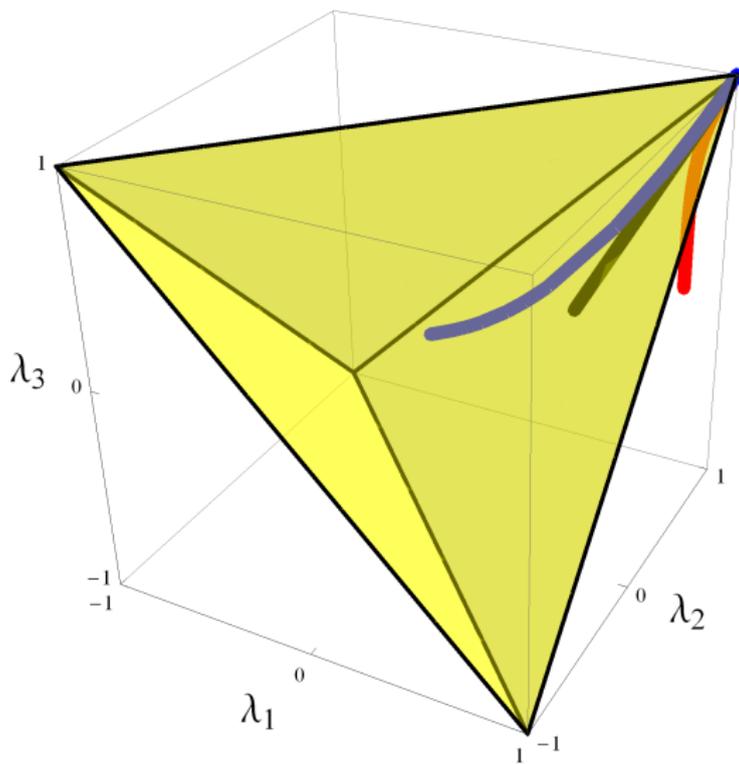
$$\tilde{\lambda}_1(t) = \tilde{\lambda}_2(t) = e^{-\alpha t} \cosh^\alpha(t), \quad \tilde{\lambda}_3(t) = e^{-2\alpha t}$$

$\tilde{\Phi}(t)$  is CP (satisfies the Fujiwara-Algoet condition  $|\tilde{\lambda}_1(t) + \tilde{\lambda}_2(t)| \leq 1 \pm \tilde{\lambda}_3(t)$ ) if and only if  $\alpha \geq 1^2$ . Thus, if  $0 < \alpha < 1$ , then the map  $\tilde{\Phi}(t)$  is not CP.

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<sup>2</sup>F. Benatti, D. Chruściński, S. F., Phys. Rev. A **95**, 012112 (2017).

$$\Phi[X] = \frac{1}{2} \left( \text{tr}[X]I + \sum_{k=1}^3 \lambda_k \text{tr}[\sigma_k X] \sigma_k \right)$$



A time deformation

$$\frac{d\rho(t)}{dt} = \alpha(t)L(t)[\rho(t)]$$

for the above generator results in CP map if and only if

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In particular, if  $\alpha(t) = \begin{cases} 1, & \text{if } 0 \leq t < t_0, \\ a, & \text{if } t \geq t_0, \end{cases}$  the latter condition is violated when  $0 < a < 1$  regardless to  $t_0$ .

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Reason = ?

Reason = eternal CP indivisibility, i.e.,  $V(t_2, t_1)$  is not CP for any  $t_2 > t_1 > 0$ .

### Theorem <sup>(3)</sup>

*Master equation  $\frac{d\rho(t)}{dt} = L(t)[\rho(t)]$  with nonsingular generator  $L(t)$  describes CP divisible dynamics if and only if the deformed map remains CP under any time deformation  $\frac{d\rho(t)}{dt} = \alpha(t)L(t)[\rho(t)]$ .*

## Theorem <sup>(3)</sup>

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### Necessity

Suppose the process  $\Phi(t)$  is CP divisible and  $L(t)$  is not singular, then  $L(t)$  has GKSL form

$$L(t)[\rho] = -i[H(t), \rho] + \sum_k \gamma_k(t) \left( A_k(t)\rho A_k^\dagger(t) - \frac{1}{2}\{A_k^\dagger(t)A_k(t), \rho\} \right),$$

where all the rates  $\gamma_k(t) \geq 0$ . Multiplication of the Hamiltonian  $H(t)$  by  $\alpha(t)$  preserves its Hermiticity, and  $\alpha(t)\gamma_k(t) \geq 0$ , so  $\alpha(t)L(t)$  is still a valid generator of the dynamical map.

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<sup>3</sup>S. F. and D. Chruściński, Phys. Rev. A **98**, 022123 (2018). 

## Sufficiency

Let  $\alpha(t) = \begin{cases} +0, & \text{if } 0 \leq t < t_1, \\ 1, & \text{if } t \geq t_1, \end{cases}$  then the deformed map

$$\tilde{\Phi}(t) = T_{\leftarrow} \exp \left( \int_0^t \alpha(t') L(t') dt' \right) = \begin{cases} \text{Id}, & \text{if } 0 \leq t < t_1, \\ V(t, t_1), & \text{if } t \geq t_1. \end{cases}$$

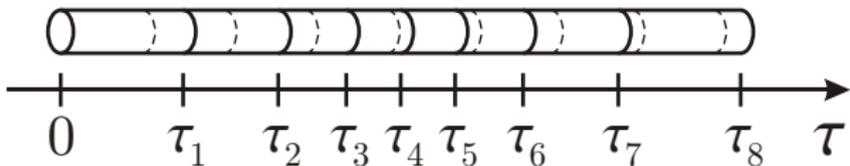
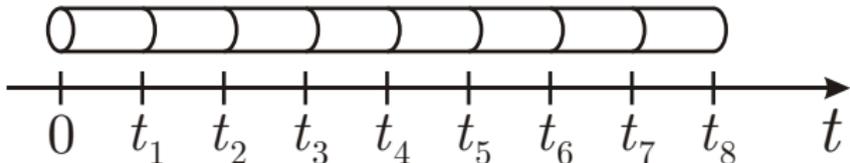
Therefore, if the deformed map  $\tilde{\Phi}(t)$  remains CP under any deformation, then  $V(t, t_1)$  is CP too for all  $t > t_1$ , i.e., the original map  $\Phi(t)$  is CP divisible.

## Sufficiency

Let  $\alpha(t) = \begin{cases} +0, & \text{if } 0 \leq t < t_1, \\ 1, & \text{if } t \geq t_1, \end{cases}$  then the deformed map

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**Remark.** Nonsingularity of generator  $L(t)$  is needed to guarantee invertibility of  $\Phi(t)$ . If  $\Phi(t)$  is not invertible, then CP divisibility of  $\Phi(t)$  does not require positivity of rates  $\gamma_k(t)$ <sup>45</sup>. However, the generator is not uniquely defined by the dynamical map  $\Phi(t)$  in this case. In particular, if the process is CP divisible, then there exists a corresponding (possibly singular) time-local generator with non-negative rates. Theorem holds true for such generators too.

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<sup>4</sup>E. Andersson, J. D. Cresser, and M. J. W. Hall, J. Mod. Opt. **54**, 1695 (2007).

<sup>5</sup>D. Chruściński, Á. Rivas, and E. Størmer, Phys. Rev. Lett. **121**, 080407 (2018).

## Convolution master equation

$$\frac{d\rho(t)}{dt} = \int_0^t \mathcal{K}(t, t')[\rho(t')] dt'$$

is defined through a memory kernel map  $\mathcal{K}(t, t') : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ .

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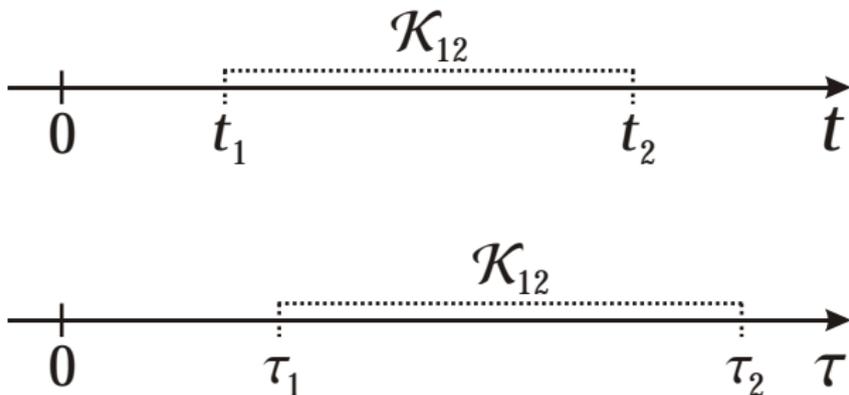
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Let us assume that the same kernel  $\mathcal{K}(t, t')$  is applied at deformed time moments  $\tau(t)$  and  $\tau(t')$ :

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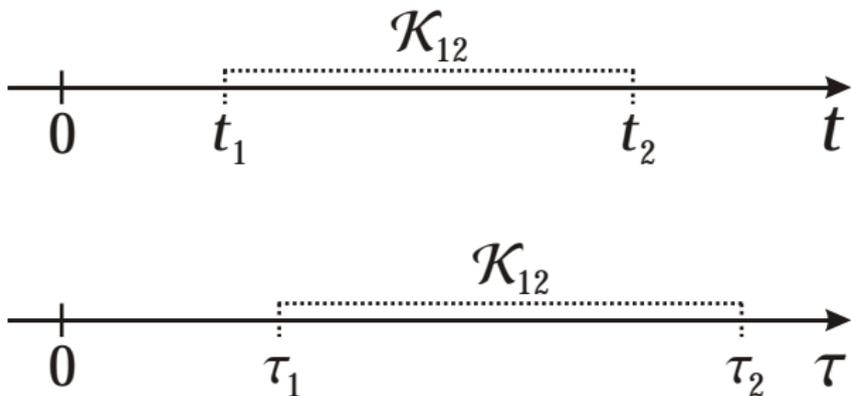
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We obtain a time deformation

$$\frac{d\tilde{\rho}(\tau(t))}{d\tau(t)} = \int_0^{\tau(t)} \mathcal{K}(t, t')[\tilde{\rho}(\tau(t'))]d\tau(t')$$

In terms of the original time  $t$

$$\frac{d\rho(t)}{dt} = \int_0^t \alpha(t)\alpha(t')\mathcal{K}(t,t')[\rho(t')]dt'$$

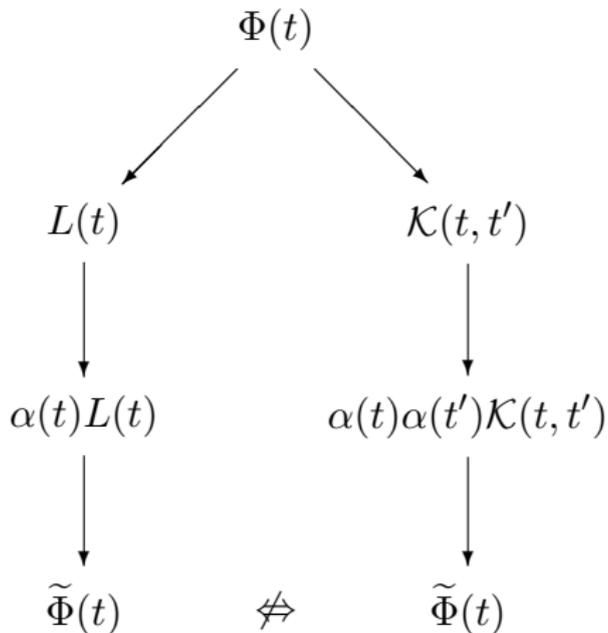
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Note that



Consider CP indivisible Pauli dynamical map  $\Phi(t)$  as in Example before, but now in terms of the *convolution* equation<sup>6</sup>

$$\frac{d\Phi}{dt} = \int_0^t K(t-t')\Phi(t')dt'$$

The uniform time deformation  $K(t) \rightarrow \alpha^2 K(t)$  leads to the deformed eigenvalues

$$\tilde{\lambda}_1(t) = \tilde{\lambda}_2(t) = \frac{1 + \alpha^2 e^{-(1+\alpha^2)t}}{1 + \alpha^2}, \quad \tilde{\lambda}_3(t) = e^{-2\alpha^2 t}$$

The deformed map  $\tilde{\Phi}(t)$  is never CP for  $t > 0$  and  $0 < \alpha < 1$  since the corresponding set of eigenvalues violates the Fujiwara-Algoet conditions for complete positivity.

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<sup>6</sup>N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, Sci. Rep. **7**, 6379 (2017).

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Denote

$$(A * B)(t) = \int_0^t A(t-t')B(t')dt'$$

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Denote

$$(A * B)(t) = \int_0^t A(t-t')B(t')dt'$$

$$\frac{d}{dt}\Phi(t) = (K * \Phi)(t)$$

## Laplace transform

$$\Phi_s = \int_0^{\infty} \Phi(t)e^{-st} dt$$

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$$\begin{aligned} \tilde{\Phi}_s &= (s\text{Id} - K_s - (\alpha^2 - 1)K_s)^{-1} = (\Phi_s^{-1} - (\alpha^2 - 1)K_s)^{-1} \\ &= (\text{Id} - (\alpha^2 - 1)K_s \Phi_s)^{-1} \Phi_s = \sum_{n=0}^{\infty} (\alpha^2 - 1)^n (K_s \Phi_s)^n \Phi_s \end{aligned}$$

In time domain:

$$\tilde{\Phi}(t) = \Phi(t) + (\alpha^2 - 1) \left( \frac{d\Phi}{dt} * \Phi \right)(t) + \dots + (\alpha^2 - 1)^n \underbrace{\left( \frac{d\Phi}{dt} * \dots * \frac{d\Phi}{dt} * \Phi \right)}_{n \text{ times}}(t) + \dots$$

Analogously:

$$\frac{d\tilde{\Phi}}{dt} = \alpha^2 \frac{d\Phi}{dt} + \alpha^2 (\alpha^2 - 1) \frac{d\Phi}{dt} * \frac{d\Phi}{dt} + \dots + \alpha^2 (\alpha^2 - 1)^n \underbrace{\frac{d\Phi}{dt} * \dots * \frac{d\Phi}{dt}}_{n+1 \text{ times}} + \dots$$

Commutative maps:  $\Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$  for all  $t_1, t_2 \geq 0$ .

$\Phi(t)$  is Hermitian if  $\Phi(t)$  coincides with its dual map  $\Phi^\dagger(t)$

P divisibility of Hermitian commutative map  $\implies \frac{d\lambda_k(t)}{dt} \leq 0^7$

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<sup>7</sup>D. Chruściński, C. Macchiavello, and S. Maniscalco, Phys. Rev. Lett. **118**, 080404 (2017).

## Proposition (<sup>8</sup>)

*Suppose the commutative Hermitian dynamical map  $\Phi(t)$  is given by a memory kernel  $K(t)$ . If the uniform time deformation  $K(t) \rightarrow \alpha^2 K(t)$  with  $0 < \alpha < 1$  and  $(1 - \alpha^2) \left\| \left( \frac{d\Phi}{dt} \right)_s \right\|_{1 \rightarrow 1} < 1$  results in a map  $\tilde{\Phi}(t)$  such that  $\frac{d\tilde{\Phi}}{dt}$  has at least one positive eigenvalue at some time  $t$ , then the original map  $\Phi(t)$  is not  $P$  divisible.*

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<sup>8</sup>S. F. and D. Chruściński, Phys. Rev. A **98**, 022123 (2018).

Pauli qubit map:

$$\Phi(t)[\rho] = \frac{1}{2} \left( \text{tr}[\rho]I + \sum_{k=1}^3 \lambda_k(t) \text{tr}[\sigma_k \rho] \sigma_k \right)$$

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### Corollary

*Suppose the Pauli map  $\Phi(t)$  is given by a memory kernel  $K(t)$ . If the uniform time deformation  $K(t) \rightarrow \alpha^2 K(t)$  with  $0 < \alpha < 1$  and  $(1 - \alpha^2)(1 - s \int_0^\infty \lambda_k(t) e^{-st} dt) < 1$  results in a map  $\tilde{\Phi}(t)$ , which is not positive, then the original map  $\Phi(t)$  is not P divisible.*

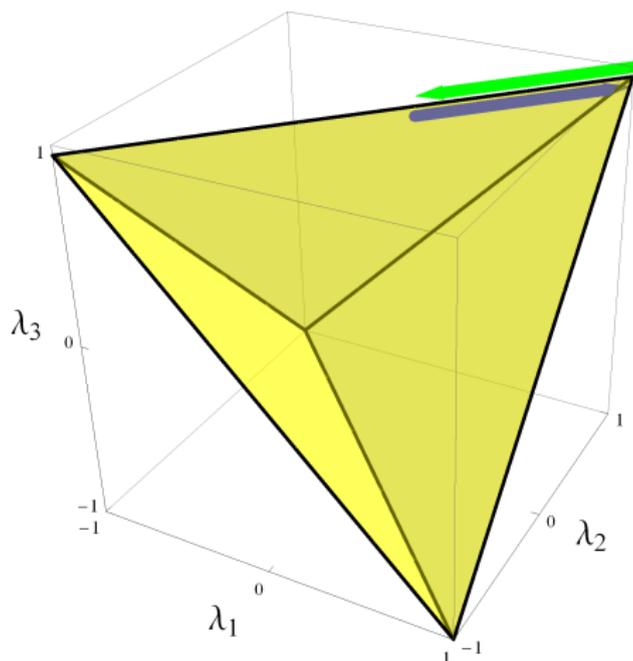
## Proof.

Condition  $(1 - \alpha^2)(1 - s \int_0^\infty \lambda_k(t)e^{-st} dt) < 1$  guarantees the validity of series expansion. Let  $\tilde{\Phi}(t)$  be non-positive. Since the Pauli map  $\tilde{\Phi}(t)$  is positive if and only if  $-1 \leq \tilde{\lambda}_k(t) \leq 1$ , either  $\tilde{\lambda}_k(t) > 1$  or  $\tilde{\lambda}_k(t) < -1$  for some time  $t$ . Note that at the initial moment  $\tilde{\lambda}_k(0) = 1$ .

Suppose  $\tilde{\lambda}_k(t) > 1$ , then there exists a time moment  $t_0 \in (0, t)$  such that  $\frac{d\tilde{\lambda}_k(t)}{dt}(t_0) > 0$ . By Proposition,  $\Phi(t)$  is not P divisible.

Suppose  $\tilde{\lambda}_k(t) < -1$ , let us show that  $\lambda_k(t) \not\geq 0$ . Using the expansion for  $\tilde{\Phi}(t)$  one finds that if  $\lambda_k(t) \geq 0$  and  $\frac{d\lambda_k}{dt} \leq 0$ , then a time deformation with  $0 < \alpha < 1$  guarantees  $\tilde{\lambda}_k(t) \geq 0$ . As we consider the case  $\tilde{\lambda}_k(t) < -1$ , this contradiction proves that  $\lambda_k(t) \not\geq 0$ . As a result, the original Pauli map  $\Phi(t)$  is not P divisible. □

**Example.**  $\Phi(t)[\rho] = \frac{1}{2} \left( \text{tr}[\rho]I + \sum_{k=1}^3 \lambda_k(t) \text{tr}[\sigma_k \rho] \sigma_k \right)$ ,  
 $\lambda_1(t) = \lambda_2(t) = 1 - 2\Gamma t e^{-\Gamma t}$  and  $\lambda_3(t) = 1$ .



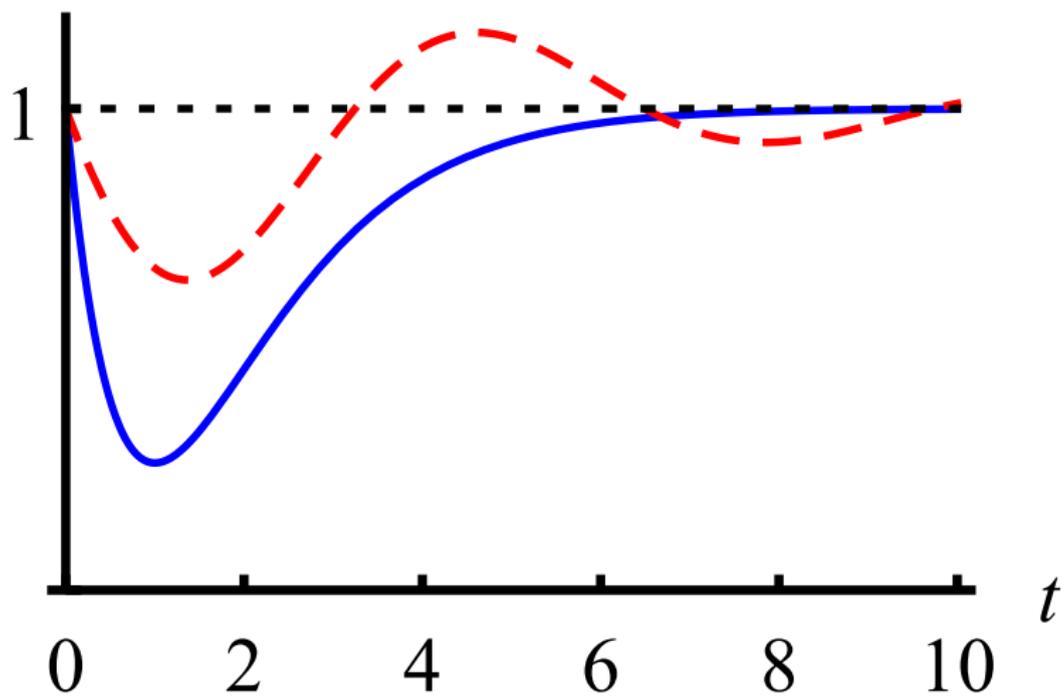
Such a map is a solution of the convolution master equation

$$\frac{d\rho(t)}{dt} = \int_0^t (\Gamma\delta(t-t') - \Gamma^2 \sin \Gamma(t-t')) \times [\sigma_z \rho(t') \sigma_z - \rho(t')] dt'.$$

Condition  $(1 - \alpha^2)(1 - s \int_0^\infty \lambda_k(t) e^{-st} dt) < 1$  is fulfilled automatically if  $0 < \alpha^2 < 1$ . Uniform deformation of the memory kernel  $K(t-t') \rightarrow \alpha^2 K(t-t')$  results in the deformed Pauli map  $\tilde{\Phi}(t)$  with

$$\tilde{\lambda}_1(t) = \tilde{\lambda}_2(t) = 1 - 2\alpha^2 e^{-\alpha^2 \Gamma t} \frac{\sin(\sqrt{1 - \alpha^4} \Gamma t)}{\sqrt{1 - \alpha^4}}$$

and  $\tilde{\lambda}_3(t) = 1$ .

$\lambda, \tilde{\lambda}$ 

### Example:

Consider a qubit evolution, where the rescaling of the memory kernel is *compatible with P divisibility* of the dynamical map.

Following <sup>9</sup>, consider a memory kernel

$$K(t)[\varrho] = \frac{1}{2} \sum_{k=1}^3 \varkappa_k(t) \sigma_k \text{tr}[\sigma_k \varrho],$$

where the time-dependent eigenvalues  $\varkappa_k(t)$  are defined (in the Laplace transform domain) via

$$(\varkappa_k)_s = \frac{-s f_s}{a_k - f_s}.$$

Positive numbers  $\{a_1, a_2, a_3\}$  satisfy triangle inequality  $a_i^{-1} + a_j^{-1} \geq a_k^{-1}$  for all permutations of  $\{i, j, k\}$ , and the real function  $f(t)$  satisfies  $f(t) \geq 0$  together with

$f_0 = \int_0^\infty f(t) dt \leq 4 (a_1^{-1} + a_2^{-1} + a_3^{-1})^{-1}$ . The corresponding eigenvalues of  $\Phi(t)$  are given by  $\lambda_k(t) = 1 - a_k^{-1} \int_0^t f(t') dt'$ .

<sup>9</sup>F. A. Wudarski, P. Należyty, G. Sarbicki, and D. Chruściński, Phys. Rev. A **91**, 042105 (2015).

The dynamical map  $\Phi(t)$  is known to be P divisible if additionally  $f(t)$  satisfies the requirement

$$f_0 = \int_0^\infty f(t)dt \leq a_{\min},$$

where  $a_{\min} = \min\{a_1, a_2, a_3\}$ . Suppose condition (36) is fulfilled, then  $f_s \leq a_{\min}$  for all  $s \geq 0$ . The deformed eigenvalue

$$(\tilde{\lambda}_k)_s = \frac{1}{s \left(1 + \frac{\alpha^2 f_s}{a_k - f_s}\right)} = \frac{1}{s} \left(1 - \frac{f_s}{a_k}\right) \sum_{n=0}^{\infty} (1 - \alpha^2)^n \left(\frac{f_s}{a_k}\right)^n$$

in time domain is a convolution of two non-negative functions: the original eigenvalue  $\lambda_k(t) \in (0, 1]$  and the inverse Laplace transform of  $\sum_{n=0}^{\infty} (1 - \alpha^2)^n \left(\frac{f_s}{a_k}\right)^n$ . Hence,  $\tilde{\lambda}_k(t) \geq 0$ . If  $0 < \alpha < 1$ , then the latter function is less or equal than the inverse Laplace transform of  $\sum_{n=0}^{\infty} \left(\frac{f_s}{a_k}\right)^n = \left(1 - \frac{f_s}{a_k}\right)^{-1}$ . Therefore,  $\tilde{\lambda}_k(t)$  is less or equal than the inverse Laplace transform of function  $(\lambda_k)_s \left(1 - \frac{f_s}{a_k}\right)^{-1} = \frac{1}{s}$ , i.e.  $\tilde{\lambda}_k(t) \leq 1$ . Thus, the deformed map is positive if  $0 < \alpha < 1$  and the original map is P divisible.

Interestingly the map  $\tilde{\Phi}(t)$  being positive and trace-preserving is in general not completely positive and hence the kernel deformation  $K(t) \rightarrow \alpha^2 K(t)$  does not lead to the legitimate dynamical map. In fact, consider the behavior of  $\tilde{\lambda}_k(t)$  when  $t \rightarrow \infty$  and  $\alpha \rightarrow +0$ . By the final value theorem

$$\begin{aligned} \lim_{t \rightarrow \infty} \tilde{\lambda}_k(t) &= \lim_{s \rightarrow 0} s(\tilde{\lambda}_k)_s \\ &= \frac{1}{1 + \alpha^2 \frac{f_0}{a_k - f_0}} = 1 - \alpha^2 \frac{f_0}{a_k - f_0} + o(\alpha^2). \end{aligned} \quad (1)$$

Suppose  $a_1 \leq a_2 \leq a_3$ . The deformed map is CP if the condition  $\tilde{\lambda}_i + \tilde{\lambda}_j \leq 1 + \tilde{\lambda}_k$  is fulfilled for permutations of indices  $\{i, j, k\}$ . In the limit  $\alpha \rightarrow +0$  this condition reduces to inequality

$$\frac{1}{a_2 - f_0} + \frac{1}{a_3 - f_0} \geq \frac{1}{a_1 - f_0},$$

which is surprisingly equivalent to CP divisibility of the original map  $\Phi(t)$ . Thus, the dynamical map given by the memory kernel is CP divisible if and only if the deformed map is CP for all  $0 < \alpha \leq 1$ .

The considered examples allows us to make a *conjecture* that a general Pauli dynamical map  $\Phi(t)$ , defined by a convolution master equation, is CP divisible if and only if the deformed map  $\tilde{\Phi}(t)$  is CP for all  $0 < \alpha < 1$ .

## Summary

- ▶ The original *convolutionless* master equation is shown to describe a completely positive divisible dynamics if and only if the deformed map is completely positive under any time deformation; if the deformed map is not completely positive, then the original dynamics is at least weakly non-Markovian.
- ▶ We have found a necessary condition for positive divisibility of a Hermitian commutative dynamical map given by a *convolution* master equation.
- ▶ Proper time deformations of the memory kernel for Pauli dynamical qubit maps preserve positivity of the deformed map if the original convolution master equation describes a positive divisible process; if the deformed map is not positive, then the original qubit dynamics is essentially non-Markovian.
- ▶ Physical implementation of time deformations via initial system-environment correlations?

Thank you for attention!

**Equivalent** master equation reads

$$\frac{d\tilde{\varrho}(\tau(t))}{d\tau(t)} = \frac{dt}{d\tau(t)} \frac{d\tilde{\varrho}(\tau(t))}{dt} = \frac{1}{\alpha(t)} L(t)[\tilde{\varrho}(\tau(t))]$$

or

$$\frac{d\tilde{\varrho}(\tau)}{d\tau} = \frac{1}{\alpha(t(\tau))} L(t(\tau))[\tilde{\varrho}(\tau)]$$

