The Friedrichs-Lee model and its singular coupling limit

Davide Lonigro

University of Bari & INFN

Joint work with Paolo Facchi and Marilena Ligabò

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Outline
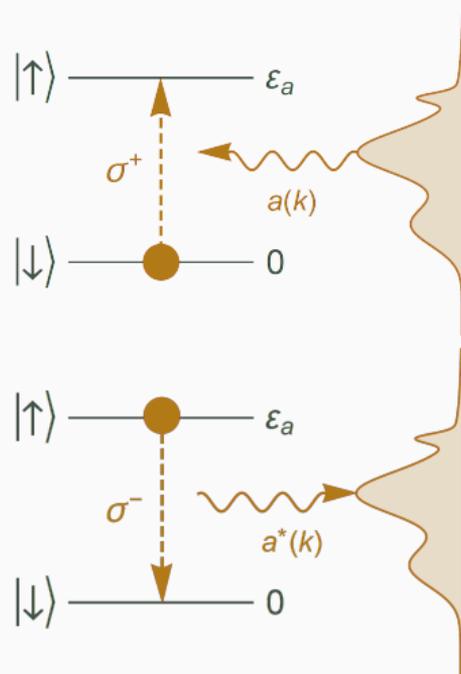
The Friedrichs-Lee model

Mathematical model for two-level system and field:

- Atom with excitation energy ε_a, ground state $|\downarrow\rangle$ and excited state $|\uparrow\rangle$;

- Bosonic field: measure space (X, μ) as momentum space, and $\omega : X \to \mathbb{R}$ as dispersion relation.

Lee field theory:[1] $H_{\text{Lee}} = (H_{\text{atom}} \otimes I) + (I \otimes H_{\text{field}}) + V_g$, with

- $H_{\text{atom}} = \varepsilon_a |\uparrow\rangle \langle\uparrow|$;

- $H_{\text{field}} = \int_X \omega(k)\, a^*(k)\, a(k)\, d\mu$;

- $V_g = \int_X \left(\sigma^+ \otimes g(k)\, a(k) + \sigma^- \otimes \overline{g(k)}\, a^*(k) \right) d\mu$,

where the form factor $g \in L^2_\mu(X)$ weights the coupling.

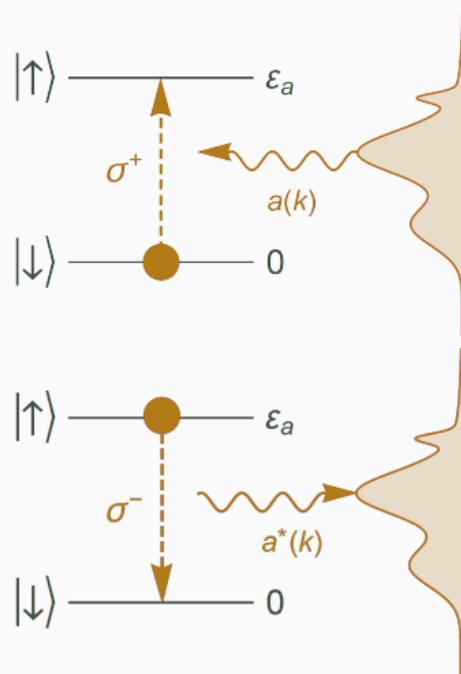[1]T. Lee (1954), Phys. Rev., 95(5), pp. 1329–1334.

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The Friedrichs-Lee model

The single-excitation sector is $\mathbb{C} \oplus L^2_\mu(X)$. Its generic normalized state is

$$\Psi = \begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad x \in \mathbb{C}, \quad \xi \in L^2_\mu(X),$$

with $|x|^2 + \int_X |\xi(k)|^2 \, d\mu = 1$.

Physical interpretation

$|x|^2$ is the probability of measuring the atom in its excited state Ψ_0, and $\xi(k)$ is the boson wavefunction in the momentum representation.

Defining $(\Omega\xi)(k) = \omega(k)\xi(k)$, here our Hamiltonian (Friedrichs-Lee model) acts as follows:[2]

$$\text{Dom}\, H_g = \left\{ \begin{pmatrix} x \\ \xi \end{pmatrix} : x \in \mathbb{C},\ \xi \in \text{Dom}\, \Omega \right\}, \quad H_g = \begin{pmatrix} \varepsilon_a & \langle g, \cdot \rangle \\ g & \Omega \end{pmatrix}.$$

Physical interpretation

A state atom+field has finite mean value (resp. variance) of the total energy if and only if its field component has finite mean value (resp. variance) of the field energy.

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The singular coupling problem

Singular coupling

The model is well-defined provided that $g \in L^2_\mu(X)$, since

$$H_g \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \varepsilon_{\mathsf{a}} x + \langle g, \xi \rangle \\ \Omega \xi + x g \end{pmatrix}.$$

Problem

Can we generalize this model to include a singular (i.e. not normalizable) coupling?

Example: exponential decay of the survival probability of the atom's excited state Ψ_0 is prohibited since $\Psi_0 \in \mathrm{Dom}\, H_g \implies$ Zeno evolution[3] at small times. On the other hand, by formal calculations, an exponential decay may be obtained by choosing

$$\omega(k) = k, \qquad g(k) = \mathrm{const.},$$

but obviously this form factor is not normalizable!

The idea

If we want to consider a broader class of form factors, we need to change the domain.

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An easy trick: the generic state in $\text{Dom}\,\Omega$ can be equivalently written as $\xi - x\frac{\Omega}{\Omega^2+1}g$ for some $\xi \in \text{Dom}\,\Omega$. We have:

$$H_g \begin{pmatrix} x \\ \xi - x\frac{\Omega}{\Omega^2+1}g \end{pmatrix} = \begin{pmatrix} \varepsilon x + \langle g, \xi \rangle \\ \Omega\xi + x\frac{1}{\Omega^2+1}g \end{pmatrix},$$

with

$$\varepsilon = \varepsilon_{\mathsf{a}} - \left\langle g, \frac{\Omega}{\Omega^2+1}g \right\rangle.$$

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ε may be interpreted as a "dressed" (coupling-dependent) excitation energy of the atom, with ε_{a} being the "bare" one. The two quantities differ by a Lamb shift.

In this expression, our model can be generalized. How?

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Singular coupling

For $s \geq 0$, define $\mathcal{H}_s$ and $\mathcal{H}_{-s}$ as the spaces of functions g that are bounded w.r.t. the norms

$$\|g\|_s^2 := \|(|\Omega| + 1)^{s/2} g\|^2 = \int_X (|\omega(k)| + 1)^s |g(k)|^2 \, \mathrm{d}\mu;$$

$$\|g\|_{-s}^2 := \|(|\Omega| + 1)^{-s/2} g\|^2 = \int_X \frac{|g(k)|^2}{(|\omega(k)| + 1)^s} \, \mathrm{d}\mu.$$

A scale of normed spaces[4] is obtained: $\ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \equiv \mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \ldots$, with $\mathcal{H}_{-s}$ and $\mathcal{H}_s$ being dual spaces. In particular, $\mathcal{H}_2 = \mathrm{Dom}\,\Omega$.

The point is:

$g \in \mathcal{H}_{-2}$ implies

- $\frac{\Omega}{\Omega^2+1} g$, $\frac{1}{\Omega^2+1} g \in \mathcal{H}$;
- $\langle g, \xi \rangle$ is well-defined for every $\xi \in \mathrm{Dom}\,\Omega$...

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By using the dressed energy ε as an independent parameter, we can thus define a generalized Friedrichs-Lee model which admits a singular coupling up to $g \in \mathcal{H}_{-2}$:

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Note:

For the singular model ε_a is generally not defined. Physically, this happens because of the atom's excited state Ψ_0 not being in $\text{Dom}\, H_g$ for singular g:
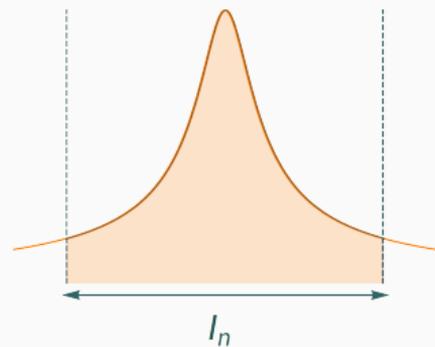
Coupling	$\langle H_g \rangle_{\Psi_0}$	$\langle H_g^2 \rangle_{\Psi_0} - \langle H_g \rangle_{\Psi_0}^2$
$g \in \mathcal{H}$	$\varepsilon_a(g)$	$\|g\|^2$
$g \in \mathcal{H}_{-1} \setminus \mathcal{H}$	$\varepsilon_a(g)$	∞
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Question

What about renormalization?

$\mathcal{H}$ is dense in $\mathcal{H}_{-2}$ $\implies$ given H_g with $g \in \mathcal{H}_{-2} \setminus \mathcal{H}$, there exists a sequence $(H_{g_n})_{n \in \mathbb{N}}$ of regular models (e.g. cutoff procedure) such that $H_{g_n} \to H_g$ in the norm resolvent sense.

I_n

Note

Norm resolvent convergence ensures convergence of both spectrum and dynamics.

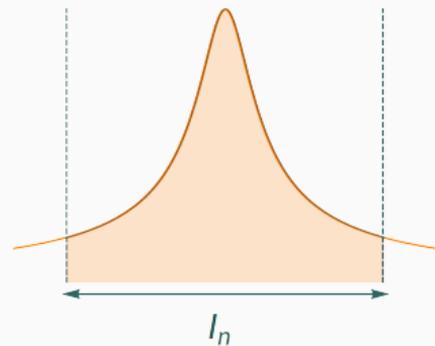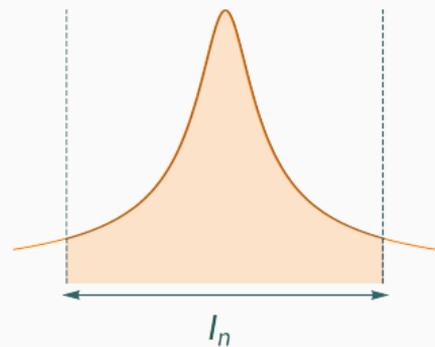Hence the model implements a natural renormalization procedure: for $n \to \infty$

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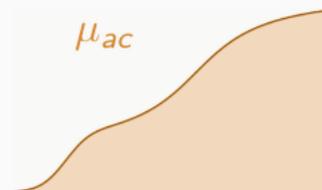- ...but the dressed energy converges to some finite value.

Spectral properties of the Friedrichs-Lee model

A brief reminder about spectra

By Lebesgue's decomposition of measures, the spectrum $\sigma(H)$ of a self-adjoint H has three disjoint components:

- pure-point spectrum $\sigma_{pp}(H) \to$ eigenvalues (and their accumulation points);
- absolutely continuous spectrum $\sigma_{ac}(H) \to$ scattering energies;
- singular continuous spectrum $\sigma_{sc}(H) \to$ all the rest.

$\sigma_{sing}(H) = \sigma_{sc}(H) \cup \sigma_{pp}(H)$ is the singular spectrum.

FL model and spectrum

The spectrum of the uncoupled Friedrichs-Lee model ($g = 0$) is simply the spectrum $\sigma(\Omega)$ of Ω plus the eigenvalue ε_a. What happens when we switch on the coupling?

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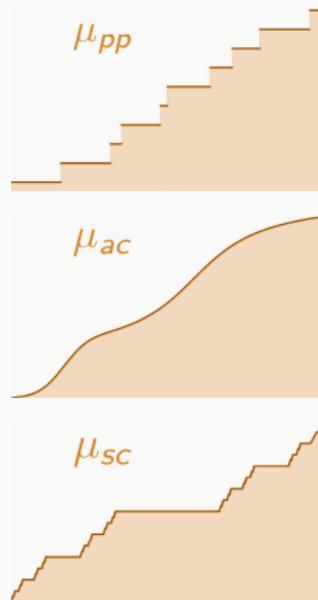
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Characterization of the spectrum

Theorem

Let H_g a FL Hamiltonian with $0 \neq g \in \mathcal{H}_{-2}$ cyclic. Then

- the a.c. spectrum of H_g is the same as the a.c. spectrum of the free field energy Ω;
- the singular spectrum of H_g is the set of solutions of the equation

$$\varepsilon - E = \Sigma_g(E + i0),$$

where $\Sigma_g(z)$ (renormalized self-energy) is

$$\Sigma_g(z) = \left\langle g, \left(\frac{1}{\Omega - z} - \frac{\Omega}{\Omega^2 + 1} \right) g \right\rangle = \int_X \left(\frac{1}{\omega(k) - z} - \frac{\omega(k)}{\omega(k)^2 + 1} \right) |g(k)|^2 \, d\mu.$$

Note: the singular spectrum is highly coupling-dependent: the singular spectra with form factors αg and $\alpha' g$ are completely disjoint for $\alpha \neq \alpha'$!

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Multi-atom extension

Multi-atom Friedrichs-Lee model

Lee field theory can be readily extended to a system of n non-interacting atoms.

The single-excitation sector is now $\mathbb{C}^n \oplus L^2_\mu(X)$.
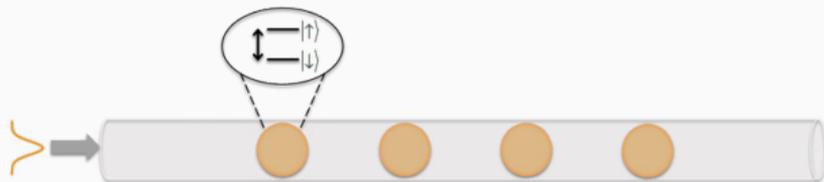Its generic normalized state is

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$|x_i|^2$ is the probability of measuring the ith atom in its excited state Ψ_i, and ξ is the wavefunction of the boson in the momentum representation.

The n-atom Friedrichs-Lee model, as well as its singular limit and renormalization, can be defined analogously. The model has been applied successfully to the study of bound states in the continuum for a regular array of emitters in a 1d geometry (see D. Pomarico's poster).

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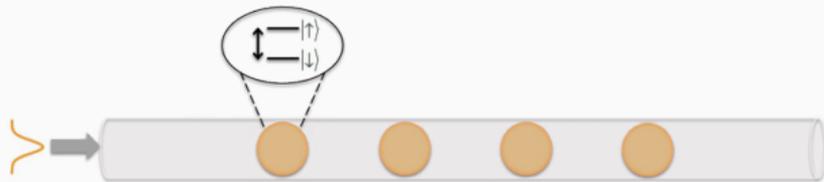Its generic normalized state is

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Conclusion and outlooks

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We have introduced a model of single-excitation atom-field interactions which accounts for

- an arbitrary momentum space $(X, \mathrm{d}\mu)$ and dispersion relation $\omega(k)$;
- a possibly singular coupling atom–field, also offering an interesting perspective on the renormalization phenomenon;
- a complete characterization of the interacting spectrum w.r.t. the free one.

$\implies$ different field theories and geometries are rigorously and effectively implementable.

Thanks for your attention!