Mathematics of efficient quantum compilers

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1. UNIVERSAL GATES

2. EFFICIENT UNIVERSAL GATES

Quantum circuit

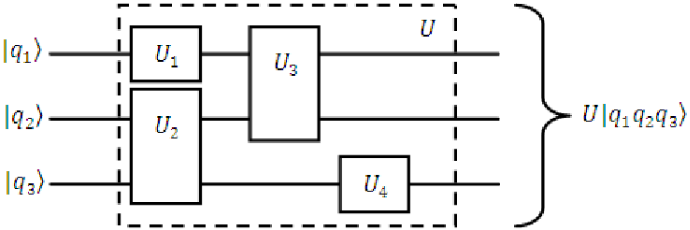
- Quantum system consisting of n qubits: $\mathcal{H} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$
- *Quantum Gates* are unitary matrices from $SU(\mathcal{H}) \simeq SU(2^n)$

$$U^\dagger U = I = UU^\dagger \text{ and } \det U = 1$$

- 1-qubit gates are unitary matrices belonging to $SU(2) \subset SU(2^n)$

$$U = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$$

- k-qubit gate is are matrices from $SU(2^k) \subset SU(2^n)$



Universal quantum gates

- $\mathcal{S} = \{U_1, \ldots, U_k\} \subset SU(\mathcal{H})$ - a finite set of quantum gates

- $\mathcal{S}_n = \{U_{a_1} U_{a_2} \cdots U_{a_n} \,:\, a_i \in \{1, \ldots, k\}\}$ - the set of all words of the length n.

- $\mathcal{S}$ is *universal* or *generates* $SU(\mathcal{H})$, iff the set:

  $$< \mathcal{S} > := \bigcup_{n=1}^{\infty} \mathcal{S}_n,$$

  is dense in $SU(\mathcal{H})$.

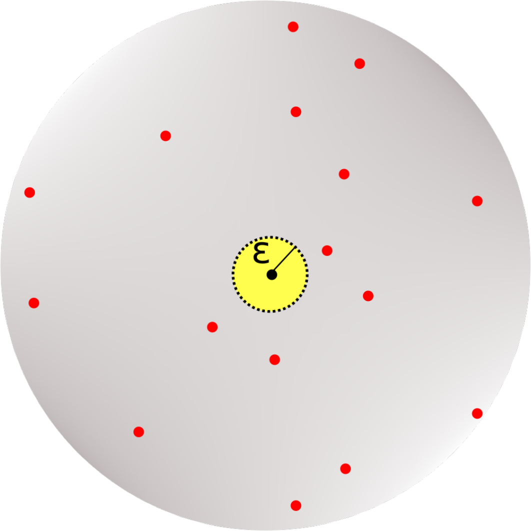
- To check if $< \mathcal{S} >$ is dense in $SU(\mathcal{H})$ we need a measure of distance:

  $$\|U - V\| = \sqrt{\mathrm{Tr}(U - V)(U^\dagger - V^\dagger)}$$

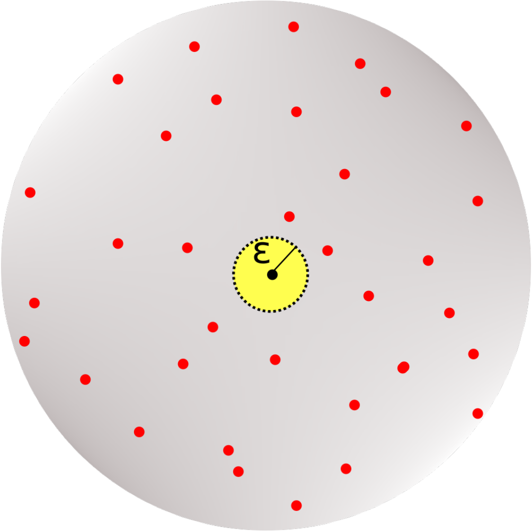
- $\mathcal{S}$ is universal iff for every $U \in SU(\mathcal{H})$ and $\epsilon > 0$ there is $n \in \mathbb{N}$ such that for some $w \in \mathcal{S}_n$

  $$\|U - w\| < \epsilon$$

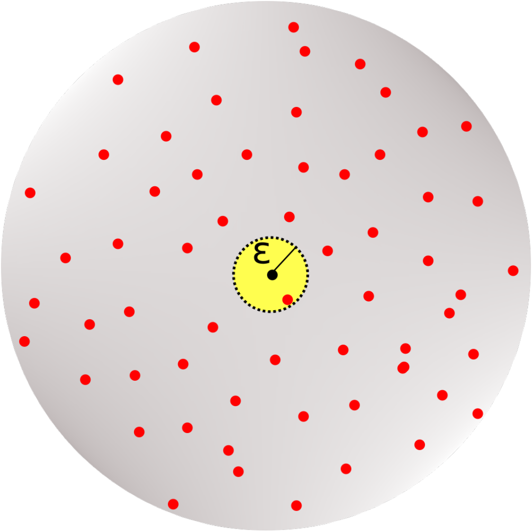
Universal gates



Universal gates



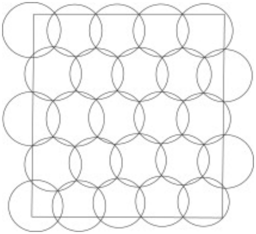
Universal gates



Universal gates and ϵ-nets

- X-finite subset of $SU(\mathcal{H})$ is an ϵ-net iff for every $U \in SU(\mathcal{H})$ there is $U_n \in X$ such that
$$||U - U_n|| < \epsilon$$

- $\mathcal{S}$ is universal iff for every $\epsilon > 0$ there is n such that $\mathcal{S}_n$ is ϵ-net



Universal sets for n-qubit quantum computation

- Quantum system consisting of n qubits: $\mathcal{H} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$
- **Theorem** A universal set for n-qubit quantum computing consists of all 1-qubit gates ($SU(2)$) and an additional 2-qubit gate E that does not map simple tensors onto simple tensors (entangling gate).
- Typically we have access to a finite set $\mathcal{S}$ of 1-qubit gates
- **Fact**: If $\mathcal{S}$ is universal for $SU(2)$ then $\mathcal{S} \cup \{E\}$ is universal for $SU(2^n)$.

Properties of universal sets

- **Theorem** (Kuranishi '49): Let $\mathcal{S} = \{U_1, \ldots, U_k\} \subset SU(2)$. Universal sets of cardinality $k = |\mathcal{S}|$ form an open and dense set in $SU(2)^{\times k}$.
- The probability that randomly chosen set of gates is universal is equal to 1!
- Universality checking algorithm – A.S., K. Karnas, Ann. Henri Poincar, 11, vol. 18, 3515-3552, (2017), A.S., K. Karnas, Phys. Rev. A 95, 062303 (2017)

Properties of universal sets

- How fast can we approximate gates?
- $\mathcal{S} = \{U_1, \ldots, U_k, U_1^{-1}, \ldots, U_k^{-1}\}$ symmetric set of qubit gates
- **Theorem**(Solovay-Kitaev): Assume $\mathcal{S}$ is an universal set. For every $U \in SU(2)$, $\epsilon > 0$ and

$$n > A \log^3 \left(\frac{1}{\epsilon} \right)$$

there is $U_n \in \mathcal{S}_n$ such that $\|U - U_n\| < \epsilon$, where A depends on $\mathcal{S}$.

- All universal sets are rather efficient.

- EFFICIENT UNIVERSAL GATES

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  there is $U_n \in \mathcal{S}_n$ such that $\|U - U_n\| < \epsilon$, where A depends on $\mathcal{S}$.
- All universal sets are roughly the same efficient.
- How A changes with $\mathcal{S}$?
- Is 3 in $\log^3\left(\frac{1}{\epsilon}\right)$ optimal?

Properties of universal sets

- V_{B_ϵ} - the volume (wrt to the normalised Haar measure) of an ϵ-ball (wrt to H-S norm) in $SU(2)$

$$a_1\epsilon^3 \leq V(B_\epsilon) \leq a_2\epsilon^3$$

- The best case: $< S >$ is free - $|\mathcal{S}_n| = |\mathcal{S}|(|\mathcal{S}|-1)^{n-1}$

$$|\mathcal{S}_n|a_2\epsilon^3 > 1$$
$$\Downarrow$$
$$n > \frac{3}{\log(|\mathcal{S}|-1)}\log\left(\frac{1}{\epsilon}\right) - \frac{\log(|\mathcal{S}|a_2)}{\log(|\mathcal{S}|-1)} + 1$$

- c can't be smaller than 1.

Averaging operators

- $T_{SU(2)} : L^2(SU(2)) \to L^2(SU(2))$

$$T_{SU(2)}f(h) = \int_{SU(2)} f d\mu$$

- $\mathcal{S} = \{U_1, \ldots, U_k, U_1^{-1}, \ldots, U_k^{-1}\} \subset SU(2)$
- $T_{\mathcal{S}} : L^2(SU(2)) \to L^2(SU(2))$

$$T_{\mathcal{S}}f(h) = \frac{1}{|\mathcal{S}|}\left(\sum_{i=1}^{k} f(U_i h) + \sum_{i=1}^{k} f(U_i^{-1} h)\right)$$

- Powers $T_{\mathcal{S}}^n$ give averages over words of length n, $\mathcal{S}_n$
- Quantify efficiency of a universal set $\mathcal{S}$ by looking how fast

$$T_{\mathcal{S}}^n \to T_{SU(2)}$$

Averaging operators

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- $||T_{\mathcal{S}}|| = \sup_{||f||=1} ||T_{\mathcal{S}} f||$
- $T_{\mathcal{S}}$ - bounded sefladjoint operator; a constant function is the eigenfunction with the eigenvalue 1, $\|T_{\mathcal{S}}\| = 1$ hence the spectrum is in $[-1, 1]$.
- Consider $T_{\mathcal{S}}|_{L_0^2(SU(2))}$. If $\|T_{\mathcal{S}}|_{L_0^2(SU(2))}\| = \lambda_1 < 1$ then and we have a spectral gap

$$gap(\mathcal{S}) = 1 - \|T_{\mathcal{S}}|_{L_0^2(SU(2))}\|$$

$$\left|\sigma\left(T_{\mathcal{S}}|_{L_0^2(SU(d))}\right)\right|$$

Spectral gap

$$\|T_{\mathcal{S}}^n - T_{SU(2)}\| = \|(T_{\mathcal{S}} - T_{SU(2)})^n\| = \|T_{\mathcal{S}} - T_{SU(2)}\|^n =$$
$$= \|T_{\mathcal{S}}|_{L_0^2(SU(2))}\|^n = (1 - gap(\mathcal{S}))^n \leq e^{-n \cdot gap(\mathcal{S})}$$

- The speed of convergence $T_{\mathcal{S}}^n \to T_{SU(2)}$ is determined by $gap(S)$
- (Bourgain, Gamburd '11) Assume $\mathcal{S}$ is universal and matrices from $\mathcal{S}$ have algebraic entries. Then $gap(\mathcal{S}) > 0$.
- (Kesten '59) $gap(\mathcal{S}) \leq 1 - \frac{2\sqrt{|\mathcal{S}|-1}}{|\mathcal{S}|}$.
- **Conjecture**(Sarnak): For any universal set $T_{\mathcal{S}}$ has a spectral gap.

Spectral gap and efficient gates

- Theorem (Harrow et. al. '02) Assume $\mathcal{S}$ is universal and $T_{\mathcal{S}}$ has a spectral gap. For every $U \in SU(2)$, $\epsilon > 0$ and

$$n > A \log\left(\frac{1}{\epsilon}\right) + B$$

there is $U_n \in \mathcal{S}_n$ such that $\|U - U_n\| < \epsilon$, where

$$A = \frac{3}{\log\left(1/(1 - gap(\mathcal{S}))\right)}, \quad B = \frac{\log\left(8/a_1\right) + 0.5\log(3)}{\log\left(1/(1 - gap(\mathcal{S}))\right)}$$

- (Lubotzky, Phillips, Sarnak '84): Using quaternion algebras constructed $SU(2)$-gates with the optimal spectral gap for $|\mathcal{S}| + 1 = p$, where $p = 1 \mod 4$.

- **Main challenge** Construction of many qubit gates with the optimal spectral gap.

Spectral gap and efficient gates

- Calculation of $gap(\mathcal{S})$ is in general a hard problem.
- Peter-Weyl theorem: $L^2(SU(2))$ decomposes under the left regular representation as a direct sum of all irreducible representations of $SU(2)$
- $SU(2)$ irreps are indexed by one nonnegative integer m. The dimension of m-irrep of $S(2)$ is $m+1$.
- The restriction of $T_{\mathcal{S}}$ to m-irrep $\rho_m : SU(2) \to U(m+1)$ is the $m+1 \times m+1$ matrix:

$$T_{\mathcal{S},m} := \frac{1}{2k} \sum_{i=1}^{k} \left(\rho_m(U_i) + \rho_m(U_i)^{-1} \right).$$

- The spectral gap of $T_{\mathcal{S},m}$ is $gap_m(\mathcal{S}) = 1 - \|T_{\mathcal{S},m}\|_{op}$.
- The spectral gap of $T_{\mathcal{S}}$ at the resolution r by

$$gap_{\leq r}(\mathcal{S}) = \inf_{0<m\leq r} gap_m(\mathcal{S}).$$

Spectral gap and efficient gates

$$gap(\mathcal{S}) = \inf_r gap_{\leq r}(\mathcal{S})$$

- (P. Varju '13) Assume that $\mathcal{S} = \{U_1, \ldots, U_k\}$ is universal. Then for every $U \in SU(2)$, $\epsilon > 0$ and

$$n > A \log\left(\frac{1}{\epsilon}\right),$$

there is $U_n \in W_n(\mathcal{S})$ such that $\|U - U_n\| < \epsilon$, where

$$A = \frac{a}{gap_{\leq b\epsilon^{-c}}(\mathcal{S})},$$

and a, b, c are some positive consts determined by $SU(2)$.

Spectral gap and efficient gates

$$n > A \log\left(\frac{1}{\epsilon}\right), \;\; A = \frac{a}{gap_{\leq b\epsilon^{-c}}(\mathcal{S})}$$

- Relation between the efficiency of ϵ-approxiamation of $U \in SU(2)$ and $gap_{\leq r}(\mathcal{S})$ at the scale $r = b\epsilon^{-c}$
- $gap_{\leq r}(\mathcal{S})$ - can be calculated in finite time
- **To do**: Establishing values of a, b, c
- **Question**: How the spectral gap at resolution r is distributed for randomly chosen universal sets of the fixed cardinality $2k$?

Spectral gap and efficient gates

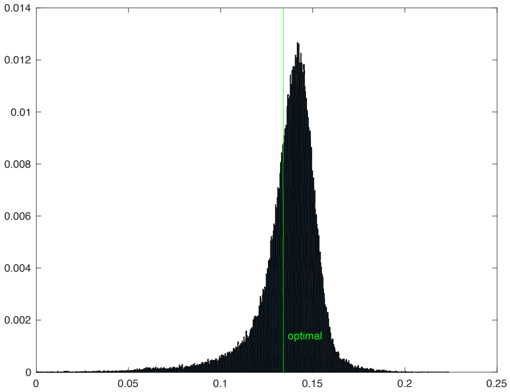

Figure : (The distribution of $gap_{100}(\mathcal{S})$ made for a sample of 10^4 randomly chosen sets $\mathcal{S} = \{U_1, U_2, U_1^{-1}, U_2^{-1}\}$. The optimal spectral gap has value $1 - \frac{\sqrt{3}}{2}$

Spectral gap and efficient gates

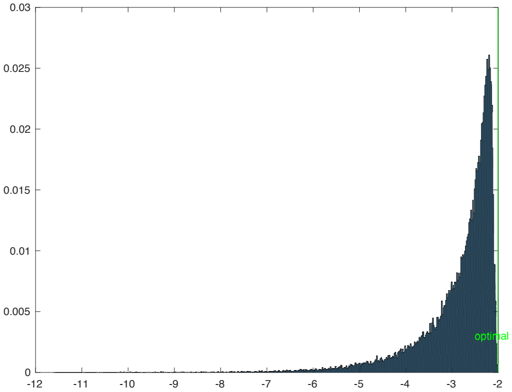

Figure : (a) The distribution of $\log(gap_{\leq 100}(\mathcal{S}))$ made for a sample of 10^4 randomly chosen sets $\mathcal{S} = \{U_1, U_2, U_1^{-1}, U_2^{-1}\}$. The optimal spectral gap has value $1 - \frac{\sqrt{3}}{2}$

Examples of gates with the optimal $gap(\mathcal{S})$

$$V_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix} \; V_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \; V_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1-2i & 0 \\ 0 & 1-2i \end{pmatrix}$$

$$H = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \exp\left(\frac{-i\pi}{8}\right) & 0 \\ 0 & \exp\left(\frac{i\pi}{8}\right) \end{pmatrix}.$$

Open problems

- Understand how the distributions of $\log(gap_{\leq r}(\mathcal{S}))$ and $gap_r(\mathcal{S})$ change when $r \to \infty$
- Contact: a.sawicki@cft.edu.pl