Membership Problem in Quantum Computing

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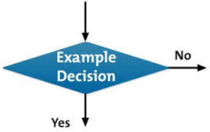
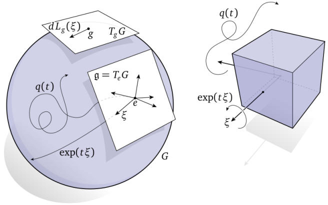
Outline



Problems

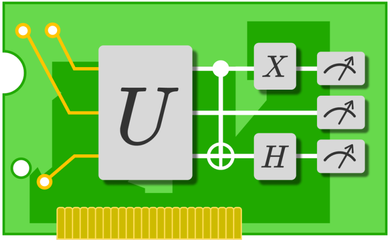
- Universality Problem
- Membership Problem



Settings

- Quantum Computing
- Lie Groups
- Lie Algebras

Universality and Membership Problems



Theorem

By composing all one-qudit gates and an *entangling* two-qudits gate we generate all gates.

Let S be a *finite* set of one-qudit gates.
By composing gates of S we generate a *countable* set of one-qudit gates.

Definition

S is *universal* when by composing gates of S we *approximate arbitrarily well* all one-qudit gates.

Universality Problem Is S universal?
Membership Problem Can we approximate arbitrarily well a given one-qudit gate?

Subgroup Generated by a Finite Set

Let G be a *compact Lie group* and $S \subset G$ be a *finite* subset.

Definition

$$\langle S \rangle = \bigcup_{n=0}^{\infty} \{g_1 \cdots g_n \mid g_k \in S\}$$

is the set of *"words"*.

We approximate arbitrarily well $\overline{\langle S \rangle}$.

Theorem

$\overline{\langle S \rangle}$ is a *compact Lie subgroup* of G.

Universality Problem $\overline{\langle S \rangle} = G$?
Membership Problem $g \in \overline{\langle S \rangle}$?
Can we solve the problems for a Lie algebra first?

Subalgebra Generated by a Finite Set

Let $\mathfrak{g}$ be a *Lie algebra* and $S \subset \mathfrak{g}$ be a *finite* subset.

Definition

The subalgebra $\langle S \rangle$ generated by S is the minimal subalgebra of $\mathfrak{g}$ such that $S \subset \langle S \rangle$.

Universality Problem $\langle S \rangle = \mathfrak{g}$?
Membership Problem $\langle S \rangle = \langle S, x \rangle$? [1]

Can we solve the problems without generating $\langle S \rangle$?

Lemma 1

Let $f : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism.

$$f\left(\langle S \rangle\right) = \langle f\left(S\right) \rangle$$

For the *adjoint representation* of $\mathfrak{g}$

$$ad_{\langle S \rangle} = \langle ad_S \rangle$$

[1]

$$\langle S \rangle \subseteq \langle S, x \rangle$$
$$\langle S \rangle = \langle S, x \rangle \iff x \in \langle S \rangle$$

Centralizer of a Set

Definition

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra and $X \subset \mathfrak{g}$ be a subset.

$$C_{\mathfrak{h}}(X) = \{y \in \mathfrak{h} \mid \forall\, x \in X\, [x, y] = 0\}$$

is the *centralizer* of X in $\mathfrak{h}$.
$C_{\mathfrak{g}}(\mathfrak{g})$ is the *center* of $\mathfrak{g}$.

By the *Jacobi Identity*

Theorem

$C_{\mathfrak{h}}(X)$ is a subalgebra of $\mathfrak{h}$.

Lemma 2

$$C_{\mathfrak{h}}(\langle S \rangle) = C_{\mathfrak{h}}(S)$$

Let's focus on the Membership Problem...

First Condition

By the lemmas

Theorem

$$C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_{\langle S\rangle}\right) = C_{\mathfrak{gl}(\mathfrak{g})}\left(\langle ad_S\rangle\right) = C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_S\right)$$

The equations of the centralizer are

Theorem

$$ad_x C - C ad_x = 0 \ \forall\ x \in S \iff \left(I \otimes ad_x - {ad_x}^T \otimes I\right) vec\left(C\right) = vec\left(0\right)\ \forall\ x \in S$$

A necessary condition is

Theorem

$$\langle S\rangle = \langle S, x\rangle \implies C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_{\langle S\rangle}\right) = C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_{\langle S,x\rangle}\right)$$

Is it sufficient too?

Reductive and ad-semisimple Lie Algebras

Definition

$\mathfrak{g}$ is *reductive* when the radical (maximal solvable ideal) of $\mathfrak{g}$ is equal to the center of $\mathfrak{g}$.

Definition

$\mathfrak{g}$ is *ad-semisimple* when $\forall\ x \in \mathfrak{g}\ ad_x$ is diagonalizable over $\mathbb{C}$.

Theorem

$\mathfrak{g}$ is reductive $\implies \mathfrak{g} = \mathfrak{g}' \oplus C_{\mathfrak{g}}(\mathfrak{g})$ and $\mathfrak{g}'$ is semisimple

Theorem

$\mathfrak{g}$ is ad-semisimple $\iff$ all subalgebras of $\mathfrak{g}$ are reductive

By the theorem

$$\langle S \rangle = \langle S \rangle' \oplus C_{\langle S \rangle}(\langle S \rangle)$$
$$\langle S, x \rangle = \langle S, x \rangle' \oplus C_{\langle S, x \rangle}(\langle S, x \rangle)$$

Second Condition

It is *almost* sufficient!

Theorem

$$C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_{\langle S\rangle}\right)=C_{\mathfrak{gl}(\mathfrak{g})}\left(ad_{\langle S,x\rangle}\right) \implies \langle S\rangle'=\langle S,x\rangle' \implies C_{\langle S\rangle}\left(\langle S\rangle\right)\subseteq C_{\langle S,x\rangle}\left(\langle S,x\rangle\right)$$

$$C_{\langle S\rangle}\left(\langle S\rangle\right)=C_{\langle S,x\rangle}\left(\langle S,x\rangle\right)?$$

Can we rewrite this problem?

Theorem

Let P be the projector onto $C_{\mathfrak{g}}\left(\langle S\rangle\right)$ along $\mathfrak{g}'\cap C_{\mathfrak{g}}\left(\langle S\rangle\right)^{\perp}$. [a]

$$\langle P(S)\rangle = C_{\langle S\rangle}\left(\langle S\rangle\right)$$
$$\langle P(S,x)\rangle = C_{\langle S,x\rangle}\left(\langle S,x\rangle\right)$$

[a] If $\mathfrak{g}$ is *semisimple* P is just the *orthogonal projector* onto $C_{\mathfrak{g}}\left(\langle S\rangle\right)$.

Membership Problem $\langle P(S)\rangle = \langle P(S,x)\rangle$?

We can solve this problem since $\langle P(S)\rangle$ is just the subspace spanned by $P(S)$!

Future Directions

We have solved the Membership Problem for $\mathfrak{g}$.

- Can we find an equivalent condition?
- Can we use the adjoint representation of G?
- Can we solve the Membership Problem for G?

Bibliography

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