Reachability in Infinite-Dimensional Unital Open Quantum Systems with Switchable GKS-Lindblad Generators

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Joint work with Gunther Dirr (University of Würzburg),

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Motivation

Open dynamical system:

$$\dot{\rho}(t) = -i[H(t), \rho(t)] - \mathbf{\Gamma}(\rho(t))$$

with $H(t)$ self-adjoint and $\mathbf{\Gamma} = \sum_k \frac{1}{2}(V_k^\dagger V_k(\cdot) + (\cdot)V_k^\dagger V_k) - V_k(\cdot)V_k^\dagger$ of Kossakowski-Lindblad form

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$\rightarrow$ characterize solutions given $\rho(0) = \rho_0 \in \mathbb{D}(\mathcal{H})$ [1]

[1] $\mathbb{D}(\mathcal{H})$, "quantum states": all $\rho : \mathcal{H} \to \mathcal{H}$ linear, $\rho \geq 0$, (trace class and) trace 1

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$$\begin{aligned} \mathfrak{reach}_\Sigma(\rho_0) &:= \mathcal{S}_\Sigma \rho_0 \\ \text{if all operators are bounded} \quad &= \{\rho(t) \,|\, \rho(\cdot) \text{ is a solution of } (1), t \geq 0\} \end{aligned}$$

where $\mathcal{S}_\Sigma$: semigroup generated by

$$\Big(\mathrm{e}^{-t(i\,\mathrm{ad}_{H_0} + i\sum_{j=1}^{m} u_j\,\mathrm{ad}_{H_j} + u_0\mathbf{\Gamma}_V)}\Big)_{t\in\mathbb{R}_+} \quad \text{with } u_1,\ldots,u_m \in \mathbb{R}\,,\; u_0 \in \{0,1\}\,.$$

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System is operator controllable if $\mathfrak{reach}(\mathrm{id}) = \mathcal{U}(\mathcal{H})$ because then

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In finite dim. equivalent to the "Lie algebra rank condition":

$$\langle iH_0, iH_j \mid j = 1, \ldots, m\rangle_{\mathrm{Lie}} = \mathfrak{u}(\mathcal{H})$$

Starting Point

Theorem (Reachability for normal noise)

Given $V \in \mathbb{C}^{n\times n}$, $(\forall_{\lambda\in\mathbb{C}}\ V \neq \lambda\,\mathrm{id})$ *normal and the control system* Σ_V

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Majorization: $A, B \in \mathbb{C}^{n\times n}$ hermitian with eigenval. $\lambda_1(A) \geq \lambda_2(A) \geq \ldots$ (similarly for B), B is said to majorize A (denoted $A \prec B$) if

$$\sum\nolimits_{j=1}^{k} \lambda_j(A) \leq \sum\nolimits_{j=1}^{k} \lambda_j(B)$$

for all $k = 1, \ldots, n-1$ and $\sum_{j=1}^{n} \lambda_j(A) = \sum_{j=1}^{n} \lambda_j(B)$.

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- closure of $\mathfrak{reach}$, what topology?
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- majorization $\rightarrow$ *defined analogously on $\mathbb{D}(\mathcal{H})$ (because eigenvalues form a non-negative null sequence)*

Operator Controllability [1]

[1]: M. Keyl: "Quantum control in infinite dimensions and Banach-Lie algebras: Pure point spectrum" (2018), arXiv:1812.09211

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Problem: Neither exact nor (norm) approximate operator controllability is possible because $\mathfrak{reach}(\mathrm{id})$ is norm-separable, but $\mathcal{U}(\mathcal{H})$ is not (Ball, Marsden, Slemrod 1982)

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The unitary system $\dot{U}(t) = -iH(t)U(t)$, $U(0) = \text{id}$ is called strongly approximately operator controllable if the strong closure [a] of $\mathfrak{reach}(\text{id})$ (in $\mathcal{U}(\mathcal{H})$) coincides with $\mathcal{U}(\mathcal{H})$

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[a] Strong topology: $(A_n)_n \in \mathcal{B}(\mathcal{H})$ converges strongly to $A \in \mathcal{B}(\mathcal{H})$ if for all $\psi \in \mathcal{H}$: $\lim_{n\to\infty} A_n\psi = A\psi$ in $\mathcal{H}$

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$$\|U\rho U^\dagger - \tilde U \rho \tilde U^\dagger\|_1 < \varepsilon$$

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and thus (for all $\rho_0 \in \mathbb{D}(\mathcal{H})$)

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(1) *H_0, H_j hermitian*

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(iii) For non-increasing, non-negative sequences $x, y \in \ell_1(\mathbb{N})$ one has $x \prec y$ if and only if $\exists U \in \mathcal{U}(\mathcal{H})$ s.t. $U\,\mathrm{diag}(y)U^\dagger$ has diag. entries x → Gohberg [2]

[2]: I. Gohberg and A. Markus, "Some relations between eigenvalues and matrix elements of linear operators", Amer. Math. Soc. Transl. Ser. 2 52 (1966)

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Rough Proof Idea

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$$\rho_0 \xrightarrow{\text{unitary}} W \operatorname{diag}(y) W^\dagger \xrightarrow{\text{noise}} \operatorname{diag}(x) \xrightarrow{\text{unitary}} \rho$$

Approximations: (a) noise sufficiently long, (b) on a finite block (if some eigenvalues coincide) (c) after possibly reordering entries of x, y

Summary

We showed: Given the Markovian control system Σ_V

$$\dot{\rho}(t) = -i\Big[H_0 + \sum\nolimits_{j=1}^{m} u_j(t)H_j, \rho\Big] - \gamma(t)\big(\tfrac{1}{2}(V^\dagger V\rho + \rho V^\dagger V) - V\rho V^\dagger\big)\,, \text{ where}$$

(1) H_0 self-adjoint, controls H_j self-adjoint and bounded,

(2) the closed part: strongly (approximately) operator controllable,

(3) $V(\neq 0)$ compact, normal and switchable by $\gamma(t) \in \{0,1\}$.

Then for all $\rho(0) = \rho_0 \in \mathbb{D}(\mathcal{H})$: $\overline{\mathfrak{reach}_{\Sigma_V}(\rho_0)}^1 = \{\rho \in \mathbb{D}(\mathcal{H}) \,|\, \rho \prec \rho_0\}$.

Open questions:

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(1) what if V bounded & normal?
(2) introducing more unbounded operators ($\rightarrow$ common dense domain)
(3) more than one $V \in \mathcal{K}(\mathcal{H}) \setminus \{0\}$ normal (non-commuting!)